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Spectral analysis of a higher-order self-adjoint Differential operator with unbounded operator coefficients

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ABSTRACT: In contrast to the setting considered by Adıgüzelov and Sezer [4], where the differential operator involves classical scalar derivatives followed by multiplication with a self-adjoint unbounded operator, this study investigates a structurally distinct operator–differential model. The dual appearance of the unbounded operator both inside the highest–order derivatives and as an independent power term has not been systematically investigated in the literature. This structural feature induces a fundamentally different functional–analytic framework, leading to novel spectral properties and domain regularity requirements. Specifically, we examine expressions of the form

$$L_o(y(x)) := (-1)^m \big(Ay(x)\big)^{(2m)} + A^m(y(x)),$$

where the operator A appears both inside the highest-order derivatives and as a power term. This formulation modifies the spectral characteristics and imposes distinct regularity conditions on the domain. Although the analytical techniques employed are analogous to those in [4], the operator structure considered here falls into a different class, requiring boundary conditions directly on $Ay(x)$. The paper establishes the fundamental spectral framework for this setting, including explicit eigenvalue–eigenfunction formulas, symmetry, self-adjointness, and lower semi-boundedness of the associated operator.

Key Words: Hilbert space, self-adjoint operator, closed operator, spectrum.

Contents

1. Introduction

The spectral theory of higher-order differential operators with unbounded operator coefficients has been extensively studied in the literature, with significant contributions addressing both classical and operator-valued coefficient settings (see, e.g., [7,8,15]). In [4], the primary focus was on operators of the form

$$L_o(y(x)) := (-1)^m \frac{d^{2m}y(x)}{dx^{2m}} + A(y(x)),$$

where A is a positive self-adjoint operator in a Hilbert space H, appearing only as a multiplicative term after differentiation.

In the present paper, we consider an alternative formulation given by

$$L_o y(x) := (-1)^m \big(Ay(x)\big)^{(2m)} + A^m(y(x)),$$

in which the highest-order derivatives are applied directly to $Ay(x)$, and the operator A also appears in an iterated composition form A^m. In contrast to classical models, the present formulation requires boundary

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conditions directly on $Ay(x)$, thereby altering the functional analytic setting and the associated spectral structure. This departure from the standard framework motivates a detailed spectral analysis within this alternative operator model. The dual placement of both within the highest–order derivatives and as an independent power term has not been systematically addressed in the literature. This configuration imposes domain regularity conditions and spectral characteristics that are fundamentally different from those arising in the classical framework. This change in structure leads to a distinct functional–analytic framework: the domain requires higher regularity for both y and Ay, and the boundary conditions are imposed directly on $Ay(x)$ rather than on $y(x)$.

Although the methods employed (spectral decomposition, eigenfunction expansions, and coercivity estimates) share similarities with those in [4], the model considered here leads to a distinct eigenvalue structure,

$$\lambda_{k,j} = k^{2m}\delta_j + \delta_j^m,$$

where $\{\delta_j\}$ are the eigenvalues of A.

The purpose of this study is to establish the self-adjointness, lower semi-boundedness, and explicit spectral representation for this operator-differential problem, thereby providing a rigorous analytical foundation for further developments, such as regularized trace formulas and asymptotic spectral analysis, within this alternative operator setting.

The spectral theory of differential operators with operator-valued (and potentially unbounded) coefficients has deep roots in classical analysis. The seminal works of Gelfand and Levitan [7,8] introduced spectral transformation techniques and laid the groundwork for trace identities, which were later extended to non-selfadjoint settings by Lidskii [15,16] through the development of generalized trace formulas.

In more recent decades, significant progress has been made in abstract frameworks involving operator-differential equations. In particular, the work of Adıgüzelov and collaborators [1,3,4,5,2] has conducted comprehensive analyses on spectral properties and established regularized trace identities for higher-order differential expressions with unbounded self-adjoint operator coefficients in Hilbert spaces. While this paper does not derive explicit regularized trace formulas, it establishes the foundational spectral framework by rigorously proving self-adjointness, lower semi-boundedness, and characterizations of the discrete spectrum for L_o.

This study also builds upon the prior contributions of Gül [10,12,13], who developed spectral decompositions and coercive estimates in the context of abstract elliptic and operator-differential systems. The operator-theoretic formulation adopted here is intended to both complement and extend this line of research.

Furthermore, recent developments by Sezer and Bakşı [17] have examined regularized traces and spectral completeness for operator-differential systems under generalized or non-classical boundary conditions. Classical analytical treatments, such as those of Smirnov [19], continue to provide essential tools in the functional analytic setting employed in this work.

2. Preliminaries

We begin by establishing the operator framework and functional setting necessary to analyze differential expressions with unbounded operator coefficients in an infinite-dimensional separable Hilbert space.

