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New Rational-Type Contractions and their Applications in b-Metric Spaces

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ABSTRACT: In this paper, we analyze new rational-type contractions in the context of b-metric spaces, establish a theoretical foundation for these contractions, and explore their applications in Fredholm integral inclusions. The paper also provides examples to verify the validity of the results.

Key Words: Fixed point, b-metric space, contraction, integral equation.

Contents

1. Introduction and Preliminaries

Fixed point theory is one of the most important areas in mathematics and nonlinear analysis, with widespread applications in several scientific fields, such as computer science, engineering, chemistry [1], biology, economics [2], medical sciences, and telecommunications. The study of metric spaces was initiated by French mathematician Maurice Frechet in 1906 [3]. In 1922, Banach [4] introduced his theory of the Banach contraction mapping. The Banach fixed point theorem, often referred to as the contraction principle, is a pivotal concept in metric spaces. Over time, the theory has seen numerous developments and generalizations. The literature on fixed point theory contains many extensions of both metric spaces and the Banach contraction principle [5,6,7,8,9,10,11,12,13,14,15,16].

In the domain of fixed point theory, Banach's result was later extended by Juliusz Schauder [17] in 1930. Furthermore, in 1968, Kannan [18] introduced a version of Banach's contraction, known as the Kannan-type contraction. Reich [19] contributed further by generalizing both Banach and Kannan's contraction principles, employing b-metric spaces and generalized metric spaces, and introducing Reich-type contraction, a new contraction type that generalized Kannan's fixed point results. Geraghty [20] significantly broadened the scope of the Banach contraction principle. In 1973, 'Ciri'c [21,22] established the well-known 'Ciri'c-type fixed point theorem, which replaced the constant with an auxiliary function and demonstrated a fixed point result for mappings in the context of complete metric spaces. This is regarded as one of the most important generalizations of the Banach contraction principle. Many authors have continued to extend and generalize both Banach and Kannan contractions in the fixed point theory literature.

The concept of b-metric spaces was introduced by Bakhtin [23] and Czerwik [24,25] as a generalization of metric spaces, where the triangular inequality is relaxed. Many fixed point theorems specific to b-metric spaces have been established [26,27]. For those looking for more detailed information, [32,33,34, 35,36,37,38,39,40,41] can be consulted for further studies.

This work introduces a few fixed point results for a novel class of generalized contractions in b-metric spaces. We also include examples to highlight the relevance of the conclusions derived from our research. One of our significant results is applied to solve an integral problem. In order to validate our findings, we rely on the following key concepts and results from the existing literature.

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Definition 1.1 [23,24,25] Consider the non-empty set Ω. If there is a number $b \geq 1$, then a mapping $\sigma : \Omega \times \Omega \to [0, +\infty)$ is referred to as a b-metric. In this way, for any $r, s, w \in \Omega$,

1. $\sigma(r, s) = 0$ if and only if $r = s$;
2. $\sigma(r, s) = \sigma(s, r)$;
3. $\sigma(r, w) \leq b(\sigma(r, s) + \sigma(s, w))$.

Then the pair (Ω, σ) is then referred to as a b-metric space. Every metric space is obviously a b-metric space with $b = 1$, although this is not always the case. Indeed, compared to the class of metric spaces, the class of b-metric spaces is bigger.

Definition 1.2 [28] Let (Ω, σ) be a b-metric space with $b \geq 1$. Next, the following sequence is called in Ω:
1. If $\epsilon > 0$ for all $n_0 \in N$ such that $\sigma(r_n, r_m) < \epsilon$ for all $n, m \geq n_0$, then the sequence is Cauchy.
2. Convergent if $l \in \Omega$ exists such that for every $\epsilon > 0$, there exists $n_0 \in N$ such that $\sigma(r_n, l) \in \epsilon$ for all $n \geq n_0$. The sequence $\{r_n\}$ is said to converge to l in this case.

Definition 1.3 [28] If every Cauchy sequence converges in a b-metric space (Ω, σ) with $b \geq 1$, the space is considered complete.

The following lemma is useful in proving all main results.

