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Matroids and Blockers

Talal Ali Al-Hawary

ABSTRACT: Let r and n be nonnegative integers such that $r \leq n$. A uniform matroid of rank r and size n is a matroid on an $n-$element set where every subset of size r is a basis. In this paper we study kernels of pointed strong maps, modular flats and characterize uniform matroids using blocker notation.

Key Words: Blocker, Clutter, Kernel, Modular flat, Uniform matroid.

Contents

1. Introduction

We begin with some background material, which follows the terminology and notation in [16]. Let $M = (E, \mathfrak{F})$ denote the matroid on the ground set E with closed sets $\mathfrak{F}$. The **uniform matroid** of rank r and size n is denoted by $U_{r,n}$ where $r = 0, 1, \dots, n$. When $r = n$, the matroid is called **free** and when $r = n = 0$, the matroid is called the **empty matroid**.

A subset $\{x\} \subseteq E$ is called a **loop** of a matroid $M = (E, \mathfrak{F})$ if x is in every flat of M. If $\circ \in E$ and $\{\circ\}$ is the only loop in $M(E,\mathfrak{F})$, then $M(E,\mathfrak{F})$ is called a **pointed matroid**. We denote by $U^{\circ}_{m,n}$ the uniform pointed matroid of size $n+1$ and rank m. Let $M_1 = M(E_1,\mathfrak{F}_1)$ and $M_2 = M(E_2,\mathfrak{F}_2)$ be two pointed matroids. Then a **strong map** f from M_1 to M_2, abbreviated as $M_1 \xrightarrow{f} M_2$, is a map f with $E_1 \xrightarrow{f} E_2$ such that $\circ = f\circ$ and the inverse image of any flat of M_2 is a flat of M_1. A strong map f is an **isomorphism** if it has a two sided inverse, call it f^{-1}, that is also a strong map.

A **clutter** is a collection of sets none of which is a proper subset of another. Some examples of clutters are sets of bases, circuits and cocircuits. Let $\mathcal{A}$ be a clutter of subsets of a set S, then the **blocker** $b(\mathcal{A})$ of $\mathcal{A}$ is the set of minimal subsets of S that have a non-empty intersection with every member of $\mathcal{A}$. Obviously, $b(\mathcal{A})$ is a clutter. For more on matroids, see [1,2,3,4,5,6,7,8,10,11,12,13,14,16].

Our goal in this paper is to study some properties of uniform matroids, modular flats, kernels of strong maps and to prove the following well known theorem using blocker notation.

Theorem 1 *A matroid is uniform if and only if every circuit and every cocircuit meet in at least two elements.*

2. Uniform Matroids and Modular Flats

We begin this section by defining the notion of kernel for strong maps and then show that this notion coincides with the categorical notion of kernel.

Let $M_1 = M(E_1,\mathfrak{F}_1)$ and $M_2 = M(E_2,\mathfrak{F}_2)$ be two pointed matroids and $M_1 \xrightarrow{f} M_2$ be a strong map. Then **kernel** of f, denoted by $\ker(f)$, is the pointed submatroid $K = M(E_K,\mathfrak{F}_K)$ where

$$\begin{aligned} E_K &= \{x \in E_1 \,:\, f(x) = \circ\} \text{ and} \\ \mathfrak{F}_K &= \{F \cap E_K \,:\, F \in \mathfrak{F}_1\}. \end{aligned}$$

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Theorem 2 *The kernel of a strong map satisfies the categorical definition of kernel. That is to say, if* $M_1 \xrightarrow{f} M_2$ *is a strong map and* $\circ_{12}$ *is the trivial map from* M_1 *to* M_2*, then there exists a strong map* $K \xrightarrow{g} M_1$ *such that* $fg = \circ_{12}g$ *and so that if* M *is a pointed matroid and* $M \xrightarrow{h} M_1$ *is a strong map with* $fh = \circ_{12}h$*, then there exists a unique strong map* $M \xrightarrow{\overline{h}} K$ *such that* $h = g\overline{h}$*.*

Proof: Let g be the inclusion map. Then clearly g is a strong map such that $fg = \circ_{12}g$. For every pointed matroid M and every strong map $M \xrightarrow{h} M_1$ with $fh = \circ_{12}h$, consider the map $M \xrightarrow{\overline{h}} K$ that maps each element x to $h(x)$. Then for every flat $F \cap E_K$ of K,

$$\overline{h}^{-1}(F \cap E_K) = h^{-1}(F) \cap h^{-1}(E_K) = h^{-1}(F),$$

and as h is a strong map, $h^{-1}(F)$ is a flat of M and hence $\overline{h}$ is a strong map. The uniqueness of the map $\overline{h}$ is obvious. □

Let M be a matroid with rank function r and let X be a flat of M such that

$$r(X) + r(F) = r(X \cup F) + r(X \cap F),$$

for all flats F of M. Then X is called a **modular flat**. Modular flats, in general, relatively well-behaved and the property of modularity is particularly important in certain matroid constructions such as amalgams and direct sums. In the next example we see that the ground set of the kernel of a strong map need not be a modular flat. Not even when the domain and codomain matroids are pointed uniform matroids.

