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On the Structure and Generalization of Bihyperbolic Leonardo Sequences

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ABSTRACT: In this paper, we give some properties of the bihyperbolic Leonardo numbers, among others the Binet formula, generating function formula and the general bilinear index-reduction formula which implies d'Ocagne, Vajda, Halton, Catalan, and Cassini identities. We also give the matrix representation and some sum formulas of the bihyperbolic Leonardo numbers. Moreover, we present a one-parameter generalization of the bihyperbolic Leonardo numbers and their properties.

Key Words: generalized Fibonacci-Leonardo numbers, Leonardo numbers, recurrence relations, bihyperbolic numbers.

Contents

1. Introduction

Let $n \geq 0$ be an integer. The nth Leonardo number Le_n is defined recursively by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \text{ for } n \geq 2$$

with $Le_0 = 1,\ Le_1 = 1$. This sequence is the sequence of the on-line encyclopedia of integers sequences [14]. The first ten Leonardo numbers are 1, 1, 3, 5, 9, 15, 25, 41, 67, 109. The sequence of Leonardo numbers can also be defined recursively by the formula

$$Le_n = 2Le_{n-1} - Le_{n-3}, \text{ for } n \geq 3$$

with $Le_0 = 1,\ Le_1 = 1$ and $Le_2 = 3$. Properties of Leonardo numbers we can find in [5]. This sequence has applications in the theory of hypercomplex numbers. Many authors studied Leonardo numbers in the context of complex numbers, dual numbers, hybrid numbers, quaternions, and others. Complex Leonardo numbers were introduced and studied in [10]. Dual Leonardo numbers were examined in [9], dual hyperbolic generalised Leonardo numbers in [6], and bicomplex Leonardo numbers in [15]. Kara and Yilmaz [8] studied Gaussian Leonardo numbers and hybrid numbers with Gaussian Leonardo coefficients. Hybrid Leonardo numbers were investigated in [2]. Spreafico and Catarino introduced hybrid hyper Leonardo numbers in [12]. In [13], a new class of Leonardo hybrid numbers was presented. In [11], defined generalized bronze Leonardo sequence. In [7], hyperbolic Leonardo and Francois quaternions were presented and worked. Hyperbolic k-Leonardo and k-Leonardo Lucas quaternions were studied in [1].

In this paper, we define bihyperbolic Leonardo numbers and give some of their properties. We also consider one-parameter generalization of Leonardo numbers and their bihyperbolic version.

Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3,$$

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where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1,\ j_2,\ j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1j_2 = j_2j_1 = j_3,\ j_1j_3 = j_3j_1 = j_2,\ j_2j_3 = j_3j_2 = j_1. \tag{1.1}$$

Note that the multiplication of bihyperbolic numbers can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients. The addition and multiplication in $\mathbb{H}_2$ are commutative and associative, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. For the algebraic properties of bihyperbolic numbers, see [4].

Let $n \geq 0$ be an integer. The nth bihyperbolic Leonardo number $BhLe_n$ is defined as

$$BhLe_n = Le_n + Le_{n+1}j_1 + Le_{n+2}j_2 + Le_{n+3}j_3,$$

where Le_n is the nth Leonardo number and $j_1,\ j_2,\ j_3$ are units which satisfy (1.1). The bihyperbolic Leonardo numbers starting from $n = 0$ can be written as

$$\begin{aligned}
BhLe_0 &= 1 + 1j_1 + 3j_2 + 5j_3,\\
BhLe_1 &= 1 + 3j_1 + 5j_2 + 9j_3,\\
BhLe_2 &= 3 + 5j_1 + 9j_2 + 15j_3,\\
BhLe_3 &= 5 + 9j_1 + 15j_2 + 25j_3,\\
BhLe_4 &= 15 + 25j_1 + 41j_2 + 67j_3,\\
BhLe_5 &= 25 + 41j_1 + 67j_2 + 109j_3.
\end{aligned}$$

2. Bihyperbolic Leonardo numbers

In this section, we will start by giving the properties of bihyperbolic Leonardo numbers.