Lemma 1.1 *[29] Every sequence$\{r_n\}$ of elements from a b-metric space (Ω, σ) with $b \geq 1$, with the condition that there exists $\lambda \in [0, 1)$ such that $\sigma(r_n, r_{n+1}) \leq \lambda\sigma(r_{n-1}, r_n)$ for every $n \in N$,*

Example 1.1 [30] Let (Ω, σ) be a metric space and let the mapping $\sigma : \Omega \times \Omega \to [0, \infty)$ be defined by

$$\sigma(r, s) = (\sigma(r, s))^\eta, \forall r, s \in \Omega$$

where $\eta > 1$ is a fixed real number. Then (Ω, σ) is a b-metric space with $b = 2^{\eta-1}$. In particular, if $\Omega = \mathbb{R}, \sigma(r, s) = |r - s|$ is the usual Euclidean metric and

$$\sigma(r, s) = (r - s)^2, \forall r, s \in \mathbb{R}.$$

then $(\mathbb{R}, \sigma)$ is a b-metric with $b = 2$. However, $(\mathbb{R}, \sigma)$ is not a metric space on $\mathbb{R}$ since the axiom 3 in Definition 1.1 does not hold. Indeed,

$$\sigma(-2, 2) - 16 > 8 - 4 + 4 - \sigma(-2, 0) + \sigma(0, 2).$$

Example 1.2 [31] Let Ω be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |r(t)|^2 dt < \infty.$$

Define $\sigma : \Omega \times \Omega \to [0, \infty)$ by

$$\sigma(r, s) = \int_0^1 |r(t) - s(t)|^2 dt.$$

Then σ satisfies the following properties

1. $\sigma(r, s) = 0$ if and only if $r = s$,
2. $\sigma(r, s) = \sigma(s, r)$, for any $r, s \in \Omega$
3. $\sigma(r, s) \leq 2(\sigma(r, w) + \sigma(w, s))$, for any points $r, s, w \in \Omega$.

Clearly, (Ω, σ) is a b-metric space with $b = 2$ but is not a metric space. For example, take $r(t) = 0, s(t) = 1$ and $w(t) = \frac{1}{2}$, for all $t \in [0, 1]$. Then

$$\sigma(0, 1) = 1 > \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \sigma\left(0, \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, 1\right).$$

2. Main Results

This section presents a novel class of contractions involving rational expressions, which we use to derive various fixed point theorems in the setting of b-metric spaces.

Theorem 2.1 *Let (Ω, σ) be a complete b-metric space with $b \geq 1$. We say that $F : \Omega \to \Omega$ is a contraction mapping such that for every $r, s \in \Omega$ satisfied*

$$\sigma(Fr, Fs) \leq \lambda \max\left\{\sigma(r,s), \sigma(r, Fr), \sigma(s, Fs)\sigma(s, Fr), \frac{\sigma(r, Fr)\sigma(s, Fr)}{\sigma(r,s)+\sigma(s,Fr)}\right\}, \tag{2.1}$$

where $\lambda \in [0,1), \sigma(r,s)+\sigma(s,Fr) \neq 0$, then F has a unique fixed point.

Proof: Let $s_0 \in \Omega$. Specify a sequence $\{s_n\}$ in Ωl as $s_n = Fs_{n-1}$ for all $n \geq 1$. Assume that any two consecutive terms of the sequence $\{s_n\}$ are distinct, otherwise, F has a fixed point. For thus, let $n \in N$. Consider,

$$\sigma(s_n, s_{n+1}) \leq \sigma(Fs_{n-1}, Fs_n).$$

Utilizing (2.1), we have

$$\begin{aligned}
&\leq \lambda \max\left\{\sigma(s_{n-1}, s_n), \sigma(s_{n-1}, Fs_{n-1}), \sigma(s_n, Fs_n)\sigma(s_n, Fs_{n-1}), \frac{\sigma(s_{n-1}, Fs_{n-1})\sigma(s_n, Fs_{n-1})}{\sigma(s_{n-1}, s_n)+\sigma(s_n, F_{n-1})}\right\}\\
&= \lambda \max\left\{\sigma(s_{n-1}, s_n), \sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\sigma(s_n, s_n), \frac{\sigma(s_{n-1}, s_n)\sigma(s_n, s_n)}{\sigma(s_{n-1}, s_n)+\sigma(s_n, s_n)}\right\}\\
&= \lambda \max\left\{\sigma(s_{n-1}, s_n), \sigma(s_{n-1}, s_n)\right\}.
\end{aligned} \tag{2.2}$$

