Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.77457

Fixed point methodologies for $\psi_{\mathbb{A}}$-contraction mappings over $\mathbf{C}^*$-algebra with applications

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ABSTRACT: This paper delves into the fundamental and versatile field of fixed point theory within functional analysis. Given its wide-ranging applications, numerous researchers have explored various generalizations and extensions of distance spaces using this theory. Our primary objective is to establish novel fixed point results for $\psi_{\mathbb{A}}$-contraction mappings in Banach spaces over $\mathbf{C}^*$-algebras. Our theorems unify and extend several existing results in the literature. To illustrate the practical significance of our theoretical findings, we provide illustrative examples and applications to nonlinear fractional differential equations. These applications demonstrate the versatility of our approach in solving a broad spectrum of problems.

Key Words: Fixed point, $\psi_{\mathbb{A}}$-contraction mapping, Banach spaces, $\mathbf{C}^*$-algebra, fractional differential equation.

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1. Scientific Introduction

Fixed point (**FP**) theory is a fascinating branch of mathematics that deals with the study of mappings or functions that possess points that remain unchanged when the mapping is applied to them. These special points are called fixed points (**FPs**). The theory of fixed points has significant applications across various fields of mathematics and beyond, making it a fundamental and essential concept in modern mathematics. It has numerous benefits and uses in mathematics across a wide range of fields, including optimization, topology, geometry, game theory, economics, engineering, computer science and algorithms. Overall, **FP** theory is a versatile and powerful tool in mathematics and its applications. Its principles and results find applications in a wide range of fields, providing elegant solutions to diverse problems and shedding light on the behavior of functions and systems in various contexts.

The Banach Contraction Principle (**BCP**), also known as the Banach **FP** Theorem [1], is a fundamental result in mathematical analysis and functional analysis. Its significance lies in its simplicity and broad applicability. **BCP** provides a constructive method for finding fixed points of certain functions, making it invaluable in various mathematical and applied domains. Crucially, **BCP** underpins iterative methods for solving equations and problems, such as finding roots of nonlinear equations or solving differential equations numerically. It provides a framework for solving a wide range of equations, including ordinary differential equations (**ODEs**), partial differential equations (**PDEs**), integral equations (**IEs**), fractional differential equations (**FDEs**), and problems in optimization, feasibility, and variational inequalities (see [2,4,3,5]). The concept of a metric, fundamental to **BCP**, has been extended in various

2020 *Mathematics Subject Classification:* 47H10, 46L07.
Submitted June 23, 2025. Published September 01, 2025

ways, leading to generalizations such as quasi-metrics, dislocated quasi-metrics, 2-metrics, b-metrics, D-metrics, G-metrics, S-metrics, and partial metrics [6,8,7,9,10]. These extensions further enrich the scope of **FP** theory and its applications. In conclusion, the **BCP** remains a pivotal result in **FP** theory, driving advancements in both theoretical research and practical applications in diverse fields.

Banach spaces [1] are a key idea in functional analysis, which is a discipline of mathematics that studies vector spaces equipped with a norm (a measure of the length or size of vectors). Banach spaces are considered complete normed vector spaces, which mean that every Cauchy sequence of vectors in a Banach space converges to only one limit (vector). Many famous mathematical spaces are examples of Banach spaces. For instance, Euclidean n-space with the standard norm, L^p spaces with L^p norm, $C[a,b]$ spaces with the sup-norm, $C(K)$ spaces with the sup-norm, l^p spaces with l^p norm, $C[a,b]$ spaces, Hilbert Spaces and Function Spaces. Consider that $\mathbb{B}$ is a Banach space with the norm $\|.\|$ and $\mathbb{D}$ is a closed subset of $\mathbb{B}$. The mapping $\mathcal{T}:\mathbb{D}\longrightarrow\mathbb{D}$ is referred to as a contraction if there exists a constant $L\in[0,1)$ such that for any $\nu,\varpi\in\mathbb{D}$ the inequality $\|\mathcal{T}\nu-\mathcal{T}\varpi\|\leq L\,\|\nu-\varpi\|$ verifys. It is said to be non-expansive if the same assumption satisfys when $L=1$. According to **BCP**, each contraction of $\mathbb{D}$ has precisely one **FP**. The same is true if we suppose that only some powers of $\mathcal{T}$ are contractions, but it does not apply to mappings that are non-expansive. Theoretically, any non-expansive mapping of a closed, bounded, convex subset of a uniformly convex Banach space has at least one **FP**, according to Browder [11]. Similar theorem was displayed by Kirk [12] in the space based on normal structure.

In 2014, Ma et al. [13] established the notion of $\mathbf{C}^*$-algebra valued metric space ($\mathbf{C}^*$**-AVMS**) which is more general than metric space by exchanging the codomain set of real numbers with other codomain set of all positive elements of unital $\mathbf{C}^*$-algebra ($\mathbf{C}^*$**-A**). Under appropriate contractive conditions, they studied the topological characteristics of this space and produced some exceptional **FP** results; for more information [14,15]. In 2015, Ma et al. [16] presented the notion of $\mathbf{C}^*$-algebra valued b-metric space as a generalization of $\mathbf{C}^*$**-AVMS** and showed some **FP** results that used as an application for assessing the existence and uniqueness of a solution for the system of an integral equation for more articles (see [17,18]).

Fractional calculus (**FC**), a generalization of ordinary calculus, was pioneered by Leibniz in 1695 [19]. Its significance has grown dramatically, finding crucial applications in fields like fluid mechanics, entropy, engineering, and physics [20,21,22,23]. **FC** offers a more accurate and insightful framework for interpreting and modeling various phenomena. For instance, **FC**-based entropies, potentially surpassing Shannon's entropy in applicability [24], have been extensively researched due to their widespread use [25]. Fractional differential equations (**FDEs**), which consider a system's entire history, not just its current state, provide a more realistic representation of physical reality compared to integer-order counterparts [26]. This has led to their prominence in modeling complex behaviors observed in viscoelastic materials, control systems, signal processing, and anomalous diffusion [27]. The literature is rich with research on **FC** theory and its applications [28,29,30,31,32]. Recent advancements have focused on fractional functional analysis and the exploration of applications in fractional ordinary and **PDEs** [33,34].

This paper is organized into five sections. Section 1 provides an introduction and overview. In Section 2, we present essential definitions, constructions, remarks, and lemmas that support our main results. Section 3 introduces and proves **FP** theorems in Banach spaces over $\mathbf{C}^*$-algebras, accompanied by illustrative examples and corollaries. Sections 5 and 6 demonstrate applications to integral equations and **FDEs**, highlighting the existence and uniqueness of solutions. Finally, we conclude with a summary of our findings.

2. Basic facts

This section consists of an overview of fundamental principles that play a role in achieving our main objectives.

Definition 2.1. [35] Let $\mathbb{A}$ be a unital algebra with unit $\mathcal{I}_{\mathbb{A}}$. Then, for every $\hbar,\ell\in\mathbb{A}$

(i) A conjugate linear mapping $h\mapsto\hbar^*$ on $\mathbb{A}$ such that $\hbar=\hbar^{**}$ and $(\hbar\ell)^*=\ell^*\hbar^*$, $(\lambda\ l)^*=\lambda\ l^*$ and $(l+\hbar)^*=l^*+\hbar^*$ are considered an involution on $\mathbb{A}$, then $(\mathbb{A},*)$ is said to be a $*$-algebra;

(ii) A complete sub-multiplicative norm such that $\|\hbar^*\| = \|\hbar\|$ with $*$-algebra $\mathbb{A}$ is said to be Banach $*$-algebra;

(iii) A $\mathbf{C}^*$-algebra is said to be Banach $*$-algebra such that $\|\hbar^*\hbar\| = \|\hbar\|^2$.

Remark 2.2. [13] There are many examples of $\mathbf{C}^*$-algebra like the set of all bounded linear operators on a Hilbert space $\mathcal{H}, \mathcal{L}(\mathcal{H})$, the set of complex numbers, and the set of $n \times n$-matrices, $\mathcal{M}_n(\mathbb{C})$.