Theorem 1 [5] (*Binet formula for Leonardo numbers*) *For* $n \geq 0$

$$Le_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1 = \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2},\quad \beta = \frac{1-\sqrt{5}}{2}. \tag{2.1}$$

Theorem 2 (*Binet formula for bihyperbolic Leonardo numbers*) *Let* $n \geq 0$ *be an integer. Then*

$$BhLe_n = \frac{2\alpha^{n+1}}{\alpha - \beta}\hat{\alpha} - \frac{2\beta^{n+1}}{\alpha - \beta}\hat{\beta} - \hat{\mathbf{1}}, \tag{2.2}$$

where α, β *are given by* (2.1) *and*

$$\hat{\alpha} = 1 + \alpha j_1 + \alpha^2 j_2 + \alpha^3 j_3,\ \hat{\beta} = 1 + \beta j_1 + \beta^2 j_2 + \beta^3 j_3,\ \hat{\mathbf{1}} = 1 + j_1 + j_2 + j_3. \tag{2.3}$$

Proof: By Theorem 1, we get

$$\begin{aligned}
BhLe_n &= \left(2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1\right) + \left(2\left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right) - 1\right) j_1\\
&\quad + \left(2\left(\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right) - 1\right) j_2 + \left(2\left(\frac{\alpha^{n+4} - \beta^{n+4}}{\alpha - \beta}\right) - 1\right) j_3\\
&= \frac{2\alpha^{n+1}}{\alpha - \beta}\left(1 + \alpha j_1 + \alpha^2 j_2 + \alpha^3 j_3\right) - \frac{2\beta^{n+1}}{\alpha - \beta}\left(1 + \beta j_1 + \beta^2 j_2 + \beta^3 j_3\right)\\
&\quad - \left(1 + j_1 + j_2 + j_3\right),
\end{aligned}$$

which ends the proof.

Using (2.2), we can prove the following theorem.

Theorem 3 *(General bilinear index-reduction formula for bihyperbolic Leonardo numbers) Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$\begin{aligned}
&BhLe_a \cdot BhLe_b - BhLe_c \cdot BhLe_d = \\
&= \frac{4}{5}\left(\alpha^a\beta^b + \beta^a\alpha^b - \alpha^c\beta^d - \beta^c\alpha^d\right)\hat{\alpha}\hat{\beta} \\
&+ \frac{2}{\sqrt{5}}\left(-\alpha^{a+1} - \alpha^{b+1} + \alpha^{c+1} + \alpha^{d+1}\right)\hat{\alpha}\hat{\mathbf{1}} \\
&+ \frac{2}{\sqrt{5}}\left(\beta^{a+1} + \beta^{b+1} - \beta^{c+1} - \beta^{d+1}\right)\hat{\beta}\hat{\mathbf{1}},
\end{aligned}$$

where α, β and $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mathbf{1}}$ are given by (2.1) and (2.3), respectively.

Proof: By (2.2), we have

$$\begin{aligned}
&BhLe_a \cdot BhLe_b - BhLe_c \cdot BhLe_d = \\
&= \left(\frac{2\alpha^{a+1}}{\alpha-\beta}\hat{\alpha} - \frac{2\beta^{a+1}}{\alpha-\beta}\hat{\beta} - \hat{\mathbf{1}}\right)\cdot\left(\frac{2\alpha^{b+1}}{\alpha-\beta}\hat{\alpha} - \frac{2\beta^{b+1}}{\alpha-\beta}\hat{\beta} - \hat{\mathbf{1}}\right) \\
&\quad - \left(\frac{2\alpha^{c+1}}{\alpha-\beta}\hat{\alpha} - \frac{2\beta^{c+1}}{\alpha-\beta}\hat{\beta} - \hat{\mathbf{1}}\right)\cdot\left(\frac{2\alpha^{d+1}}{\alpha-\beta}\hat{\alpha} - \frac{2\beta^{d+1}}{\alpha-\beta}\hat{\beta} - \hat{\mathbf{1}}\right) \\
&= \frac{-4\alpha^{a+1}\beta^{b+1} - 4\beta^{a+1}\alpha^{b+1} + 4\alpha^{c+1}\beta^{d+1} + 4\beta^{c+1}\alpha^{d+1}}{(\alpha-\beta)^2}\hat{\alpha}\hat{\beta} \\
&+ \frac{-2\alpha^{a+1} - 2\alpha^{b+1} + 2\alpha^{c+1} + 2\alpha^{d+1}}{\alpha-\beta}\hat{\alpha}\hat{\mathbf{1}} \\
&+ \frac{2\beta^{a+1} + 2\beta^{b+1} - 2\beta^{c+1} - 2\beta^{d+1}}{\alpha-\beta}\hat{\beta}\hat{\mathbf{1}} \\
&= \frac{4}{5}\left(\alpha^a\beta^b + \beta^a\alpha^b - \alpha^c\beta^d - \beta^c\alpha^d\right)\hat{\alpha}\hat{\beta} \\
&+ \frac{2}{\sqrt{5}}\left(-\alpha^{a+1} - \alpha^{b+1} + \alpha^{c+1} + \alpha^{d+1}\right)\hat{\alpha}\hat{\mathbf{1}} \\
&+ \frac{2}{\sqrt{5}}\left(\beta^{a+1} + \beta^{b+1} - \beta^{c+1} - \beta^{d+1}\right)\hat{\beta}\hat{\mathbf{1}}
\end{aligned}$$