Since $\max\left\{\sigma(s_{n-1}, s_n), \sigma(s_{n-1}, s_n)\right\} = \sigma(s_{n-1}, s_n)$, then inequality (2.2) becomes,

$$\sigma(s_n, s_{n+1}) \leq \sigma(s_{n-1}, s_n).$$

On repeating this process, we obtain

$$\sigma(s_n, s_{n+1}) \leq \lambda^n \sigma(s_0, s_1) \qquad \forall n \geq 1. \tag{2.3}$$

As, $\lambda \in [0,1)$, then the sequence $\{s_n\}$ is a Cauchy sequence according to the Lemma (1.1). Since (Ω, σ) is complete, then there exists some $p \in \Omega$ such that $s_n \to p$ as $n \to \infty$.
By (2.1), it is easy to see that

$$\begin{aligned}
\sigma(s_{n+1}, Fp) &= \sigma(Fs_n, Fp)\\
&\leq \lambda \max\left\{\sigma(s_n, p), \sigma(s_n, Fs_n), \sigma(p, Fp)\sigma(p, Fs_n), \frac{\sigma(s_n, Fs_n)\sigma(p, Fs_n)}{\sigma(s_n, p)+\sigma(p, Fs_n)}\right\}\\
&= \lambda \max\left\{\sigma(s_n, p), \sigma(s_n, s_{n+1}), \sigma(p, Fp)\sigma(p, s_{n+1}), \frac{\sigma(s_n, s_{n+1})\sigma(p, s_{n+1})}{\sigma(s_n, p)+\sigma(p, s_{n+1})}\right\}.
\end{aligned} \tag{2.4}$$

Taking the limit as $n \to \infty$ on both sides of (2.4), we have

$$\lim_{n\to\infty} \sigma(s_{n+1}, Fp) = 0. \tag{2.5}$$

That is, $s_{n+1} \to Fp$. Hence, $Fp = p$, so p is a fixed point of F.
Finally, we prove the uniqueness of the fixed point. Indeed, if there is additional point q, then by (2.1),

$$\begin{aligned}
\sigma(p, q) &= \sigma(Fp, Fq)\\
&\leq \lambda \max\left\{\sigma(p,q), \sigma(p, Fp), \sigma(q, Fq)\sigma(q, Fp), \frac{\sigma(p, Fp)\sigma(q, Fp)}{\sigma(p,q)+\sigma(q, Fp)}\right\}\\
&= \lambda \max\left\{\sigma(p,q), \sigma(p,p), \sigma(q,q)\sigma(q,p), \frac{\sigma(p,p)\sigma(q,p)}{\sigma(p,q)+\sigma(q,p)}\right\}\\
&= \lambda \max\left\{\sigma(p,q), 0, 0, 0\right\}\\
&= \lambda\sigma(p,q).
\end{aligned}$$

Since $0 \leq \lambda < 1$, so we obtain that $\sigma(p,q) = 0$. Thus, $p = q$ and we conclude that F has a unique fixed point. □

Corollary 2.1 *Let (Ω, σ) be a complete b-metric space with $b \geq 1$. We say that $F : \Omega \to \Omega$ is a contraction mapping such that for every $r, s \in \Omega$ satisfied*

$$\sigma(Fr, Fs) \leq h \max\Big\{\sigma(r,s), \sigma(r,Fr), \sigma(s,Fs)\sigma(s,Fr)\Big\}, \tag{2.6}$$

for all $r, s \in \Omega$, where $0 \leq h < 1$ is a constant. Then F has a unique fixed point in Ω.

Proof: The proof can be obtained from Theorem 2.1 by taking $\lambda(t) = ht$ for each $t \geq 0$, where $h \in [0,1)$ and $\frac{\sigma(r,Fr)\sigma(s,Fr)}{\sigma(r,s)+\sigma(s,Fr)} = 0$. □

Theorem 2.2 *Let (Ω, σ) be a complete b-metric Space with $b \geq 1$. We say that $F : \Omega \to \Omega$ is a contraction mapping such that every $r, s \in \Omega$ satisfied*

$$\sigma(Fr, Fs) \leq \lambda \max\Big\{\sigma(r,s), \frac{\sigma(r,Fr)\sigma(s,Fr)}{1+\sigma(s,Fr)}, \frac{\sigma(s,Fs)\sigma(r,Fs)}{\sigma(r,Fs)}, \sigma(s,Fr)\Big\}, \tag{2.7}$$

where $\lambda \in [0,1), \sigma(r,Fs) \neq 0$, then F has a unique fixed point.