The following results presented in the work of [36]:

Theorem 2.3. Assume that $\mathbb{A}$ is a $\mathbf{C}^*$**-A**, then:
(1) The set $\mathbb{A}_+$ is a closed cone in $\mathbb{A}$, i.e., a cone $\mathbf{C}$ in a complex is a subset closed under scalar multiplication and addition by $\mathbb{C}$).
(2) The set $\mathbb{A}_+$ is equal to $\{\hbar^*\hbar : \hbar \in \mathbb{A}\}$.
(3) $\|\hbar\| \preceq \|\ell\|$ if $0_{\mathbb{A}} \preceq \hbar \preceq \ell$.
(4) If $\mathbb{A}$ is unital and all of $\hbar$ and ℓ are positive invertible elements, then $0_{\mathbb{A}} \preceq \hbar \preceq \ell \Rightarrow 0_{\mathbb{A}} \preceq \ell^{-1} \preceq \hbar^{-1}$.

Lemma 2.4. Let $\mathbb{A}$ be a unital $\mathbf{C}^*$**-A** with a unit $\mathcal{I}_{\mathbb{A}}$, then:
(1) $\mathcal{I}_{\mathbb{A}} - \hbar$ is invertible and $\left\|\hbar(\mathcal{I}_{\mathbb{A}} - \hbar)^{-1}\right\| < 1$ if $\hbar \in \mathbb{A}_+$ with $\|\hbar\| < 1/2$.
(2) Let $\hbar, \ell \in \mathbb{A}$ with $\hbar, \ell \succeq 0_{\mathbb{A}}$ and $\hbar\ell = \ell\hbar$; then $\hbar\ell \succeq 0_{\mathbb{A}}$.
(3) Define $\mathbb{A}' = \{\hbar \in \mathbb{A} : \hbar\ell = \ell\hbar, \ \forall\ \ell \in \mathbb{A}\}$. Assume that $\hbar \in \mathbb{A}_+$, then

$$(\mathcal{I}_{\mathbb{A}} - \hbar)^{-1}\,\ell \quad \succeq \quad (\mathcal{I}_{\mathbb{A}} - \hbar)^{-1}\,u.$$

if $\ell, u \in \mathbb{A}$ with $\ell \succeq u \succeq 0_{\mathbb{A}}$ and $\mathcal{I}_{\mathbb{A}} - \hbar \in \mathbb{A}'_+$ is invertible operator.

Remark 2.5. [35] For every $\varpi \in \mathbb{A}_+$ we have $\varpi \preceq \mathcal{I}_{\mathbb{A}} \Leftrightarrow \|\varpi\| \leq 1$ when $\mathbb{A}$ is a unital $\mathbf{C}^*$**-A**.

Definition 2.6. For a Banach space $\mathcal{V}$, the mapping $\|.\|_{\mathbb{A}} : \mathcal{V} \longrightarrow \mathbb{A}_+$ and verify the following conditions:
(i) $\|v\|_{\mathbb{A}} \succeq 0_{\mathbb{A}}, \quad \|v\|_{\mathbb{A}} = 0_{\mathbb{A}} \quad \Leftrightarrow \quad v = 0;$
(ii) $\|\lambda\, v\|_{\mathbb{A}} = |\lambda|\, \|v\|_{\mathbb{A}}, \quad \lambda \in \mathbb{C};$
(iii) $\|a\, v\|_{\mathbb{A}} = \|a\|_{\mathbb{A}} \|v\|_{\mathbb{A}}, \quad a \in \mathbb{A}, \quad v \in \mathcal{V};$
(iv) $\|v_1 + v_2\|_{\mathbb{A}} \preceq \|v_1\|_{\mathbb{A}} + \|v_2\|_{\mathbb{A}}, \quad v_1, v_2 \in \mathcal{V}.$
Then, $\|.\|_{\mathbb{A}}$ is called $\mathbb{A}$-valued norm on $\mathcal{V}$ and $(\mathcal{V}, \|.\|_{\mathbb{A}}, \mathbb{A}_+)$ is Banach space over $\mathbf{C}^*$**-A**.

Definition 2.7. $(\mathcal{V}, \|.\|_{\mathbb{A}}, \mathbb{A}_+)$ is Banach space over $\mathbf{C}^*$**-A** if it is complete with $\|.\|_{\mathbb{A}}$.

Lemma 2.8. The following nonlinear two-term **FDEs**:

$${}^cD^{\alpha}\varpi(t) \ + \ p\,[{}^cD^{\beta}\varpi(t)] \quad = \quad f(t, \varpi(t)), \qquad p \in \mathbb{R}, \quad t \in [0, 1],$$

with initial value condition

$$\varpi(0) \ = \ \varpi_0.$$

has a solution

$$\varpi(t) \quad = \quad \varpi_0 \ + \ \frac{p\, t^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)}\, \varpi_0 \ - \ \frac{p}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha-\beta-1}\, \varpi(s)\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(t, \varpi(s)).$$

Proof. Taking the Riemann–Liouville (**R**–**L**) integral operator as

$${}^{RL}I^{\alpha}\,\left[f(t, \varpi(t))\right] \quad = \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\, f(t, \varpi(s))\, ds.$$

Then,

$$ {}^{RL}I^{\alpha}\left[{}^{c}D^{\alpha}\,\varpi(t)\right]\;+\;p\;\;{}^{R-L}I^{\alpha}\left[{}^{c}D^{\beta}\,\varpi(t)\right]\;=\;{}^{R-L}I^{\alpha}\left[f(t,\varpi(t))\right]. $$

From the properties of fractional calculus

$$ {}^{RL}I^{\alpha}\left[{}^{c}D^{\alpha}\,\varpi(t)\right]\;=\;\varpi(t)\;-\;\sum_{k=0}^{n-1}\varpi^{k}(0)\,\frac{t^{k}}{k!},\qquad t>0. $$

Consequently, we get

$$ \varpi(t)\;-\;\sum_{k=0}^{n-1}\varpi^{k}(0)\,\frac{t^{k}}{k!}\;+\;\frac{p}{\Gamma(\alpha-\beta)}\int_{0}^{t}(t-s)^{\alpha-\beta-1}\,\varpi(s)\,ds\;-\;p\sum_{k=0}^{n-1}\frac{t^{k+\alpha-\beta}}{\Gamma(k+\alpha-\beta+1)}\,\varpi^{k}(0) $$
$$ =\;\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}\,f(t,\varpi(s))\,ds. $$

Since, there exist one initial condition, then $n=1$. Therefore,

$$ \varpi(t)\;=\;\varpi_{0}\;+\;\frac{p\,t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\,\varpi_{0}\;-\;\frac{p}{\Gamma(\alpha-\beta)}\int_{0}^{t}(t-s)^{\alpha-\beta-1}\,\varpi(s)\,ds $$
$$ +\;\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}\,f(t,\varpi(s))\,ds. $$

□

3. FP results in Banach spaces over C*-A

We begin this part with the following concepts:

Definition 3.1. [37] Let $\Psi_{\mathbb{A}}$ be the set of all positive continuous functions $\psi_{\mathbb{A}}:\mathbb{A}_{+}\longrightarrow\mathbb{A}_{+}$ fulfill the axioms below:

(i) $\psi_{\mathbb{A}}(\nu_{*})=(\psi_{\mathbb{A}}(\nu))^{*}$;

(ii) $\psi_{\mathbb{A}}(a\nu+b\varpi)=a\;\psi_{\mathbb{A}}(\nu)+b\;\psi_{\mathbb{A}}(\varpi),\quad a,b\in\mathbb{C}$;

(iii) $\psi_{\mathbb{A}}(\nu\varpi)=\psi_{\mathbb{A}}(\nu)\,\psi_{\mathbb{A}}(\varpi)$;

(iv) $\lim_{n\to+\infty}\psi^{n}_{\mathbb{A}}(\nu)=0_{\mathbb{A}}$ for all $\nu>0_{\mathbb{A}}$ where $\psi^{n}_{\mathbb{A}}(\nu)=\psi^{n-1}_{\mathbb{A}}\circ\psi_{\mathbb{A}}(\nu)$;

(v) $\psi_{\mathbb{A}}(\nu)=0_{\mathbb{A}}$ iff $\nu=0_{\mathbb{A}}$.