Let H be an infinite-dimensional separable Hilbert space, and let $A : D(A) \subset H \to H$ be a self-adjoint operator such that $A \geq I$, $A^{-1} \in \sigma_\infty(H)$. Define $H_1 := L^2([0, \pi]; H)$ with the inner product

$$(f, g)_{H_1} = \int_0^\pi (f(x), g(x))_H dx.$$

The differential operator of interest is defined by

$$L_o y(x) := (-1)^m (Ay(x))^{2m} + A^m y(x),$$

subject to boundary conditions ensuring the self-adjointness of the associated operator

3. The Operator L_o and Its Spectral Properties

We define the differential operator L_o in the Hilbert space H_1 and conduct a detailed examination of its spectral structure under suitable boundary conditions. The eigenvalues and eigenfunctions are explicitly derived and shown to form a complete spectral decomposition. Let H be an infinite-dimensional separable Hilbert space, where the inner product and the norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We denote the set of kernel operators from H to H as $\sigma_1(H)$. Let $H_1 = L_2([0,\pi];H)$ denote the space of all strongly measurable functions f defined on the interval $[0,\pi]$ with values in H, such that for every $g \in H$, the scalar function $(f(x),g)$ is measurable on the interval $[0,\pi]$, and

$$\int_0^\pi \|f(x)\|^2 dx < \infty.$$

The space H_1 is also a separable Hilbert space with the inner product

$$(f,g)_{H_1} = \int_0^\pi (f(x),g(x))dx, \quad f,g \in H_1.$$

We consider the operator L_o in H_1, generated by the differential expression

$$\ell_o(y) = (-1)^m (Ay(x))^{2m} + A^m y(x),$$

where $A : D(A) \to H$ is a densely defined operator on H and $A^m = A \circ A \circ \cdots \circ A$ (with m applications of A). A^* is the adjoint operator of A.

We assume that A is self-adjoint, satisfies $A \geq I$ (where I is the identity operator), and that $A^{-1} \in \sigma_\infty(H)$, meaning that A^{-1} is a compact operator.

Let $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n \leq \ldots$ be the eigenvalues of the operator A, and let $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ be the orthonormal eigenvectors corresponding to these eigenvalues. Each eigenvalue is repeated according to its multiplicity.

Let $D(L_o)$ denote the set of functions $y(x)$ and $A(y(x))$ of the space H_1 satisfying the following conditions:

1. y(x) has a continuous derivative of order $2m$ with respect to the norm in H on the interval $[0,\pi]$.

2. Ay(x) has a continuous derivative of order $2m$ with respect to the norm in H on the interval $[0,\pi]$.

3. For every $x \in [0,\pi]$, $y(x) \in D(A)$ is continuous with respect to the norm in H.

4. $(Ay(0))^{2i-1} = (Ay(\pi))^{2i-1} = 0$ for $i = 1,2,\ldots m$.

The operator L_o is symmetric, and its domain is dense $\overline{\text{in ;}}$ its closure in H_1 will also be shown to be symmetry. The operator L_o is symmetric, and its closure $\overline{D(L_o)} = H_1$. Let us consider the linear operator $L_0 y = \ell_o(y)$ from $D(L_o)$ to H_1. The eigenvalues and eigenvectors of L_o are given by:

$$k^{2m}\delta_j + \delta_j^m, \quad M_k \cos(kx)\nu_j,$$

where $k = 0,1,2,\ldots$ and $j = 1,2,\ldots$, with the normalization constants M_k defined as:

$$M_k = \begin{cases} \frac{1}{\sqrt{\pi}}, & \text{if } k = 0, \\ \sqrt{\frac{2}{\pi}}, & \text{if } k = 1,2,\ldots \end{cases}$$

These eigenvalues and eigenvectors form the spectral decomposition of the operator L_o

4. Main Results

We now present the principal theoretical findings of the paper, including rigorous statements and proofs of key properties of the operator L_o and its relation to the underlying operator A.

$$L_o y(x) := (-1)^m (Ay(x))^{2m} + A^m y(x)$$

defined in the Hilbert space $H_1 = L^2([0,\pi]; H)$, where A is a self-adjoint, positive, and unbounded operator acting in an infinite-dimensional separable Hilbert space H, satisfying $A \geq I$ and $A^{-1} \in \sigma_\infty(H)$. Under these conditions, we establish the following main result.

To construct higher-order differential expressions involving unbounded operator coefficients, it is crucial to establish the domain and well-definedness of powers of the underlying operator.