Proof: Let $s_0 \in \Omega$. Define a sequence $\{s_n\}$ in Ω as $s_{n+1} = Fs_n$ for all $n \geq 1$. Assume that any two consecutive terms of the sequence $\{s_n\}$ are distinct, otherwise, F has a fixed point. For thus, let $n \in N$. Consider,

$$\begin{aligned}
\sigma(s_n, s_{n+1}) &\leq \sigma(Fs_{n-1}, Fs_n) \\
&\leq \lambda \max\Big\{\sigma(s_{n-1}, s_n), \frac{\sigma(s_{n-1}, Fs_{n-1})\sigma(s_n, Fs_{n-1})}{1+\sigma(s_n, Fs_{n-1})}, \frac{\sigma(s_n, Fs_n)\sigma(s_{n-1}, Fs_n)}{\sigma(s_{n-1}, Fs_n)+\sigma(s_n, Fs_{n-1})}\Big\} \\
&= \lambda \max\Big\{\sigma(s_{n-1}, s_n), \frac{\sigma(s_{n-1}, s_n), \sigma(s_n, s_n)}{1+\sigma(s_n, s_n)}, \frac{\sigma(s_n, s_{n+1})\sigma(s_{n-1}, s_{n+1})}{\sigma(s_{n-1}, s_{n+1})+\sigma(s_n, s_n)}\Big\} \\
&= \lambda \max\Big\{\sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\Big\}.
\end{aligned} \tag{2.8}$$

If $\max\Big\{\sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\Big\} = \sigma(s_n, s_{n+1})$ then from (2.8), we obtain that

$$\begin{aligned}
\sigma(s_n, s_{n+1}) &\leq \lambda\sigma(s_n, s_{n+1}) \\
&< \sigma(s_n, s_{n+1}),
\end{aligned}$$

a contradiction. This means that $\max\Big\{\sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\Big\} = \sigma(s_{n-1}, s_n)$. Hence, we obtain that

$$\sigma(s_n, s_{n+1}) \leq \lambda\sigma(s_{n-1}, s_n).$$

Repeating the process we get

$$\sigma(s_n, s_{n+1}) \leq \lambda^n \sigma(s_0, s_1) \qquad \forall n \geq 1. \tag{2.9}$$

Since, $\lambda \in [0,1)$, then the sequence $\{s_n\}$ is a Cauchy sequence according to the Lemma 1.1. Since Ω is complete, then there exists some $p \in \Omega$ such that $s_n \to p$ as $n \to \infty$.
By (2.7), it is easy to see that

$$\begin{aligned}
\sigma(s_{n+1}, Fp) &= \sigma(Fs_n, Fp) \\
&\leq \lambda \max\Big\{\sigma(s_n, p), \frac{\sigma(s_n, Fs_n)\sigma(p, Fs_n)}{1+\sigma(p, Fs_n)}, \frac{\sigma(p, Fp)\sigma(s_n, Fp)}{\sigma(s_n, Fp)+\sigma(p, Fs_n)}\Big\} \\
&= \lambda \max\Big\{\sigma(s_n, p), \frac{\sigma(s_n, s_{n+1})\sigma(p, s_{n+1})}{1+\sigma(p, s_{n+1})}, \frac{\sigma(p, Fp)\sigma(s_n, Fp)}{\sigma(s_n, Fp)+\sigma(p, s_{n+1})}\Big\}.
\end{aligned} \tag{2.10}$$