Now, we deal with continuous function $\psi_{\mathbb{A}}$ in the contraction condition to get the following theorem:

Theorem 3.2. Let $(\varDelta,\|.\|_{\mathbb{A}},\mathbb{A}_{+})$ be a Banach space over **C*-A**. Let $\mathcal{T}:\varDelta\longrightarrow\varDelta$, and $\|.\|_{\mathbb{A}}:\varDelta\longrightarrow\mathbb{A}_{+}$ be functions satisfy: for all $\nu,\varpi\in\varDelta$,

$$ \|\mathcal{T}\nu-\mathcal{T}\varpi\|_{\mathbb{A}}\;\preceq\;\frac{\|\nu-\varpi\|_{\mathbb{A}}}{3}+\psi_{\mathbb{A}}\Big(\varUpsilon_{1,\mathbb{A}}(\nu,\varpi)\Big), \tag{3.1} $$

where

$$ \varUpsilon_{1,\mathbb{A}}(\nu,\varpi)\;=\;\frac{\|\mathcal{T}\nu-\varpi\|_{\mathbb{A}}+\|\mathcal{T}\varpi-\nu\|_{\mathbb{A}}+\|\mathcal{T}\nu-\nu\|_{\mathbb{A}}+\|\mathcal{T}\varpi-\varpi\|_{\mathbb{A}}}{3}, $$

with $\psi_{\mathbb{A}}:\mathbb{A}'_{+}\longrightarrow\mathbb{A}'_{+}$ and $\psi_{\mathbb{A}}\prec\frac{1}{2}\,\mathcal{I}_{\mathbb{A}}$. Then, $\mathcal{T}$ has a unique **FP** in $\varDelta$.

Proof. Let an arbitrary element ν_0 in $\varDelta$. Define a sequence $\{\nu_n\}$ by

$$\mathcal{T}\nu_n = \nu_{n+1}, \quad n \geq 0.$$

From our contraction condition, we get

$$\begin{aligned}\|\nu_{n+1} - \nu_n\|_{\mathbb{A}} &= \|\mathcal{T}\nu_n - \mathcal{T}\nu_{n-1}\|_{\mathbb{A}} \\ &\preceq \frac{\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}}{3} + \psi_{\mathbb{A}}\Big(\varUpsilon_{1,\mathbb{A}}(\nu_n, \nu_{n-1})\Big),\end{aligned}$$

where

$$\begin{aligned}\varUpsilon_{1,\mathbb{A}}(\nu_n, \nu_{n-1}) &= \frac{\|\mathcal{T}\nu_n - \nu_{n-1}\|_{\mathbb{A}} + \|\mathcal{T}\nu_{n-1} - \nu_n\|_{\mathbb{A}} + \|\mathcal{T}\nu_n - \nu_n\|_{\mathbb{A}} + \|\mathcal{T}\nu_{n-1} - \nu_{n-1}\|_{\mathbb{A}}}{3} \\ &= \frac{\|\nu_{n+1} - \nu_{n-1}\|_{\mathbb{A}} + \|\nu_n - \nu_n\|_{\mathbb{A}} + \|\nu_{n+1} - \nu_n\|_{\mathbb{A}} + \|\nu_n - \nu_{n-1}\|_{\mathbb{A}}}{3} \\ &= \frac{\|\nu_{n+1} - \nu_{n-1}\|_{\mathbb{A}} + \|\nu_{n+1} - \nu_n\|_{\mathbb{A}} + \|\nu_n - \nu_{n-1}\|_{\mathbb{A}}}{3} \\ &\preceq \frac{2}{3}\ \|\nu_{n+1} - \nu_n\|_{\mathbb{A}} \ + \ \frac{2}{3}\ \|\nu_n - \nu_{n-1}\|_{\mathbb{A}}.\end{aligned}$$

This implies that

$$\|\nu_{n+1} - \nu_n\|_{\mathbb{A}} \preceq \frac{\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}}{3} + \frac{2}{3}\ \psi_{\mathbb{A}}\Big(\|\nu_{n+1} - \nu_n\|_{\mathbb{A}}\Big) \ + \ \frac{2}{3}\ \psi_{\mathbb{A}}\Big(\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}\Big).$$

Then,

$$(\mathcal{I}_{\mathbb{A}} - \frac{2}{3}\ \psi_{\mathbb{A}})\ \big(\|\nu_{n+1} - \nu_n\|_{\mathbb{A}}\big) \preceq (\frac{1}{3}\ \mathcal{I}_{\mathbb{A}} + \frac{2}{3}\ \psi_{\mathbb{A}})\ \big(\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}\big).$$

Consequently,

$$\|\nu_{n+1} - \nu_n\|_{\mathbb{A}} \preceq \frac{1}{3}\ (\ \mathcal{I}_{\mathbb{A}} + 2\psi_{\mathbb{A}})\ (\mathcal{I}_{\mathbb{A}} - \frac{2}{3}\ \psi_{\mathbb{A}})^{-1}\ \big(\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}\big).$$

Taking $\phi_{\mathbb{A}} = \frac{1}{3}\Big(\ \mathcal{I}_{\mathbb{A}} + 2\psi_{\mathbb{A}}\Big)\Big(\mathcal{I}_{\mathbb{A}} - \frac{2}{3}\ \psi_{\mathbb{A}}\Big)^{-1}$. Thus,

$$\|\nu_{n+1} - \nu_n\|_{\mathbb{A}} \preceq \phi_{\mathbb{A}}^{n}\Big(\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}\Big).hg$$

Let $m, n \in \mathbb{N}$ with $m > n$, we get

$$\|\nu_{n+1} - \nu_n\|_{\mathbb{A}} \preceq \sum_{k=n}^{m-1} \phi_{\mathbb{A}}^{k}\Big(\|\nu_n - \nu_{n-1}\|_{\mathbb{A}}\Big) \to 0_{\mathbb{A}}, \quad as \quad m, n \to \infty.$$

Hence, $\{\nu_n\}$ is a Cauchy sequence (**CS**) in $\varDelta$. Based on $\varDelta$ is complete, there exists $\nu_* \in \varDelta$ such that $\lim_{n\to\infty} \nu_n = \nu_*$. Now, we will show that ν_* is a **FP** of $\mathcal{T}$. Therefore, from (3.1), we get

$$\begin{aligned}\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} &\preceq \|\mathcal{T}\nu_* - \mathcal{T}\nu_n\|_{\mathbb{A}} \ + \ \|\mathcal{T}\nu_n - \nu_*\|_{\mathbb{A}} \\ &\preceq \frac{\|\nu_* - \nu_n\|_{\mathbb{A}}}{3} \ + \ \psi_{\mathbb{A}}\Big(\varUpsilon_{1,\mathbb{A}}(\nu_*, \nu_n)\Big) \ + \ \|\nu_{n+1} - \nu_*\|_{\mathbb{A}},\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\varUpsilon_{1,\mathbb{A}}(\nu_*, \nu_n) &= \frac{\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} + \|\mathcal{T}\nu_n - \nu_n\|_{\mathbb{A}} + \|\mathcal{T}\nu_n - \nu_*\|_{\mathbb{A}} + \|\mathcal{T}\nu_* - \nu_n\|_{\mathbb{A}}}{3} \\ &= \frac{\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} + \|\nu_{n+1} - \nu_n\|_{\mathbb{A}} + \|\nu_{n+1} - \nu_*\|_{\mathbb{A}} + \|\mathcal{T}\nu_* - \nu_n\|_{\mathbb{A}}}{3}.\end{aligned}$$

Substituting in (3.2), we obtain

$$\begin{aligned}\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} \quad \preceq \quad & \frac{\|\nu_* - \nu_n\|_{\mathbb{A}}}{3} \;+\; \frac{1}{3}\;\psi_{\mathbb{A}}\Big(\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}}\Big)\\ & + \frac{1}{3}\;\psi_{\mathbb{A}}\Big(\|\nu_{n+1} - \nu_n\|_{\mathbb{A}}\Big) \;+\; \frac{1}{3}\;\psi_{\mathbb{A}}\Big(\|\nu_{n+1} - \nu_*\|_{\mathbb{A}}\Big)\\ & + \frac{1}{3}\;\psi_{\mathbb{A}}\Big(\|\mathcal{T}\nu_* - \nu_n\|_{\mathbb{A}}\Big) \;+\; \|\nu_{n+1} - \nu_*\|_{\mathbb{A}}.\end{aligned}$$

Taking $n \to \infty$, we get

$$\Big(\,\mathcal{I}_{\mathbb{A}} - \frac{2}{3}\,\psi\,\Big)^{-1}\,\big(\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}}\big) \quad \preceq \quad 0_{\mathbb{A}}.$$

which is a contradiction where $\mathcal{I}_{\mathbb{A}} - \dfrac{2}{3}\,\psi \;\neq\; 0$. Then, $\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} \;=\; 0_{\mathbb{A}}$, i.e., $\mathcal{T}\nu_* \;=\; \nu_*$ is a **FP** of $\mathcal{T}$.