Theorem 4.1 *Let $A : D(A) \subset H \to H$ be a densely defined linear operator on a Hilbert space H such that $\overline{D(A)} = H$. For a fixed integer $m \geq 1$, define the operator A^m recursively by*

$$A^1 = A, \quad A^{k+1} = A \circ A^k \quad \textit{for } k = 1, 2, \ldots, m-1.$$

Then the operator power A^m is well-defined and densely defined on the domain

$$D(A^m) := \{y \in D(A) \mid A^k y \in D(A) \textit{ for all } k = 1, \ldots, m-1\}.$$

The domain $D(A^m)$ is dense in H, ensuring the operator A^m is densely defined.

Proof: Let $y : [0,\pi] \to H$ be a function such that $y \in C^{2m}([0,\pi]; H))$ and $y(x) \in D(A)$ for all $x \in [0,\pi]$. Under the given assumption, $Ay(x) \in C^{2m}([0,\pi]; H)$, so that $Ay(x) \in H$ and is sufficiently regular to ensure $Ay(x) \in D(A)$ for all x. We conclude that

$$Ay(x) \in D(A) \; for \; all \; x \in [0,\pi]$$

and

$$A^2(y(x)) := A(A(y(x))).$$

Continuing inductively, assume that $A^k(y(x)) \in D(A)$ for some $k \leq m-1$. Then the composition

$$A^{k+1}(y(x)) := A(A^k(y(x))).$$

By the principle of mathematical induction, we conclude that $A^m y(x) \in H$ and $A^m(y(x))$ is well-defined on $D(A^m)$.

To prove that $D(A^m)$ is dense in H, we use the spectral properties of A. Since A is self-adjoint with compact inverse $A^{-1} \in \sigma_\infty(H)$, it admits a countable orthonormal basis $\{\nu_j\}_{j=1}^{\infty} \subset D(A)$ consisting of eigenvectors corresponding to eigenvalues $\delta_j > 0$. For each j,

$$A\nu_j = \delta_j \nu_j \quad \Rightarrow \quad A^k \nu_j = \delta_j^k \nu_j \in D(A), \quad \text{for all } k = 1, \ldots, m.$$

Thus, $\nu_j \in D(A^m)$ for all $j \in \mathbb{N}$, and since $\{\nu_j\}$ is an orthonormal basis of H, it follows that

$$\overline{D(A^m)} \supset \overline{\text{span}\{\nu_j\}} = H.$$

Therefore, the domain $D(A^m)$ is dense in H. □

Analyzing the transformation behavior of eigenvalues and eigenvectors under operator powers is fundamental in spectral theory.

Theorem 4.2 *Let $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n \leq \ldots$ be the eigenvalues of the operator A, with corresponding orthonormal eigenvectors $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ Then the eigenvalues of the operator A^m are*

$$\delta_1^m \leq \delta_2^m \leq ... \leq \delta_n^m \leq ...$$

and the corresponding orthonormal eigenvectors are identical to those of A

Proof: Let $A : D(A) \subset H$ be a self-adjoint operatorin an infinite-dimensional separable Hilbert space with $\overline{D(A)} = H$, with compact inverse. Then its spectrum consists of a countable sequence of positive real eigenvalues $\{\delta_j\}_{j=1}^{\infty}$, and there exists a complete orthonormal set of eigenvectors $\{\nu_j\}_{j=1}^{\infty} \subset D(A)$ such that

$$A\nu_j = \delta_j \nu_j, \quad \text{for each } j \in \mathbb{N}.$$

We now show that each ν_j is also an eigenvector of A^m, with corresponding eigenvalue δ_j^m, and that $A^m \nu_j$ is well-defined for all $m \in \mathbb{N}$.

This follows from the recursive application of A:

$$A^2(\nu_i) := A(A\nu_i) = A(\delta_i \nu_i) = \delta_i A\nu_i = \delta_i^2 \nu_i,$$

$$A^3(\nu_i) := A(A^2\nu_i) = A(\delta_i^2 \nu_i) = \delta_i^2 A\nu_i = \delta_i^3 \nu_i,$$

and, in general,

$$A^m \nu_i = \delta_i^m \nu_i,$$

Moreover, since each ν_j satisfies $A^k \nu_j = \delta_j^k \nu_j \in D(A)$ for all $1 \leq k < m$, and $\nu_j \in D(A)$, it follows that $A^m \nu_j$ is well-defined and ν_j belongs to the domain of A^m.

Therefore, ν_i remains an eigenvector of A^m with eigenvalue δ_i^m. Since ν_i is an orthonormal basis of eigenvectors for A, it is also an orthonormal basis of eigenvectors for A^m, and the corresponding eigenvalues are δ_i^m. □

To ensure the self-adjointness of differential expressions involving operator powers, we must verify the self-adjointness of these powers.