Taking the limit as $n \to \infty$ by both sides of (2.10), we have

$$\lim_{n\to\infty} \sigma(s_{n+1}, Fp) = 0. \tag{2.11}$$

That is, $s_{n+1} \to Fp$. Hence, $Fp = p$, so p is a fixed point of F.
Finally, we prove the uniqueness of the fixed point. Indeed, if there is another point q, then by (2.7),

$$\begin{aligned}\sigma(p,q) &= \sigma(Fp,Fq)\\ &\le \lambda \max\Big\{\sigma(p,q), \frac{\sigma(p,Fp)\sigma(q,Fp)}{1+\sigma(q,Fp)}, \frac{\sigma(q,Fq)\sigma(p,Fq)}{\sigma(p,Fq)+\sigma(q,Fp)}\Big\}\\ &= \lambda \max\Big\{\sigma(p,q), \frac{\sigma(p,p)\sigma(q,p)}{1+\sigma(q,p)}, \frac{\sigma(q,q)\sigma(p,q)}{\sigma(p,q)+\sigma(q,p)}\Big\}\\ \sigma(p,q) &\le \lambda\sigma(p,q).\end{aligned}$$

Since $0 \le \lambda < 1$, so we obtain that $\sigma(p,q) = 0$. Thus, $p = q$ and we conclude that F has a unique fixed point. □

Corollary 2.2 *Let (Ω, ν) be a complete b-metric space with $b \ge 1$. We Say that $F : \Omega \to \Omega$ is a contraction mapping such that for every $r, s \in \Omega$ satisfied*

$$\nu(Fr,Fs) \le \lambda \max\Big\{\nu(r,s), \frac{\nu(r,Fr)\nu(s,Fr)}{1+\nu(s,Fr)}, \frac{\nu(s,Fs)\nu(r,Fs)}{\nu(r,Fs)+\nu(s,Fr)}\Big\}, \tag{2.12}$$

where $\lambda \in [0,1), \nu(r,Fs) + \nu(s,Fr) \ne 0$, then F has a unique fixed point.

Proof: If we put $\sigma = \nu$ in Theorem 2.2, we get the proof of above corollary. □

Example 2.1 Let $\Omega = [0,1]$ to be fitted using the b-metric provided by

$$\sigma(r,s) = (r+s)^2,$$

with $b = 2$. Describe the self mapping $F : \Omega \to \Omega$ by

$$F(r) = \frac{r}{2},$$

for all $r, s \in \Omega$. We have

$$\begin{aligned}\sigma(Fr,Fs) &= \Big(\frac{r}{2} - \frac{s}{2}\Big)^2\\ &= \frac{1}{4}(r-s)^2 = \frac{1}{4}\sigma(r,s)\\ &\le \lambda \max\Big\{\sigma(r,s), \frac{\sigma(r,Fr)\sigma(s,Fr)}{1+\sigma(s,Fr)}, \frac{\sigma(s,Fs)\sigma(r,Fs)}{\sigma(r,Fs)+\sigma(s,Fr)}\Big\},\end{aligned}$$

for $\frac{1}{4} \le \lambda < 1$. All conditions of Theorem 2.2 are satisfied, so clearly, $r = 0$ is the unique fixed point of F.

Theorem 2.3 *Let (Ω, σ) be a complete b-metric Space with $b \ge 1$. We say that $F : \Omega \to \Omega$ is a contraction mapping such that every $r, s \in \Omega$ satisfied*

$$\sigma(Fr,Fs) \le \lambda \max\Big\{\sigma(r,s), \frac{\sigma(r,Fr)\sigma(s,Fs)}{\sigma(r,s)}, \sigma(s,Fr)\Big\}, \tag{2.13}$$

where $\lambda \in [0,1), \sigma(r,s) \ne 0$, then F has a unique fixed point.

Proof: Let $s_0 \in \Omega$. Define a sequence $\{s_n\}$ in Ω as $s_{n+1} = Fs_n$ for all $n \geq 1$. Assume that any two consecutive terms of the sequence $\{s_n\}$ are distinct, otherwise, F has a fixed point. For this, let $n \in N$. Consider,

$$\begin{aligned}
\sigma(s_n, s_{n+1}) &\leq \sigma(Fs_{n-1}, Fs_n) \\
&\leq \lambda \max\Big\{\sigma(s_{n-1}, s_n), \frac{\sigma(s_{n-1}, Fs_{n-1})\sigma(s_n, Fs_n)}{\sigma(s_{n-1}, s_n)}, \sigma(s_n, Fs_{n-1})\Big\} \\
&= \lambda \max\Big\{\sigma(s_{n-1}, s_n), \frac{\sigma(s_{n-1}, s_n)\sigma(s_n, s_{n+1})}{\sigma(s_{n-1}, s_n)}, \sigma(s_n, s_n)\Big\} \\
&= \lambda \max\Big\{\sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\Big\}. \qquad (2.14)
\end{aligned}$$