Now, if $(\varpi_* \neq 0) \neq \nu_*$ is another **FP** of the mapping $\mathcal{T}$, then

$$\begin{aligned}\|\nu_* - \varpi_*\|_{\mathbb{A}} \quad & = \quad \|\mathcal{T}\nu_* - \mathcal{T}\varpi_*\|_{\mathbb{A}}\\ & \preceq \quad \frac{\|\nu_* - \varpi_*\|_{\mathbb{A}}}{3} \;+\; \psi_{\mathbb{A}}\Big(\Upsilon_{1,\mathbb{A}}(\nu_*, \varpi_*)\Big),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}& \Upsilon_{1,\mathbb{A}}(\nu_*, \varpi_*)\\ = \;& \frac{\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} + \|\mathcal{T}\varpi_* - \varpi_*\|_{\mathbb{A}} + \|\mathcal{T}\varpi_* - \nu_*\|_{\mathbb{A}} + \|\mathcal{T}\nu_* - \varpi_*\|_{\mathbb{A}}}{3}\\ = \;& \frac{0_{\mathbb{A}} \;+\; 0_{\mathbb{A}} \;+\; \|\nu_* - \varpi_*\|_{\mathbb{A}} \;+\; \|\varpi_* - \nu_*\|_{\mathbb{A}}}{3} \quad = \quad \frac{2}{3}\;\|\nu_* - \varpi_*\|_{\mathbb{A}}.\end{aligned}$$

Substituting in (3.3), we obtain

$$\Big(\,\mathcal{I}_{\mathbb{A}} - \psi\,\Big)^{-1}\,\big(\|\nu_* - \varpi_*\|_{\mathbb{A}}\big) \quad \preceq \quad 0_{\mathbb{A}}.$$

Again, we get a contradictory where $\mathcal{I}_{\mathbb{A}} - \psi \;\neq\; 0_{\mathbb{A}}$. Therefore,

$$\|\nu_* - \varpi_*\|_{\mathbb{A}} \quad = \quad 0_{\mathbb{A}} \quad \Longrightarrow \quad \nu_* \quad = \quad \varpi_*.$$

This implies that the **FP** is a unique. □

Corollary 3.3. Let $(\varDelta, \|.\|_{\mathbb{A}}, \mathbb{A}_+)$ be a Banach space over **C***-**A**. Let $\mathcal{T} : \varDelta \longrightarrow \varDelta$, and $\|.\|_{\mathbb{A}} : \varDelta \longrightarrow \mathbb{A}_+$ be functions satisfy: for all $\nu, \varpi \in \varDelta$, $k \in \mathbb{A}_+$,

$$\|\mathcal{T}\nu - \mathcal{T}\varpi\|_{\mathbb{A}} \quad \preceq \quad \frac{\|\nu - \varpi\|_{\mathbb{A}}}{2} + \psi_{\mathbb{A}}\Big(\Upsilon_{2,\mathbb{A}}(\nu, \varpi)\Big), \tag{3.4}$$

where

$$\Upsilon_{2,\mathbb{A}}(\nu, \varpi) \quad = \quad \frac{\|\mathcal{T}\nu - \varpi\|_{\mathbb{A}} + \|\mathcal{T}\varpi - \nu\|_{\mathbb{A}}}{2},$$

with $\psi_{\mathbb{A}} : \mathbb{A}'_+ \longrightarrow \mathbb{A}'_+$ and $\psi_{\mathbb{A}} \prec \mathcal{I}_{\mathbb{A}}$. Then, $\mathcal{T}$ has a unique **FP** in $\varDelta$.

Corollary 3.4. Let $(\mathit{\Delta}, \|.\|_{\mathbb{A}}, \mathbb{A}_{+})$ be a Banach space over **C*****-A**. Let $\mathcal{T}: \mathit{\Delta} \longrightarrow \mathit{\Delta}$ and $\|.\|_{\mathbb{A}}: \mathit{\Delta} \longrightarrow \mathbb{A}_{+}$ be functions satisfy: for all $\nu, \varpi \in \mathit{\Delta}$,

$$\|\mathcal{T}\nu - \mathcal{T}\varpi\|_{\mathbb{A}} \quad \preceq \quad \frac{\|\nu - \varpi\|_{\mathbb{A}}}{2} + \psi_{\mathbb{A}}\Big(\varUpsilon_{3,\mathbb{A}}(\nu, \varpi)\Big), \tag{3.5}$$

where

$$\varUpsilon_{3,\mathbb{A}}(\nu, \varpi) \quad = \quad \frac{\|\mathcal{T}\varpi - \varpi\|_{\mathbb{A}} + \|\mathcal{T}\varpi - \nu\|_{\mathbb{A}}}{2},$$

with $\psi_{\mathbb{A}}: \mathbb{A}_{+}^{'} \longrightarrow \mathbb{A}_{+}^{'}$ and $\psi_{\mathbb{A}} \prec \frac{1}{2}\,\mathcal{I}_{\mathbb{A}}$. Then, $\mathcal{T}$ has a unique **FP** in $\mathit{\Delta}$.

Secondly, we discuss another corollary combines continuous function $\psi_{\mathbb{A}}$ in the contraction condition as a special case of Theorem 3.2.

Corollary 3.5. Let $(\mathit{\Delta}, \|.\|, \mathbb{A}_{+})$ be a Banach space over **C*****-A**. Let $\mathcal{T}: \mathit{\Delta} \longrightarrow \mathit{\Delta}$, and $\|.\|_{\mathbb{A}}: \mathit{\Delta} \longrightarrow \mathbb{A}_{+}$ be functions satisfy: for all $\nu, \varpi \in \mathit{\Delta}$,

$$\|\mathcal{T}\nu - \mathcal{T}\varpi\|_{\mathbb{A}} \quad \preceq \quad \psi_{\mathbb{A}}\Big(\varUpsilon_{4,\mathbb{A}}(\nu, \varpi)\Big), \tag{3.6}$$

where

$$\varUpsilon_{4,\mathbb{A}}(\nu, \varpi) \quad = \quad \frac{\|\nu - \varpi\|_{\mathbb{A}} + \|\mathcal{T}\varpi - \varpi\|_{\mathbb{A}}}{2},$$

with $\psi_{\mathbb{A}}: \mathbb{A}_{+}^{'} \longrightarrow \mathbb{A}_{+}^{'}$ and $\psi_{\mathbb{A}} \prec \mathcal{I}_{\mathbb{A}}$. Then, $\mathcal{T}$ has a unique **FP** in $\mathit{\Delta}$.

Example 3.6. Let $\mathfrak{X} = [0,1]$ and $\mathbb{A}_{+} = \mathbb{R}_{+}^{3}$ with a usual norm, be a real Banach space. Let $\|.\|_{\mathbb{A}}: \mathfrak{X} \to \mathbb{A}_{+}$ be given as follows: $\|\nu - \varpi\|_{\mathbb{A}} = (|\nu - \varpi|, |\nu - \varpi|, |\nu - \varpi|)\;.\;\mathcal{I}_{\mathbb{A}}$. Then, $(\mathfrak{X}, \|.\|_{\mathbb{A}}, \mathbb{A}_{+})$ is Banach space over **C***-algebra.