Theorem 4.3 *Let $A : D(A) \subset H$ be a self-adjoint operator with $\overline{D(A)} = H$ and unbounded linear operator in an infinite-dimensional separable Hilbert space H, with compact inverse. Assume that for any $y \in D(A)$, the mapping $x \to y$ are $2m$ times continuously diffrentiable on $[0, \pi]$ in the norm of H, and that A is independent of x. Then the operator A^m, defined as the m-fold composition $A^m = AoAo...oA$, is self-adjoint on its natural domain $D(A^m)$.*

Proof: We first note that since

$$y(x) \in C^{2m}([0, \pi]; H)) \quad and \quad Ay(x) \in C^{2m}([0, \pi]; H)),$$

and A is independent of x, the operator, being independent of x, commutes with differentiation; hence, one may interchange their order. That is

$$\frac{d^k}{dx^k}(A(y(x)) = A(\frac{d^k}{dx^k} y(x)),$$

for all $0 \leq k \leq 2m$. $(Ay(x))^k \in C^{2m}([0, \pi]; H))$, for all $k = 1, ..., m$. $y(x) \in D(A^m)$ for each $x \in [0, \pi]$. Consider the operator A^m defined by the composition, where the domain is given by

$$D(A^m) := \left\{ y \in D(A) \,\middle|\, A^k y \in D(A), \text{ for all } k = 1, 2, \ldots, m-1 \right\}.$$

This recursive definition ensures that the composition $A \circ A^k$ is well-defined for each k, and the domain $D(A^m)$ is dense in H. Now we prove that A^m is self-adjoint. The self-adjointness of A^m is established via mathematical induction on m.
For $m = 1$, this is the self-adjointness of A :

$$(Ax, y) = (x, Ay).$$

Assume that the equality holds for $m = k$, i.e.,

$$(A^k x, y) = (x, A^k y).$$

Assume that A^k is self-adjoint for some $k \geq 1$. Define $A^{k+1} := AoA^k$, we aim to demonstrate that A^{k+1} inherits self-adjointness. Let $x, y \in D(A^{k+1}) \subset D(A^k) \cap D(A)$. Then

$$(A^{k+1}x, y) = (A(A^k x), y) = (A^k x, Ay) = (x, A^k(Ay)) = (x, A^{k+1}y),$$

$$(Ax, y) = (x, Ay).$$
$$(A^k x, y) = (x, A^k y).$$

Since A and A^k are self-adjoint, then for $m = k+1$: Furthermore, since A and A^k are both closed and densely defined, their composition A^{k+1} is closed and densely defined. A closed, symmetric, densely defined operator whose adjoint is an extension of itself is self-adjoint. Therefore, by induction A^m is self adjoint on its natural domain $D(A^m)$, as claimed. By induction, A^m is self-adjoint on $D(A^m)$. □

We now derive the eigenvalues and corresponding eigenfunctions of the operator, linking them to those of the generating operator.

Theorem 4.4 *Let $\{\nu_j\}_{j\in\mathbb{N}}$ be an orthonormal system in H, and let $k \in \mathbb{N}$. Suppose $\nu_j \in D(A^m)$. Then the functions*

$$\varphi_{k,j}(x) := \cos(kx)\nu_j$$

belong to $H_1 = L^2([0, \pi]; H)$, and are eigenfunctions of the operator L_o corresponding to the eigenvalues $\lambda_{k,j} := k^{2m}\delta_j + \delta_j^{m.}$

Proof: Note that $\nu_j \in D(A^m)$ implies $\cos(kx)\nu_j \in D(A^m)$ for all $x \in [0, \pi]$, since the scalar multiplier does not affect domain membership. Therefore, the operator A^m acts pointwise on $\varphi_{k,j}(x) := \cos(kx)\nu_j$, and all expressions involving A^m and higher derivatives are valid. To verify the eigenstructure, we apply the operator L_o to the functions $\cos kx\nu_j$, where ν_j is an eigenvector of A with eigenvalue δ_j, i.e, $A\nu_j = \delta_j\nu_j$, and $A^m\nu_j = \delta_j^m\nu_j$. A straightforward computation yields the following expression:

$$L_o(\cos(kx)\nu_j) = (-1)^m(A(\cos(kx)\nu_j))^{2m} + A^m(\cos(kx)\nu_j).$$

As ν_j is independent of x, differentiation acts solely on the cosine term.

$$\frac{d^{2m}}{dx^{2m}}[\cos(kx)\nu_j] = (-1)^m k^{2m}\cos(kx)\nu_j \tag{4.1}$$

and

$$\frac{d^{2m}}{dx^{2m}}(A(\cos(kx)\nu_j) = A(\frac{d^{2m}}{dx^{2m}}(\cos(kx)\nu_j)) = (-1)^m k^{2m}\cos(kx)\nu_j \tag{4.2}$$

From 4.1 and 4.2

$$\begin{aligned} L_o(\cos kx.\nu_j) &= (-1)^m(A(\cos(kx)\nu_j))^{2m} + A^m(\cos(kx)\nu_j) \\ &= (-1)^m(\cos kx)^{2m}\nu_j\delta_j + \cos(kx)\delta_j^m\nu_j \\ &= (k^{2m}\delta_j + \delta_j^m)\cos(kx)\nu_j \end{aligned}$$

Consequently, $\cos(kx)\nu_j$ is an eigenfunction of L_o, corresponding to the eigenvalue $k^{2m}\delta_j + \delta_j^m$, as required. □

This integral identity plays a crucial role in establishing the symmetric nature of the operator.