If $\max\Big\{\sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\Big\} = \sigma(s_n, s_{n+1})$ then from (2.14), we obtain that

$$\begin{aligned}
\sigma(s_n, s_{n+1}) &\leq \lambda\sigma(s_n, s_{n+1}) \\
&< \sigma(s_n, s_{n+1}),
\end{aligned}$$

a contradiction. This means that $\max\Big\{\sigma(s_{n-1}, s_n), \sigma(s_n, s_{n+1})\Big\} = \sigma(s_{n-1}, s_n)$ for all $n \geq 1$. Hence, we obtain that

$$\sigma(s_n, s_{n+1}) \leq \lambda\sigma(s_{n-1}, s_n).$$

Repeating this process we get

$$\sigma(s_n, s_{n+1}) \leq \lambda^n \sigma(s_0, s_1) \qquad \forall n \geq 1. \qquad (2.15)$$

Since, $\lambda \in [0, 1)$, then the sequence $\{s_n\}$ is a Cauchy sequence according to the Lemma 1.1. Since Ω is complete, then there exists some $p \in \Omega$ such that $s_n \to p$ as $n \to \infty$.
By (2.13), it is easy to see that

$$\begin{aligned}
\sigma(s_{n+1}, Fp) &= \sigma(Fs_n, Fp) \\
&\leq \lambda \max\Big\{\sigma(s_n, p), \frac{\sigma(s_n, Fs_n)\sigma(p, Fp)}{\sigma(s_n, p)}, \sigma(p, Fs_n)\Big\} \\
&= \lambda \max\Big\{\sigma(s_n, p), \frac{\sigma(s_n, s_{n+1})\sigma(p, Fp)}{\sigma(s_n, p)}, \sigma(p, s_{n+1})\Big\}. \qquad (2.16)
\end{aligned}$$

Taking the limit as $n \to \infty$ by both sides of (2.16), we have

$$\lim_{n\to\infty} \sigma(s_{n+1}, Fp) = 0. \qquad (2.17)$$

That is, $s_{n+1} \to Fp$. Hence, $Fp = p$, so p is a fixed point of F.
Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point q, then by (2.13),

$$\begin{aligned}
\sigma(p, q) &= \sigma(Fp, Fq) \\
&\leq \lambda \max\Big\{\sigma(p, q), \frac{\sigma(p, Fp)\sigma(q, Fq)}{\sigma(p, q)}, \sigma(q, Fp)\Big\} \\
&\leq \lambda \max\Big\{\sigma(p, q), \frac{\sigma(p, p)\sigma(q, q)}{\sigma(p, q)}, \sigma(q, p)\Big\} \\
\sigma(p, q) &\leq \lambda\sigma(p, q).
\end{aligned}$$

Since $0 \leq \lambda < 1$, so we obtain that $\sigma(p, q) = 0$. Thus, $p = q$ and we conclude that F has a unique fixed point. □

Example 2.2 Let $\Omega = [0,1]$ to be fitted using the b-metric provided by

$$\sigma(r,s) = |r-s|^3$$

with $b = 2$. Describe the self mapping $F : \Omega \to \Omega$ by.

$$F(r) = \frac{r}{3},$$

for all $r, s \in \Omega$. We have

$$\begin{aligned}\sigma(Fr, Fs) &= \left|\frac{r}{3} - \frac{s}{3}\right|^3 \\ &= \frac{1}{27}|r-s|^3 = \frac{1}{27}\sigma(r,s) \\ &\le \lambda \max\left\{\sigma(r,s), \frac{\sigma(r,Fr)\sigma(s,Fs)}{\sigma(r,s)}, \sigma(s,Fr)\right\},\end{aligned}$$

for $\frac{1}{27} \le \lambda < 1$. All conditions of Theorem 2.3 are satisfied, so clearly, $r = 0$ is the unique fixed point of F.