4. Common fixed point theorems

Theorem 4.1. Let $(\mathfrak{X}, \mathbb{A}_{+}, \|.\|_{\mathbb{A}})$ be Banach space over **C***-algebra. Suppose that two mappings $\mathcal{T}, \mathcal{S}: \mathfrak{X} \longrightarrow \mathfrak{X}$ satisfy the following condition: For all $\nu, \varpi \in \mathfrak{X}$,

$$\|\mathcal{T}\nu - \mathcal{S}\varpi)\|_{\mathbb{A}} \quad \preceq \quad \psi_{\mathbb{A}}\,\Big(\Upsilon_{5,\mathbb{A}}(\nu, \varpi)\Big). \tag{4.1}$$

Since

$$\Upsilon_{5,\mathbb{A}}(\nu, \varpi) \quad = \quad \frac{\|\mathcal{T}\nu - \varpi\|_{\mathbb{A}} + \|\nu - \mathcal{S}\varpi\|_{\mathbb{A}}}{2}, \tag{4.2}$$

where $\|.\|_{\mathbb{A}}: \mathfrak{X} \longrightarrow \mathbb{A}_{+}$. If $\mathcal{S}(\mathfrak{X})$ and $\mathcal{T}(\mathfrak{X})$ are complete in $\mathfrak{X}$ with $\psi_{\mathbb{A}}: \mathbb{A}_{+}^{'} \longrightarrow \mathbb{A}_{+}^{'}$ and $\psi_{\mathbb{A}} \prec \mathcal{I}_{\mathbb{A}}$. Then, $\mathcal{T}$ and $\mathcal{S}$ have a unique common **FP** in $\mathfrak{X}$.

Proof. Let $\nu \in \mathit{\Delta}$ and construct a sequence $\{\nu_n\}_0^\infty \subseteq \mathit{\Delta}$ by the following way: $\nu_{2n+1} = \mathcal{T}\nu_{2n}$, $\nu_{2n+2} = \mathcal{S}\nu_{2n+1}$. From (4.1), we get

$$\begin{aligned}
\|\nu_{2n+2} - \nu_{2n+1}\|_{\mathbb{A}} &= \|\mathcal{S}\nu_{2n+1} - \mathcal{T}\nu_{2n}\|_{\mathbb{A}} \\
&\preceq \psi_{\mathbb{A}}\left(\frac{\|\mathcal{T}\nu_{2n} - \nu_{2n+1}\|_{\mathbb{A}} + \|\nu_{2n} - \mathcal{S}\nu_{2n+1}\|_{\mathbb{A}}}{2}\right) \\
&= \psi_{\mathbb{A}}\left(\frac{\|\nu_{2n+1} - \nu_{2n+1}\|_{\mathbb{A}} + \|\nu_{2n} - \nu_{2n+2}\|_{\mathbb{A}}}{2}\right) \\
&= \frac{1}{2}\;\psi_{\mathbb{A}}\big(\|\nu_{2n} - \nu_{2n+1}\|_{\mathbb{A}}\big) + \frac{1}{2}\;\psi_{\mathbb{A}}\Big(\|\nu_{2n+1} - \nu_{2n+2}\|_{\mathbb{A}}\Big).
\end{aligned}$$

Then,

$$(\mathcal{I}_{\mathbb{A}} - \frac{1}{2}\ \psi_{\mathbb{A}})\ \ \|\nu_{2n+2} - \nu_{2n+1}\|_{\mathbb{A}} \ \ \preceq\ \ \frac{1}{2}\ \ \psi_{\mathbb{A}}\Big(\|\nu_{2n+1} - \nu_{2n}\|_{\mathbb{A}}\Big).$$

Consequently,

$$\|\nu_{2n+2} - \nu_{2n+1}\|_{\mathbb{A}} \ \ \preceq\ \ \frac{1}{2}\ \Big(\ \mathcal{I}_{\mathbb{A}} - \frac{1}{2}\ \psi_{\mathbb{A}}\ \Big)^{-1}\ \psi_{\mathbb{A}}\ \Big(\|\nu_{2n+1} - \nu_{2n}\|_{\mathbb{A}}\Big),$$

i.e.,

$$\|\nu_{n+1} - \nu_{n}\|_{\mathbb{A}} \ \ \preceq\ \ \frac{1}{2}\ \Big(\ \mathcal{I}_{\mathbb{A}} - \frac{1}{2}\ \psi_{\mathbb{A}}\Big)^{-1}\ \psi_{\mathbb{A}}\ \Big(\|\nu_{n} - \nu_{n-1}\|_{\mathbb{A}}\Big).$$

Taking $\phi \ = \ \frac{1}{2}\ \Big(\ \mathcal{I}_{\mathbb{A}} - \frac{1}{2}\ \psi_{\mathbb{A}}\Big)^{-1}\ \psi_{\mathbb{A}}\ \prec\ \mathcal{I}_{\mathbb{A}}.$ Thus,

$$\|\nu_{n+1} - \nu_{n}\|_{\mathbb{A}} \ \ \preceq\ \ \phi^{n}\ \Big(\|\nu_{n} - \nu_{n-1}\|_{\mathbb{A}}\Big).$$

Let $m, n \in \mathbb{N}$ with $m > n,$ we get

$$\|\nu_{n+1} - \nu_{n}\|_{\mathbb{A}} \ \ \preceq\ \ \sum_{k=n}^{m-1} \phi^{k}\ \Big(\|\nu_{n} - \nu_{n-1}\|_{\mathbb{A}}\Big) \ \ \rightarrow\ \ 0_{\mathbb{A}}, \quad \text{as} \quad m, n \rightarrow \infty.$$

Hence, $\{\nu_n\}$ is a **CS** in $\varDelta$. Based on $\varDelta$ is complete, there exists $\nu_* \in \varDelta$ such that $\lim_{n\to\infty} \nu_n = \nu_*$. Now, we will show that ν_* is a **FP** of $\mathcal{T}$. Therefore, from (4.1), we get

$$\begin{aligned} \|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} \ &\preceq\ \|\mathcal{T}\nu_* - \mathcal{S}\nu_{2n+1}\|_{\mathbb{A}} + \|\mathcal{S}\nu_{2n+1} - \nu_*\|_{\mathbb{A}} \\ &\preceq\ \psi_{\mathbb{A}}\Big(\Upsilon_{5,\mathbb{A}}(\nu_*, \nu_{2n+1})\Big) \ + \ \|\nu_{2n+2} - \nu_*\|_{\mathbb{A}}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \Upsilon_{5,\mathbb{A}}(\nu_*, \nu_{2n+1}) \ &= \ \frac{\|\mathcal{T}\nu_* - \nu_{2n+1}\|_{\mathbb{A}} + \|\nu_* - \mathcal{S}\nu_{2n+1}\|_{\mathbb{A}}}{2} \\ &= \ \frac{1}{2}\ \|\mathcal{T}\nu_* - \nu_{2n+1}\|_{\mathbb{A}} \ + \ \frac{1}{2}\ \|\nu_* - \nu_{2n+2}\|_{\mathbb{A}}. \end{aligned}$$

Substituting in (4.3), we have

$$\begin{aligned} \|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} \ \preceq\ &\frac{1}{2}\ \ \psi_{\mathbb{A}}\Big(\|\mathcal{T}\nu_* - \nu_{2n+1}\|_{\mathbb{A}}\Big) \\ &+\ \frac{1}{2}\ \ \psi_{\mathbb{A}}\Big(\|\nu_* - \nu_{2n+2}\|_{\mathbb{A}}\Big) \ + \ \|\nu_{2n+2} - \nu_*\|_{\mathbb{A}}, \end{aligned}$$

Taking $n \rightarrow \infty,$ we get

$$\Big(\mathcal{I}_{\mathbb{A}} - \frac{1}{2}\ \ \psi_{\mathbb{A}}\Big)^{-1}\ \big(\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}}\big) \ \ \preceq\ \ 0_{\mathbb{A}}.$$

which is a contradiction where $\mathcal{I}_{\mathbb{A}} - \frac{1}{2}\ \ \psi_{\mathbb{A}} \ \neq\ 0_{\mathbb{A}}.$

Then, $\|\mathcal{T}\nu_* - \nu_*\|_{\mathbb{A}} \ = \ 0_{\mathbb{A}},$ i.e., $\mathcal{T}\nu_* \ = \ \nu_*$ is a **FP** of $\mathcal{T}$.