Lemma 4.1 *Let $y(x), z(x) \in D(L_0)$. Then the following identity holds:*

$$\int_0^\pi ((Ay(x))^{2m}, z(x))dx = \int_0^\pi (y(x), (Az(x))^{2m})dx.$$

Proof: The proof proceeds by successive integration by parts, leveraging the boundary conditions

$$(Ay(0))^{2i-1} = (Ay(\pi))^{2i-1} = 0, \quad i = 1, \dots, m,$$

which guarantee the vanishing of boundary terms. The identity is established through successive applications of integration by parts and the self-adjointness of the operator A. Let us denote the derivatives with respect to x by primes. Starting with the left-hand side:

$$\int_0^\pi ((Ay(x))^{2m}, z(x))dx.$$

Applying integration by parts, we obtain:

$$= ((Ay(x))^{2m-1}, z(x)) \,|_0^\pi - \int_0^\pi ((Ay(x))^{2m-1}, z^{|}(x))dx$$

Repeating this process for each derivative order:

$$= -((Ay(x))^{2m-2}, z^{|}(x)) \,|_0^\pi + \int_0^\pi ((A(y(x))^{2m-2}, z^{||}(x))dx,$$

After carrying out the integration procedure iteratively, we obtain:

$$= \int_0^\pi (Ay(x), z^{2m}(x))dx.$$

Since A is self-adjoint

$$(Ay(x), z^{2m}(x)) = (y(x), Az^{2m}(x)),$$

and from linear operator of A, and we have:

$$(Ay(x), z^{2m}(x)) = (y(x), (Az(x))^{2m}).$$

Thus,

$$\int_0^\pi ((Ay(x))^{2m}, z(x))dx = \int_0^\pi (y(x), (Az(x))^{2m})dx.$$

□

Before establishing self-adjointness, we first verify the symmetry of the operator.

Theorem 4.5 *Suppose that $y, z \in D(L_o)$and for every $x \in [0, \pi]$, $y(x), z(x) \in D(A^m)$. Then the operator L_o is symmetric on the Hilbert space H_1, i.e.,*

$$(L_o y, z)_{H_1} = (y, L_o z)_{H_1}.$$

The operator L_o is symmetric on the Hilbert space H_1.

Proof: We further assume that for every $x \in [0, \pi]$, both $y(x)$ and $z(x)$ possess sufficient regularity under the operator A, so that all iterated compositions up to order m are defined. In particular, $A^k y(x), A^k z(x) \in D(A)$ for all $k < m$, ensuring that all terms involving $(Ay(x))^{2m}$, $A^m y(x)$, and similarly for $z(x)$, are well-defined in H. We aim to prove that for all $y, z \in D(L_o)$, the symmetric condition

$$(L_o y, z)_{H_1} = (y, L_o z)_{H_1},$$

holds.

By the definition of the inner product in H_1*, we have :*

$$(L_o y, z)_{H_1} = \int_0^\pi (L_o y, z)_H dx = \int_0^\pi (\ell_o(y), z)_H dx$$

substituting this into the inner product the definition of L_o,

$$= \int_0^\pi ((-1)^m (Ay(x))^{2m} + A^m y(x), z(x)) dx$$

$$= (-1)^m \int_0^\pi ((Ay(x))^{2m}, z(x)) dx + \int_0^\pi (A^m y(x), z(x)) dx$$

By applying Lemma 4.1 in conjunction with Theorem 4.3

$$= (-1)^m \int_0^\pi (y(x), (Az(x))^{2m}) dx + \int_0^\pi (y(x), A^m z(x)) dx$$

$$= \int_0^\pi (y(x), (-1)^m (Az(x))^{2m} + A^m z(x)) dx$$

$$= \int_0^\pi (y(x), \ell_o(z)) dx$$

$$= \int_0^\pi (y(x), L_o z)_H dx.$$

Hence,

$$(L_o y, z)_{H_1} = (y, L_o z)_{H_1}, \forall y, z \in D(L_o).$$

□

The following example illustrates the abstract theory in a concrete Hilbert space setting, clarifying the spectral structure.