Example 1 *Consider the matroids* $U^{\circ}_{3,4}$ *and* $U^{\circ}_{1,1}$ *with ground sets* $\{\circ, a, b, c, d\}$ *and* $\{\circ, e\}$*, respectively. Let* $U^{\circ}_{3,4} \xrightarrow{f} U^{\circ}_{1,1}$ *be the map defined as* $f(a) = f(b) = f\circ = \circ$ *and* $f(c) = f(d) = e$*. Then* f *is a strong map and ker(*f*) is isomorphic to*

$$K = M(E_K = \{\circ, a, b\}, \mathfrak{F}_K = \{\{\circ\}, \{\circ, a\}, \{\circ, b\}, \{\circ, a, b\}\}).$$

If we take $F = \{\circ, c, d\}$ *which is a flat of* $U^{\circ}_{3,4}$*, we find that*

$$r(E_K) + r(F) = 2 + 2$$

while

$$r(E_K \cup F) + r(E_K \cap F) = 3 + 0.$$

Hence E_K *is not a modular flat.*

Next, we look at three cases where the ground set of the kernel is a modular flat.

Theorem 3 *Let* $M_1 = M(E_1, \mathfrak{F}_1)$ *and* $M_2 = M(E_2, \mathfrak{F}_2)$ *be two pointed matroids. If* $M_1 \xrightarrow{f} M_2$ *is a strong map such that* $f(x) = \circ$ *only if* $x = \circ$*, then the ground set of ker(*f*) is a modular flat.*

□

Obviously, ker(f) is isomorphic to $U^{\circ}_{0,0}$. Thus, for all flats F of M_1,

$$\begin{aligned} r(E_K \cup F) + r(E_K \cap F) &= r(\{\circ\} \cup F) + r(\{\circ\} \cap F) \\ &= r(F) + r(\{\circ\}). \end{aligned}$$

Hence E_K is a modular flat. □

Corollary 4 *If* $M_1 \xrightarrow{f} M_2$ *is an injective strong map, then the ground set of ker(*f*) is a modular flat.*

Lemma 5 [Oxley [15], pp. 230] *Every free matroid is modular; that is every flat in a free matroid is a modular flat.*

Corollary 6 *If $U^{\circ}_{m,m} \xrightarrow{f} M$ is a strong map, then the ground set of ker(f) is a modular flat.*

We end this section with three results related to properties of pointed uniform matroids.

Lemma 7 *The circuits of a uniform matroid $U^{\circ}_{m,n}$ with ground set E are*

$$\mathcal{C}(U^{\circ}_{m,n}) = \begin{cases} \{\circ\} & \text{if } m = n \\ \{X \subseteq E \,:\, |X| = m+1 \text{ or } X = \{\circ\}\} & \text{if } m < n \end{cases}.$$

Proof: Obvious. □

Theorem 8 *Submatroid of pointed uniform matroids are either pointed free matroids or full rank pointed uniform matroids.*

Proof: Let N be a nontrivial submatroid of $U^{\circ}_{m,n}$, defined on the ground set $E(N) \subseteq E(U^{\circ}_{m,n})$ where $E(U^{\circ}_{m,n})$ is the ground set of $U^{\circ}_{m,n}$. If $\mathcal{C}(N) = \{\circ\}$, then N is free; so suppose $\mathcal{C}(N) \neq \{\circ\}$. By Lemma 7: $\mathcal{C}(N) = \{C \subseteq E(N) \,:\, C \in \mathcal{C}(U^{\circ}_{m,n})\}$. If $X \in \mathcal{C}(N) - \{\circ\}$, then $X \in \mathcal{C}(U^{\circ}_{m,n})$ and $|X| = m+1$. Conversely, if $X \subseteq E(N)$ with $|X| = m+1$, then since $X \subseteq E(U^{\circ}_{m,n})$, we have $X \in \mathcal{C}(U^{\circ}_{m,n})$ and by definition, $X \in \mathcal{C}(N)$. Thus N is a pointed uniform matroid of rank m. □