Similarly, we can get $\mathcal{S}\nu_* \ = \ \nu_*.$

Now, if $(\varpi_* \neq 0) \neq \nu_*$ is another **FP** of the mapping $\mathcal{T}$, then

$$\begin{aligned} \|\nu_* - \varpi_*\|_{\mathbb{A}} &= \|\mathcal{T}\nu_* - \mathcal{T}\varpi_*\|_{\mathbb{A}} \\ &\preceq \psi_{\mathbb{A}}\Big(\Upsilon_{5,\mathbb{A}}(\nu_*, \varpi_*)\Big), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \Upsilon_{5,\mathbb{A}}(\nu_*, \varpi_*) &= \frac{\|\mathcal{T}\nu_* - \varpi_*\|_{\mathbb{A}} + \|\nu_* - \mathcal{S}\varpi_*\|_{\mathbb{A}}}{2} \\ &= \frac{\|\nu_* - \varpi_*\|_{\mathbb{A}} + \|\varpi_* - \nu_*\|_{\mathbb{A}}}{2} = \|\nu_* - \varpi_*\|_{\mathbb{A}}. \end{aligned}$$

Substituting in (4.4), we obtain

$$\Big(\mathcal{I}_{\mathbb{A}} - \psi_{\mathbb{A}}\Big)^{-1} \left(\|\nu_* - \varpi_*\|_{\mathbb{A}}\right) \preceq 0_{\mathbb{A}}.$$

Again, we get a contradictory where $\mathcal{I}_{\mathbb{A}} - \psi_{\mathbb{A}} \neq 0_{\mathbb{A}}$. Therefore,

$$\|\nu_* - \varpi_*\|_{\mathbb{A}} = 0_{\mathbb{A}} \implies \nu_* = \varpi_*.$$

This implies that the **FP** is a unique. □

5. Applications to nonlinear two-term fractional differential equations

5.1. Application I

The theory of **FDEs** has gained significant attention due to its diverse applications in engineering and science [38,39,40,41,42]. This work focuses on investigating the existence and uniqueness of solutions to Caputo fractional boundary value problems (**BVPs**) with specific boundary conditions. Recent years have witnessed substantial progress in the study of multi-term **FDEs** [43,44,45,46,47,48,49,50]. Nonlocal **BVPs** are particularly intriguing due to their greater naturalness and broader applicability compared to local **BVPs**. Notably, the local conditions $\varpi(0) = 0$ and $\varpi(1) = 0$ can be considered a special case of the more general boundary condition (5.2) when $\gamma = 1$. This study draws upon existing knowledge from various scientific publications on nonlocal **BVPs**, contributing to the ongoing advancement of this critical area [43,50,51,52,53].

Before delving into the existence and uniqueness results, it is crucial to recall the definition of the Caputo fractional derivative (**CFD**) and associated concepts. Let ϖ be a continuous function and α be a positive real number. The Caputo fractional derivative of order α for $\varpi(t)$ is defined as:

$${}^{c}D^{\alpha}\varpi = I^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}\varpi,$$

where $\lceil\alpha\rceil$ is the smallest integer, which is greater than α and I^{α} is the Riemann-Liouville fractional integral operator (**R-LFIO**) of order $\alpha \geq 0$ characterized by:

$$I^{\alpha}\varpi(t) = \int_0^t \frac{\varpi(s)}{\Gamma(\alpha)\,(t-s)^{1-\alpha}}\,ds.$$

Observing that when $\alpha = 0$, the operator I^0 is called an identity operator. The fractional integral verifies the following equations:

$$\begin{aligned} I^{\alpha}I^{\beta}\varpi(t) &= I^{\alpha+\beta}\varpi(t), \qquad \alpha, \beta \geq 0, \\ I^{\alpha}t^{k} &= \frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)}\, t^{k+\alpha}, \qquad k > -1. \end{aligned}$$

Furthermore, according to the **CFD** of α-order and its integer-ordered, we obtain

$$I^{\alpha}\ {}^{c}D^{\alpha}\varpi(t) \;=\; \varpi(t)-\sum_{k=0}^{m-1}\frac{t^k}{\Gamma(k+1)}\,\varpi^{(k)}(0), \qquad m-1<m\le\alpha.$$

Inspired by [44,48,49], we study the nonlinear two-term **FDEs** in the following form:

$${}^{c}D^{\alpha}\varpi(t)\;+\;p\,[{}^{c}D^{\beta}\varpi(t)] \;=\; f(t,\varpi(t)), \qquad p\in\mathbb{R},\;\; t\in[0,1], \tag{5.1}$$

with the initial condition

$$\varpi(0)\;=\;a_0, \tag{5.2}$$

where α,β are arbitrary real constants with $0\le\beta\le1<\alpha\le2$, and $f:[0,1]\times\mathbb{A}\longrightarrow\mathbb{A}$ is a continuous function, where $\mathbb{A}$ is a $\mathbf{C}^*$-algebra.

Now, assume that $\Delta=C([0,1],\mathbb{A})$ is the space of all continuous functions defined on $[0,1]$. Under the norm $\|\nu\|_{\mathbb{A}} \;=\; \left(\max_{t\in[0,1]}\ \frac{|\nu(t)|}{e^{\sigma t}}\right)\mathcal{I}_{\mathbb{A}}$, for all $\sigma>0,\ t\in[0,1]$. Clearly, Δ is a Banach space.

To accomplish our main goal in this section, we consider the assertions below:

(A_1) the function f is continuous on $[0,1]\times\mathbb{A}\longrightarrow\mathbb{A}$,

(A_2) there exits a positive constant M such that

$$|f(t,\nu)-f(t,\varpi)|\;\le\;M\;|\nu-\varpi|\,,\quad \text{for all}\quad t\in[0,1],\quad \nu,\varpi\in C([0,1],\mathbb{A}),$$

(A_3) for all $t\in[0,1]$, there exists a mapping $\mathcal{T}:C([0,1],\mathbb{A})\longrightarrow C([0,1],\mathbb{A})$ such that

$$\begin{aligned}\mathcal{T}\varpi(t) \;&=\; \Bigg[a_0+\frac{p\,t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\;a_0-\frac{p}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\varpi(s)\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\;f(s,\varpi(s))\;\;ds\Bigg].\mathcal{I}_{\mathbb{A}},\end{aligned}$$

(A_4) there exists a constant $\sigma>0$ such that

$$\|\nu-\varpi\|_{\mathbb{A}}\;=\;\left(\max_{t\in[0,1]}\;\frac{|\nu(t)-\varpi(t)|}{e^{\sigma t}}\right)\mathcal{I}_{\mathbb{A}},$$

for all $t\in[0,1]$, $\;\nu,\varpi\in C([0,1],\mathbb{A})$.

Now, we present our main theorems in this part.

Theorem 5.1. According to hypotheses $(A_1)-(A_4)$, the fractional problem (5.1) with the condition (5.2) has a solution on $\mathbb{A}$.