3. Application

This section discusses the existence of a solution to the Fredholm integral equation using Theorem 2.2. Let $\Omega = C[a,b]$ be the set of all conditions real valued functions define on $[a,b]$. Note that Ω is complete b-metric space by considering $\sigma(r,s) = \sup_{t\in[a,b]}(r(t)-s(t))^{\eta}$, with $b = 2^{\eta-1}$ and where, $\eta > 1, \sigma : \Omega\times\Omega \to [0,+\infty)$. Now, consider the Fredholm integral equation as:

$$r(t) = \int_a^b N(t,\tau,r(\tau))d\tau + f(t), \qquad t,\tau \in [a,b] \tag{3.1}$$

where $f : [a,b] \to R, r \in C[a,b]$ is the unknown functions, $\lambda \in R$, and $N : [a,b]\times[a,b]\times R \to R$ are given continuous functions.

Theorem 3.1 *Suppose that all $r, s \in C([a,b], R)$ satisfy the following requirements,*
1. There exists a continuous function $\phi : [a,b]\times[a,b] \to R$ such that for all $r, s \in \Omega, \lambda \in R$ and $t, \tau \in [a,b]$, we have

$$|N(t,\tau,r(\tau)) - N(t,\tau,s(\tau))|^{\eta} \le \phi(t,\tau)M(r,s),$$

where

$$M(r,s) \le \lambda \max\left\{\sigma(r,s), \frac{\sigma(r,Fr)\sigma(s,Fr)}{1+\sigma(s,Fr)}, \frac{\sigma(s,Fs)\sigma(r,Fs)}{\sigma(r,Fs)+\sigma(s,Fr)}\right\}.$$

2. $|\lambda| < 1$, and $\sup_{t\in[a,b]}\int_a^b \phi(t,\tau)d\tau \le \frac{1}{2^{\eta-1}(b-a)^{\eta-1}}$.

Proof: The Fredholm integral equation (3.1) allows us to define an operator $F : \Omega \to \Omega$ as follows:

$$Fr(t) = \lambda\int_a^b N(t,\tau,r(\tau))d\tau + f(t), \qquad \forall t \in [a,b].$$

Therefore, the existence of a solution to (3.1) is equivalent to the existence of a fixed point and its uniqueness in F. Given $\alpha \in R$, let $\frac{1}{\eta} + \frac{1}{\alpha} = 1$. Applying conditions 1 and 2, we have

$$\begin{aligned}
\sigma(Fr, Fs) &= \sup_{t\in[a,b]} |Fr(t) - Fs(t)|^{\eta} \\
&\le |\lambda|^{\eta} \sup_{t\in[a,b]} \left(\int_a^b |N(t,\tau,r(\tau)) - N(t,\tau,s(\tau)))| d\tau \right)^{\eta} \\
&\le \sup_{t\in[a,b]} \left[\left(\int_a^b 1^{\alpha} d\tau \right)^{\frac{1}{\alpha}} \left(\int_a^b |(N(t,\tau,r(\tau)) - N(t,\tau,s(\tau)))^{\eta} d\tau \right)^{\frac{1}{\eta}} \right]^{\eta} \\
&\le (b-a)^{\frac{\eta}{\alpha}} \sup_{t\in[a,b]} \left(\int_a^b |N(t,\tau,r(\tau) - N(t,\tau,s(\tau)|^{\eta} d\tau \right) \\
&\le (b-a)^{\eta-1} \sup_{t\in[a,b]} \left(\int_a^b \phi(t,\tau) d\tau M(r,s) \right) \\
&\le (b-a)^{\eta-1} \sup_{t\in[a,b]} \left(\int_a^b \phi(t,\tau) d\tau \right) M(r,s) \\
&\le \frac{1}{2^{\eta-1}} M(r,s).
\end{aligned}$$

Ultimatly we get

$$\sigma(Fr, Fs) \le \lambda \max\left\{ \sigma(r,s), \frac{\sigma(r,Fr)\sigma(s,Fr)}{1+\sigma(s,Fr)}, \frac{\sigma(s,Fs)\sigma(r,Fs)}{\sigma(r,Fs)+\sigma(s,Fr)} \right\}.$$

The requirements of Theorem 2.2 are thus all met. Thus, a solution to the integral equation (3.1) exists. □

4. Conclusions

In this study, we introduced a novel form of contractions with rational expressions by applying the concept of b-metric spaces. Subsequently, we proved several fixed point theorems in this setting. To validate the applicability of our results, two examples are provided. Additionally, an application is included to demonstrate the relevance of one of our key findings.

Competing interests

The authors declare that they have no competing interests.

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