Example 4.1 *Let* $H = L^2([0,1])$*, and consider the differential operator*

$$A := -\frac{d^2}{dx^2}$$

with domain

$$D(A) := \left\{ f \in H^2([0,1]) \,\middle|\, f(0) = f(1) = 0 \right\}.$$

Then A *is a positive self-adjoint operator with compact inverse in* H*. The eigenvalues of* A *are given by*

$$\delta_n = (n\pi)^2, \quad n \in \mathbb{N},$$

and the corresponding orthonormal eigenfunctions are

$$\nu_{n(x)} = \sqrt{2}\sin(n\pi x), \quad n \in \mathbb{N}.$$

We define the operator L_o *in the Hilbert space* $H_1 = L^2([0,\pi]; H)$*, generated by the expression*

$$L_o y(x) = (-1)^m (Ay(x))^{2m} + A^m y(x),$$

subject to boundary conditions

$$(Ay(0))^{2i-1} = (Ay(\pi))^{2i-1} = 0, \quad i = 1, \ldots, m.$$

In this setting, the eigenfunctions of L_o *are*

$$\varphi_{k,n}(x) = \cos(kx)\nu_n, \quad k \in \mathbb{N}_0,\, n \in \mathbb{N},$$

and the corresponding eigenvalues are

$$\lambda_{k,n} = k^{2m}\delta_n + \delta_n^m = k^{2m}(n\pi)^2 + (n\pi)^{2m}.$$

This example demonstrates how the abstract theory applies to a classical differential operator with well-known spectral properties. The operator A satisfies all required conditions: it is self-adjoint, positive, has compact inverse, and possesses a discrete spectrum. The construction of L_o follows naturally from this setting, and the spectral data of L_o can be explicitly described in terms of the eigenvalues and eigenfunctions of A

We establish that the powers of the operator are bounded below by the identity, which is crucial for coercivity.

Theorem 4.6 *Let A be a self-adjoint operator in a Hilbert space H, and suppose that $A \geq I$ where I is the identity operator and $\overline{D(A)} = H$. Assume further that for a given positive integer m, the iterated compositions*

$$A^1 := A, \quad A^{k+1} := A \circ A^k$$

are well-defined on the domain

$$D(A^m) := \left\{ y \in D(A) \;\middle|\; A^j y \in D(A) \text{ for all } j = 0, 1, \ldots, m-1 \right\},$$

where $A^0 := I$ and $D(A)$ is dense in H. Then the operator A^m, defined on the domain $D(A^m)$, is self-adjoint and satisfies the inequality:

$$A^m \geq I \quad \text{on } D(A^m).$$

Proof: The iterated powers A^k are considered to act on elements $y \in D(A)$ such that the successive compositions are well-defined, i.e., $A^j x \in D(A)$ for all $j < k$. This structure guarantees that $A^k y \in D(A) \subset H$ for all $k \leq m$, and in particular, that the inequality $A^m \geq I$ is meaningful on its natural domain. We proceed by induction on the positive integer m. For $m = 1$, the assumption implies that $A \geq I$. Assume that $A^k \geq I$ for some $k \in \mathbb{N}$. Since A is a positive self-adjoint operator, it follows that A^k is also positive and self-adjoint. Then

$$A^{k+1} = A^k A \geq A^k I = A^k \geq I.$$

Therefore, by induction, $A^m \geq I$ holds for all $m \in N$.

Hence, the operator A^m is bounded below by the identity operator. □

Theorem 4.7 *[9,14] Let A be a self-adjoint operator on a Hilbert space H. Then the spectrum of A lies entirely within the set of real numbers, i.e.,*

$$\sigma(A) \subset \mathbb{R}.$$

Theorem 4.8 *[11] Let A be a self-adjoint operator on a Hilbert space H such that $A \geq I$. Then every eigenvalue δ_j of A satisfies $\delta_j \geq 1$, and the operator A^{-1} exists and is bounded.*

This result consolidates key spectral properties of the operator, including positivity and invertibility.

Corollary 4.1 *Let A be a self-adjoint operator such that $A \geq I$ and let $A^m = A \circ A \circ \cdots \circ A$ (m times). Then:*

1. *The set of eigenvalues of the operator is contained in the real line, i.e., $\sigma(A^m) \subset \mathbb{R}$.*
2. *The eigenvalues of A^m are greater than or equal to 1, i.e., $\sigma(A^m) \subset [1, \infty)$.*
3. *The inverse operator A^{-m} exists and is bounded.*

4. *The eigenvalues of the operator* $\ell_o(y) = (-1)^m (Ay(x))^{2m} + A^m y(x)$ *are all positive.*

Proof:

1. Since A^m is self-adjoint and by Theorem 5.3, we have $\sigma(A^m) \subset \mathbb{R}$.

2. From Theorem 4.6, we have $A^m \geq I$, which implies that for all $x \in H$,

$$(A^m x, x) \geq (Ix, x).$$

   Since A has purely discrete spectrum, it follows that $\sigma(A^m) = \{\delta_j^m\}_{j=1}^{\infty}$, where each $\delta_j^m \geq 1$.