Theorem 9 *Let $M_1 = M(E_1,\mathfrak{F}_1)$ be a pointed matroid and $M_2 = M(E_2,\mathfrak{F}_2) \cong U^{\circ}_{n,n}$. If $M_1 \xrightarrow{f} M_2$ is an injective strong map, then $M_1 \cong U^{\circ}_{m,m}$ where $m = |E_1|$.*

Proof: For every subset F of E_1, $F = f^{-1}(A)$ for some $A \subseteq E_2$. As A is a flat of $U^{\circ}_{n,n}$ and as f is a strong map, $F \in \mathfrak{F}_1$. Therefore, $M_1 \cong U^{\circ}_{m,m}$ where $m = |E_1|$. □

3. Characterizing Uniform Matroids via Blockers

In this section we begin by recalling some results to prove Theorem 1.

Lemma 10 [Oxley [15], Proposition 2.1.16 and Corollary 2.1.17] *Let M be a matroid and $\mathcal{B}(M)$, $\mathcal{B}^*(M)$, $\mathcal{C}(M)$, $\mathcal{C}^*(M)$ and $\mathcal{H}(M)$ be the clutters of bases, cobases, circuits, cocircuits and hyperplanes of M, respectively. Then*

$$\mathcal{C}^*(M) = \mathcal{H}(\acute{M}), \tag{3.1}$$

$$\mathcal{C}^*(M) = b(\mathcal{B}(M)),\ b(\mathcal{C}^*(M)) = \mathcal{B}(M), \tag{3.2}$$

and

$$\mathcal{C}(M) = b(\mathcal{B}^*(M)),\ b(\mathcal{C}(M)) = \mathcal{B}^*(M). \tag{3.3}$$

Lemma 11 [Oxley [15], Proposition 2.1.20] *If C is a circuit and C^* is a cocircuit of a matroid M, then $|C \cap C^*| \neq 1$.*

Therefore, to prove Theorem 1, we need only show a matroid is uniform if and only if every circuit and every cocircuit have a non-empty intersection.

Proof: (The proof of Theorem 1). Let $M = U_{r,n}$ be a matroid having ground set E. Then every circuit has size $r+1$ and hence every hyperplane has size $r-1$. Then by Equation (3.1) in Lemma 10, every cocircuit has size $n-(r-1) = n-r+1$. If $C \in \mathcal{C}(M)$ and $C^* \in \mathcal{C}^*(M)$ such that $C \cap C^* = \varnothing$, then $|C| + |C^*| = n+2 > n = |E|$ which is impossible. Therefore, $C \cap C^* \neq \varnothing$.

Conversely, Let M be a matroid of rank r on an $n-$element ground set such that every circuit meets every cocircuit. If a subset T of E is a basis, then

$$|T| = r \tag{3.4}$$

It remains to show if T is a subset of E with cardinality r, then T is a basis. This together with Equation (3.4) implies that M is a uniform matroid of rank r and size n. Suppose there exist $T \subseteq E$ such that $|T| = r$ and T is not a basis. Then there exists $C \in \mathcal{C}(M)$ such that $C \subseteq T$ and then, by Equation (3.3) in Lemma 10, $C \in b(\mathcal{B}^*(M))$. Thus C is minimal such that $C \cap B^* \neq \varnothing$ for every $B^* \in \mathcal{B}^*(M)$, i.e. $C \cap (E - B) \neq \varnothing$ for every $B \in \mathcal{B}(M)$. Now if $B \subseteq C$ for some $B \in \mathcal{B}(M)$, then, as $|B| = r$ and $C \subseteq T$, $|C| \leq |T| = r = |B|$. Thus $C = B$, i.e. C is a basis which is impossible. Hence, $B \nsubseteq C$ for every $B \in \mathcal{B}(M)$. So for every $B \in \mathcal{B}(M)$, there exists $x_B \in B - C$. Let $D = \{x_B \in B - C \,|\, B \in \mathcal{B}(M)\}$. We show that D is a cocircuit and as $D \cap C = \varnothing$, this contradicts the assumption and hence completes the proof. If G is a subset of D such that $G \in b(\mathcal{B}(M))$, then by Equation (3.2) in Lemma 10 G is a cocircuit of M. But $G \cap C = \varnothing$. Thus D is minimal such that $D \cap B \neq \varnothing$ for every $B \in \mathcal{B}(M)$. Hence $D \in b(\mathcal{B}(M)) = \mathcal{C}^*(M)$ and thus D is a cocircuit. □

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Talal Al-Hawary,
Department of Mathematics,
Yarmouk University,
Jordan.
E-mail address: talalhawary@yahoo.com