Proof. Assume that the conditions $(A_1)-(A_4)$ hold. Then, $\varpi\in\Delta$ is a solution of (5.1) if and only if $\varpi\in\Delta$ is a solution of the following integral equation:

$$\begin{aligned}\varpi(t)\;&=\;\Bigg[a_0+\frac{p\,t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)}\;a_0-\frac{p}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\varpi(s)\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\;f(s,\varpi(s))\;\;ds\Bigg]\;\mathcal{I}_{\mathbb{A}}.\end{aligned} \tag{5.3}$$

This means that $\mathcal{T}\varpi(t) = \varpi(t)$ for all $t \in [0,1]$. By the assumption (A_2), we have

$$\begin{aligned} |\mathcal{T}\nu(t) - \mathcal{T}\varpi(t)| &\leq \frac{|p|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}|\nu(s) - \varpi(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, |f(s,\nu(s)) - f(s,\varpi(s))| \, ds \\ &\leq \frac{|p|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}|\nu(s) - \varpi(s)|\, ds \\ &+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, |\nu(s) - \varpi(s)| \, ds. \end{aligned}$$

Then,

$$\begin{aligned} \frac{|\mathcal{T}\nu(t) - \mathcal{T}\varpi(t)|}{e^{\sigma t}} &\leq \frac{|p|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} e^{\sigma s} \, \frac{|\nu(s) - \varpi(s)|}{e^{\sigma s}}\, ds \\ &+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, e^{\sigma s} \, \frac{|\nu(s) - \varpi(s)|}{e^{\sigma s}} \, ds. \end{aligned}$$

Taking the maximum, we get

$$\begin{aligned} \|\mathcal{T}\nu(t) - \mathcal{T}\varpi(t)\|_{\mathbb{A}} &\preceq \Big[\frac{|p|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} e^{\sigma s} \, ds \\ &+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, e^{\sigma s} \, ds \Big] \, \|\nu(t) - \varpi(t)\|_{\mathbb{A}}, \end{aligned}$$

which together with the fact that

$$\int_0^t (t-s)^{\alpha-1} e^{\gamma s} \, ds \leq \frac{\Gamma(\alpha)}{\gamma^{\alpha}}, \qquad \text{for all } t > 0, \quad \gamma > 0,$$

we have estimate

$$\|\mathcal{T}\nu(t) - \mathcal{T}\varpi(t)\|_{\mathbb{A}} \preceq \Big[\frac{|p|}{\sigma^{\alpha-\beta}} + \frac{M}{\sigma^{\alpha}} \Big] \, \|\nu(t) - \varpi(t)\|_{\mathbb{A}}.$$

Taking σ is large enough such that $k = \Big[\frac{|p|}{\sigma^{\alpha-\beta}} + \frac{M}{\sigma^{\alpha}} \Big] < 1$. Hence,

$$\begin{aligned} \|\mathcal{T}\nu(t) - \mathcal{T}\varpi(t)\|_{\mathbb{A}} &\preceq k \, \|\nu(t) - \varpi(t)\|_{\mathbb{A}} \\ &\preceq \frac{\|\nu(t) - \varpi(t)\|_{\mathbb{A}}}{3} + \psi_{\mathbb{A}}\Big(\Upsilon_{1,\mathbb{A}}(\nu,\varpi)\Big). \end{aligned}$$

According to Theorem 3.2, the mapping $\mathcal{T}$ has a unique **FP**, which is a unique solution to the fractional problem (5.1) on $\mathbb{A}$. ∎

Now, we present conditions of second main theorem in this section:

Suppose that $f : [0,\infty) \times \mathbb{A} \longrightarrow \mathbb{A}$ satisfies the following assertion:
(A_5) Let $\mu > 0$ be a constant such that

$$\sup_{t \geq 0} \frac{\int_0^t (t-s)^{\alpha-1}|f(s,0)| \, ds}{e^{\mu t}} < \infty.$$

We denote by $C([0,\infty),\mathbb{A})$ the set of all continuous functions $f:[0,\infty)\times\mathbb{A}\to\mathbb{A}$ endowed with the norm

$$\|\nu\|_{\mathbb{A}} = \left(\sup_{t\geq 0} \ |\nu(t)| \ e^{-\mu t}\right) \mathcal{I}_{\mathbb{A}}.$$

Clearly, $C([0,\infty),\mathbb{A},\|.\|_{\mathbb{A}})$ is a Banach space.

Theorem 5.2. According to hypotheses $(A_1)-(A_5)$, the fractional problem (5.1) with the condition (5.2) has a solution on $\mathbb{A}$.

Proof. Define the mapping $\mathcal{T}$ by

$$\begin{aligned}\mathcal{T}\nu(t) &= \Bigg[a_0 + \frac{p\ t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\ a_0 - \frac{p}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\nu(s)\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ f(s,\nu(s))\ ds\Bigg]\ \mathcal{I}_{\mathbb{A}}.\end{aligned}$$

Now, for every $\nu\in C([0,\infty),\mathbb{A})$, we get

$$\begin{aligned}|\mathcal{T}\nu(t)| &\leq \left(1 + \frac{|p|\ t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)|a_0| + \frac{|p|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\ |\nu(s)|\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ |f(s,\nu(s))-f(s,0)|\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ |f(s,0)|\ ds \\ &\leq \left(1 + \frac{|p|\ t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)|a_0| + \frac{|p|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|\nu(s)|\ ds \\ &+ \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ |\nu(s)|\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ |f(s,0)|\ ds.\end{aligned}$$

Hence, for any $t\geq 0$, we have

$$\begin{aligned}\frac{|\mathcal{T}\nu(t)-\mathcal{T}\varpi(t)|}{e^{\mu t}} &\leq \frac{|p|}{\Gamma(\alpha-\beta)}\ .\int_0^t (t-s)^{\alpha-\beta-1}\ e^{\mu s}\ \frac{|\nu(s)-\varpi(s)|}{e^{\mu s}}\ ds \\ &+ \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ e^{\mu s}\ \frac{|\nu(s)-\varpi(s)|}{e^{\mu s}}\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\ \sup_{t\geq 0}\ \frac{\int_0^t\ (t-s)^{\alpha-1}\ |f(s,0)|\ ds}{e^{\mu t}}.\end{aligned}$$

Taking the supremum, we get

$$\begin{aligned}
\|\mathcal{T}\nu(t)-\mathcal{T}\varpi(t)\|_{\mathbb{A}} \;\preceq\;& \frac{|p|}{\Gamma(\alpha-\beta)}\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}\;\int_0^t (t-s)^{\alpha-\beta-1}\; e^{\mu s}\; ds\\
&+\frac{M}{\Gamma(\alpha)}\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}\;\int_0^t (t-s)^{\alpha-1}\; e^{\mu s}\; ds.\\
&+\frac{|f(s,0)|}{\Gamma(\alpha)}\;\sup_{t\geq 0}\;\frac{\int_0^t\;(t-s)^{\alpha-1}\;ds}{e^{\mu t}}\\
=\;& \frac{|p|}{\Gamma(\alpha-\beta)}\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}\;\int_0^t (t-s)^{\alpha-\beta-1}\; e^{\mu s}\; ds\\
&+\frac{M}{\Gamma(\alpha)}\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}\;\int_0^t (t-s)^{\alpha-1}\; e^{\mu s}\; ds.\\
&+\frac{|f(s,0)|}{\alpha\;\Gamma(\alpha)}\;\sup_{t\geq 0}\;\lim_{t\to\infty}\;\frac{t^{\alpha}}{e^{\mu t}}.\\
=\;& \frac{|p|}{\Gamma(\alpha-\beta)}\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}\;\int_0^t (t-s)^{\alpha-\beta-1}\; e^{\mu s}\; ds\\
&+\frac{M}{\Gamma(\alpha)}\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}\;\int_0^t (t-s)^{\alpha-1}\; e^{\mu s}\; ds.\\
&+\frac{|f(s,0)|}{\Gamma(\alpha+1)}\;\sup_{t\geq 0}\;\lim_{t\to\infty}\;\frac{\alpha!}{\mu^n\; e^{\mu t}}.
\end{aligned}$$

In the same lines of Theorem 5.1, we have

$$\|\mathcal{T}\nu(t)-\mathcal{T}\varpi(t)\|_{\mathbb{A}} \;\preceq\; \Big(\frac{|p|}{\mu^{\alpha-\beta}}+\frac{M}{\mu^{\alpha}}\Big)\;\|\nu(t)-\varpi(t)\|_{\mathbb{A}}.$$

Taking $\mu>0$ is large enough, then $k \;=\; \frac{|p|}{\mu^{\alpha-\beta}}+\frac{M}{\mu^{\alpha}}<1$. Hence,

$$\begin{aligned}
\|\mathcal{T}\nu-\mathcal{T}\varpi\|_{\mathbb{A}} \;&\preceq\; k\;\|\nu-\varpi\|_{\mathbb{A}}\\
&\preceq\; \frac{\|\nu-\varpi\|_{\mathbb{A}}}{2}\;+\;\psi_{\mathbb{A}}\Big(\Upsilon_{3,\mathbb{A}}(\nu,\varpi)\Big),
\end{aligned}$$

where

$$\Upsilon_{3,\mathbb{A}}(\nu,\varpi) \;=\; \frac{\|\mathcal{T}\varpi-\varpi\|_{\mathbb{A}}+\|\mathcal{T}\varpi-\nu\|_{\mathbb{A}}}{2}.$$

Thank to Corollary 3.4, the mapping $\mathcal{T}$ has a unique **FP**, which is also a unique solution to the fractional problem (5.1) on $\mathbb{A}$. □

5.2. Application II

This section investigates the solution of a system of nonlinear two-term **FDEs**, drawing connections to Green's functions and Mittag-Leffler functions. While numerous studies have explored the existence and uniqueness of similar problems (see [57,58,59,60,61]), this work employs a novel contraction condition to establish existence and uniqueness results for the following system.