3. By Theorem 4.8, since $A \geq I$, the operator A^{-1} exists and is bounded. Hence, $A^{-m} = (A^{-1})^m$ also exists and is bounded. As A^m is self-adjoint by Theorem 4.3, all points in $\mathbb{R} \setminus [1, \infty)$ are regular points, i.e., $0 \in \rho(A^m)$, implying $(A^m - 0I)^{-1} = A^{-m}$ is defined on the entire space.

4. As the eigenvalues of ℓ_o possess the same property as those of A^m, it follows that the eigenvalues of ℓ_o are positive

□

5. Closure Properties

Within the theory of unbounded operators, it is critical to confirm that the constructed differential operator is closable and that its closure retains key structural properties such as symmetry and positivity. In this study, we establish that the symmetric operator L_o admits a closure $\overline{L_o}$, which is itself symmetric and densely defined in the Hilbert space H_1. Furthermore, the quadratic form associated with $\overline{L_o}$ is shown to be coercive, satisfying

$$(\overline{L_o} y, y)H_1 \geq \|y\|_{H_1}^2, \quad \forall y \in D(\overline{L_o}).$$

This ensures that the spectrum of $\overline{L_o}$ is bounded below and lies entirely on the real axis. Consequently, the interval $(-\infty, 1)$ is included in the resolvent set $\rho(\overline{L_o})$, ensuring the closed operator is suitable for rigorous spectral analysis and further trace considerations.

Theorem 5.1 *[4] Let A be a symmetric operator. Then:*

1. $A \subset A^*$.

2. $\overline{A}$ *is symmetric.*

Theorem 5.2 *[6,18] If a closed symmetric operator has a complete set of eigenvectors, then it is self-adjoint.*

We confirm that the symmetric operator admits a closure, which is also symmetric.

Corollary 5.1

1. *Since L_o is symmetric, we have $L_o \subset L_o^*$.*

2. *Let L_o be a symmetric operator. Then its closure $\overline{L_o}$ is symmetric.*

3. *If L_o is closed and symmetric, it follows that L_o is self-adjoint.*

Proof: Direct consequence of Theorem 5.1 and Theorem 5.2 □

We denote the closure of $L_o^{'} := \overline{L_o}$. Then $L_o^{'}$ is a closed symmetric operator defined as

$$L_o^{'} : D(L_o^{'}) \to H_1.$$

Theorem 5.3 *[9,14] Let A be a self-adjoint operator on a Hilbert space H. Then the spectrum of A lies entirely within the set of real numbers, i.e.,*

$$\sigma(A) \subset \mathbb{R}.$$

We establish the positivity of the quadratic form associated with the closure of the operator.

Theorem 5.4 *Let $L_o^{'}$ be the closure of the operator L_o. Assume that for every $y \in D(L_o^{'})$, there exists a sequence $\{y_n\} \subset D(L_o)$ such that $y_n(x) \in D(A^m)$ for all $x \in [0,\pi]$. Then the following inequality holds:*

$$(L_o^{'}y, y)_{H_1} \geq (y, y)_{H_1}.$$

Proof: Since $L_o^{'} := D(\overline{L_o})$, observe that $y \in D(L_o^{'})$ implies $y \in D(\overline{L_o})$. Then there exists a sequence $(y_n) \subset D(\overline{L_o})$ such that $\lim y_n = y$ and $\lim L_o y_n = y^*$, with $\overline{L_o} y = y^*$.
We analyze

$$(L_o^{'}y, y), y)_{H_1} = \int_0^\pi (L_o^{'}y, y)_H dx = \int_0^\pi (\overline{L_o}y, y)_H .dx \tag{5.1}$$

Then,

$$(\overline{L_o}y, y) = (y^*, y) = (\lim L_o y_n, \lim y_n) = \lim (L_o y_n, y_n) = \lim_{n\to\infty} (\ell_o y_n, y_n). \tag{5.2}$$

It follows that

$$\begin{aligned}(\ell_o y_n, y_n) &= ((-1)^m (Ay_n(x))^{2m} + A^m y(x), y_n(x)) \\ &= ((-1)^m (Ay_n(x))^{2m}, y_n(x)) + (A^m y(x), y_n(x)).\end{aligned} \tag{5.3}$$

Substituting (5.3) into (5.2):

$$(\overline{L_o}y, y) = \lim_{n\to\infty} \{((-1)^m \left(Ay_n(x))^{2m}, y_n(x)\right) + (A^m y(x), y_n(x))\} \tag{5.4}$$