Consider the following system of non-linear fractional differential equations:

$$\begin{cases} {}^{c}D^{\alpha}\varpi(t)\;+\;{}^{c}D^{\beta}\varpi(t)\;=\;f_i(t,\varpi(t)),\\ \varpi(0)\;=\;0\;=\;\varpi(1), \end{cases} \tag{5.4}$$

for all $t \in [0,1]$, $0 < \beta < \alpha < 1$, $i = 1,2$. The operators ${}^cD^{\alpha}$ and ${}^cD^{\beta}$ represent the **CFDs** of order α and β, respectively and $f_i : [0,1] \times \mathbb{A} \longrightarrow \mathbb{A}$ are continuous functions. The generalized Mittag-Leffler function is defined by:

$$E^i_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) \;=\; \sum_{k=0}^{\infty} \frac{(-t^{\alpha-\beta})^k}{\Gamma[(\alpha-\beta)k+\alpha]}.$$

The Green's function related to (5.4) is defined as:

$$\begin{aligned}
G_i(t) &= t^{\alpha-1}\, E^i_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) \\
&= t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-t^{\alpha-\beta})^k}{\Gamma[(\alpha-\beta)k+\alpha]} \\
&\leq t^{\alpha-1} \sum_{k=0}^{\infty} \frac{|t^{\alpha-\beta}|^k}{\Gamma[(\alpha-\beta)k+\alpha]} \\
&= t^{\alpha-1} \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(2\alpha-\beta)}\, |t^{\alpha-1}| + \frac{1}{\Gamma(3\alpha-2\beta)}\, |t^{\alpha-1}|^2 + \ldots.. \right] \\
&\leq t^{\alpha-1} \left[1 + |t^{\alpha-1}| + |t^{\alpha-1}|^2 + \ldots.. \right] \\
&\leq t^{\alpha-1} \frac{1}{1-|t^{\alpha-1}|} \;\leq\; t^{\alpha-1}.
\end{aligned}$$

Assume that the following assertions are satisfied:

(B_1) $\mathit{\Delta} = C[0,1]$ is a Banach space of all continuous functions $f_i : [0,1] \times \mathbb{A} \longrightarrow \mathbb{A}$, where $\mathbb{A}$ is $\mathbf{C}^*$-algebra under the norm $\|\nu\|_{\mathbb{A}} \;=\; \Big(\max_{t\in[0,1]}\; |\nu(t)| \Big)\, \mathcal{I}_{\mathbb{A}}$, for all $t \in [0,1]$,

(B_2) there exits a positive constant h such that

$$|f_i(t,\nu) - f_i(t,\varpi)| \;\leq\; h\; |\nu - \varpi|\,, \quad \text{for all} \quad t \in [0,1], \quad \nu, \varpi \in C([0,1],\mathbb{A}), \quad 2h \leq \alpha,$$

(B_3) for all $t \in [0,1]$, there exists a mapping $\mathcal{T}_i : C([0,1],\mathbb{A}) \longrightarrow C([0,1],\mathbb{A})$ such that

$$\mathcal{T}_i\; \varpi(t) \;=\; \left[\int_0^t G_i(t-s)^{\alpha-1}\; f_i(s,\varpi(s))\; ds \right] \mathcal{I}_{\mathbb{A}},$$

(B_4) for all $t \in [0,1]$

$$\|\nu - \varpi\|_{\mathbb{A}} \;=\; \left(\max_{t\in[0,1]}\; |\nu(t) - \varpi(t)| \right) \mathcal{I}_{\mathbb{A}},$$

for all $t \in [0,1]$, $\nu, \varpi \in C([0,1],\mathbb{A})$,

(B_5) $\max_{t\in[0,1]}\; \int_0^t G_i(t-s)^{\alpha-1}\; ds \;\leq\; \dfrac{1}{\alpha}$.

Theorem 5.3. According to hypotheses $(B_1) - (B_5)$, the fractional problem (5.4) has a solution on $\mathbb{A}$.

Proof. Assume that the conditions $(B_1) - (B_5)$ hold. Then, $\varpi \in \mathit{\Delta}$ is a solution of (5.4) if and only if $\varpi \in \mathit{\Delta}$ is a solution of the following integral equation:

$$\varpi(t) \;=\; \left[\int_0^t G_i(t-s)^{\alpha-1}\; f_i(s,\varpi(s))\; ds \right] \mathcal{I}_{\mathbb{A}}. \tag{5.5}$$

This means that $\mathcal{T}_i\varpi(t) = \varpi(t)$ for all $t \in [0,1]$. By the assumptions (B_2) and (B_3), we have

$$\begin{aligned} |\mathcal{T}_i\nu(t) - \mathcal{T}_i\varpi(t)| &= \left| \int_0^t G_i(t-s)^{\alpha-1} f_i(s,\nu(s))\, ds - \int_0^t G_i(t-s)^{\alpha-1} f_i(s,\varpi(s))\, ds \right| \\ &\leq \int_0^t G_i(t-s)^{\alpha-1} \left| f_i(s,\nu(s)) - f_i(s,\varpi(s)) \right| ds \\ &\leq \int_0^t G_i(t-s)^{\alpha-1}\, h\, \left| \nu(s) - \varpi(s) \right| ds \\ &\leq \left(\int_0^t G_i(t-s)^{\alpha-1}\, ds \right) h\, |\nu(s) - \varpi(s)|. \end{aligned}$$

Taking the maximum and multiplying by $\mathcal{I}_{\mathbb{A}}$, we get

$$\begin{aligned} \|\mathcal{T}_i\nu(t) - \mathcal{T}_i\varpi(t)\|_{\mathbb{A}} &\preceq \left(\max_{t\in[0,1]} \int_0^t G_i(t-s)^{\alpha-1}\, ds \right) h\, \|\nu(t) - \varpi(t)\|_{\mathbb{A}} \\ &\preceq \frac{h}{\alpha}\, \|\nu(t) - \varpi(t)\|_{\mathbb{A}} \\ &\preceq \frac{\|\nu - \varpi\|_{\mathbb{A}}}{2} + \psi_{\mathbb{A}}\Big(\Upsilon_{2,\mathbb{A}}(\nu,\varpi)\Big), \end{aligned}$$

According to Corollary 3.3, the mapping $\mathcal{T}$ has a unique **FP**, which is also a unique solution to the fractional problem (5.4) on $\mathbb{A}$. □

6. Conclusion and future work

This paper extends the existing literature on contraction mappings in Banach spaces over $\mathbf{C}^*$-algebras. We introduce a new type of contraction mapping, the $\psi_{\mathbb{A}}-$contraction, and present several key definitions, theorems, and corollaries. Our results generalize and expand upon previous findings in this field. We apply $\psi_{\mathbb{A}}-$contraction mappings to nonlinear fractional differential equations (**FDEs**), demonstrating their utility in solving these equations. This work contributes significantly to the field of fractional calculus. Future research directions include investigating variations of $\psi_{\mathbb{A}}-$contractions, such as weak or mixed $\psi_{\mathbb{A}}-$contractions, to explore their properties and applications. Additionally, applying $\psi_{\mathbb{A}}-$contractions to other types of integral and differential equations could lead to further advancements in the field.

Declarations

Funding Not applicable.

Data availability Not applicable.

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval Not applicable.

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