Substituting (5.4) into (5.1):

$$\begin{aligned}(L_o^{'}y, y)_{H_1} &= \int_0^\pi (\overline{L_o}y, y)_H dx \\ &= \lim_{n\to\infty} \int_0^\pi \left\{\left((-1)^m (Ay_n(x))^{2m}, y_n(x)\right) + (A^m y(x), y_n(x))\right\} \\ &= \lim_{n\to\infty} \{\int_0^\pi ((-1)^m \left(Ay_n(x))^{2m}, y_n(x)\right)\} + \lim_{n\to\infty} \{\int_0^\pi (A^m y(x), y_n(x))\}\end{aligned}$$

The first term is evaluated by repeated integration by parts using repeated integration by parts. Due to the boundary conditions

$$(Ay(0))^{2i-1} = (Ay(\pi))^{2i-1} = 0 \quad for \quad i = 1, 2, \ldots, m,$$

the boundary terms arising from integration by parts vanish. Therefore,

$$(-1)^m \int_0^\pi \left(Ay_n(x))^{2m}, y_n(x)\right)_H dx = (-1)^m (-1)^m \int_0^\pi \left(A^m y_n(x), y_n^{(m)}(x)\right)_H dx.$$

Thus,

$$\begin{aligned}&(L_o^{'}y, y)_{H_1} = \\ &\qquad = \lim_{n\to\infty} \int_0^\pi ((Ay_n(x))^m, y_n^{(m)}(x))_H dx + \lim_{n\to\infty} \int_0^\pi (A^m y_n(x), y_n(x))_H dx. \\ &\qquad = \lim_{n\to\infty} \int_0^\pi ((Ay_n^m(x)), y_n^{(m)}(x))_H dx + \lim_{n\to\infty} \int_0^\pi (A^m y_n(x), y_n(x))_H dx.\end{aligned}$$

In Theorem 4.6, since $A \geq I$, it follows that $A^m \geq I$, due to the linearity of the operator A

$$(A^m y_n(x), y_n(x))_H \geq \|y_n(x)\|_H^2 \quad \text{for all } x \in [0,\pi],$$

and

$$((Ay_n(x))^m, y_n^m(x))_H \geq \|y_n^m(x)\|_H^2 \quad \text{for all } x \in [0,\pi].$$

Integrating both sides:

$$\int_0^\pi ((A^m y_n(x)), y_n(x)))_H dx \geq \int_0^\pi \|y_n(x)\|_H^2 dx = \|y\|_{H_1}^2,$$

and

$$\int_0^\pi ((Ay_n(x))^m, y_n^m(x)))_H dx \geq \int_0^\pi \|y_n^m(x)\|_H^2 dx = \|y^m\|_{H_1}^2.$$

Finally, since the first term (coming from $(A)^{2m}$ via integration by parts) is non-negative due to coercivity and positivity of A^m, we conclude:

$$(L_o y, y)_{H_1} \geq \|y\|_{H_1}^2 = (y,y)_{H_1}.$$

□

The spectral lower bound implies that the interval $(-\infty, 1)$ lies in the resolvent set of the operator.

Corollary 5.2 *The interval $(-\infty, 1)$ lies in the resolvent set of $L_o^{'}$, i.e.,*

$$(-\infty, 1) \subset \rho(L_o^{'}).$$

Proof: Since $L_o^{'}$ is bounded from below by the identity operator as shown in Theorem 5.4, and its spectrum is real due to symmetry, it follows that the entire negative real line $(-\infty, 1)$ and the interval lies in the resolvent set $\rho(L_o^{'})$. □

6. Conclusion

This work has examined a class of higher–order self–adjoint differential operators in which the unbounded operator appears both within the highest–order derivatives and as an independent power term . The simultaneous occurrence of the unbounded operator in these two distinct roles, together with the imposition of boundary conditions directly on, yields a functional–analytic framework that is fundamentally distinct from those treated in earlier studies, including [4].

Within this framework, we have rigorously determined the domain of the operator, established its well–posedness, and proved self–adjointness, lower semi–boundedness, and an explicit spectral decomposition. The spectral representation demonstrates that the eigenfunctions constitute a complete orthonormal basis of, and that the associated eigenvalues are strictly positive and can be expressed explicitly in terms of the spectrum of.

The results obtained here form a rigorous analytical foundation for subsequent investigations aimed at deriving regularized trace formulas, asymptotic eigenvalue distributions, and other spectral invariants within this operator setting. Such developments will enable a more comprehensive comparison with classical operator–differential models and may reveal further structural properties intrinsic to the formulation presented in this paper.

In addition to their theoretical significance, the techniques and conclusions established are applicable to a broader class of operator–differential systems in which the coefficient operator occupies a dual position, acting both inside and outside the differential expression. This structural flexibility allows for enriched modelling capacity while preserving the fundamental spectral properties required for precise mathematical analysis. The present results also serve as a groundwork for a forthcoming study in which regularized trace formulas and related spectral invariants will be derived for this class of operators.

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