since $a+b=c+d$, $\alpha\cdot\beta=-1$ and $\alpha-\beta=\sqrt{5}$.

For special values of a, b, c, d, by Theorem 3, we can obtain some identities for bihyperbolic Leonardo numbers:

- d'Ocagne type identity – for $a=n$, $b=m+1$, $c=n+1$, $d=m$,
- Vajda type identity – for $a=m+r$, $b=n-r$, $c=m$, $d=n$,
- first Halton type identity – for $a=m+r$, $b=n$, $c=r$, $d=m+n$,
- second Halton type identity – for $a=n+k$, $b=n-k$, $c=n+s$, $d=n-s$,
- Catalan type identity – for $a=n+r$, $b=n-r$, $c=d=n$,
- Cassini type identity – for $a=n+1$, $b=n-1$, $c=d=n$.

The bihyperbolic Leonardo numbers can also be written in recursive form

$$BhLe_n = BhLe_{n-1} + BhLe_{n-2} + \hat{\mathbf{1}},\ \text{for } n \geq 2$$

or

$$BhLe_n = 2BhLe_{n-1} - BhLe_{n-3},\ \text{for } n \geq 3.$$

As with any sequence defined by the recurrence relations, we can also define bihyperbolic Leonardo numbers with negative indices. Let $n \geq 0$ be an integer. The bihyperbolic Leonardo number $BhLe_{-n}$ is defined by

$$BhLe_{-n} = BhLe_{-n+2} - BhLe_{-n+1} - \hat{\mathbf{1}},$$

or

$$BhLe_{-n} = 2BhLe_{-n+2} - BhLe_{-n+3}.$$

Using Leonardo numbers, we can also write $BhLe_{-n}$ as

$$\begin{aligned} BhLe_{-n} = (-1)^n \, [&(Le_{n-2} + 1) + (-Le_{n-3} - 1)j_1 \\ &+(Le_{n-4} + 1)j_2 + (-Le_{n-5} - 1)j_3] - \hat{\mathbf{1}}. \end{aligned}$$

Theorem 4 *The generating formula for the bihyperbolic Leonardo numbers is*

$$\begin{aligned} &\sum_{n=0}^{\infty} BhLe_n t^n = \\ &= \frac{\left(1 - t + t^2\right) + \left(1 + t - t^2\right) j_1 + \left(3 - t - t^2\right) j_2 + \left(5 - t - 3t^2\right) j_3}{1 - 2t - t^3}. \end{aligned}$$

Proof: Let $h(t)$ be the generating function for the bihyperbolic Leonardo numbers as $\sum_{n=0}^{\infty} BhLe_n t^n$. We get the following equations

$$2th(t) = 2\sum_{n=0}^{\infty} BhLe_n t^{n+1}$$

and

$$t^3 h(t) = \sum_{n=0}^{\infty} BhLe_n t^{n+3}.$$

After the needed calculations, the generating function for the bihyperbolic Leonardo numbers is obtained as

$$\sum_{n=0}^{\infty} BhLe_n t^n = \frac{(BhLe_2 - 2BhLe_1)\, t^2 + (BhLe_1 - 2BhLe_0)\, t + BhLe_0}{1 - 2t - t^3},$$

and consequently

$$\begin{aligned} &\sum_{n=0}^{\infty} BhLe_n t^n = \\ &= \frac{\left(1 - t + t^2\right) + \left(1 + t - t^2\right) j_1 + \left(3 - t - t^2\right) j_2 + \left(5 - t - 3t^2\right) j_3}{1 - 2t - t^3}. \end{aligned}$$

Theorem 5 *Let $n > 0$ be an integer. The following equality holds*

$$a) \begin{bmatrix} BhLe_{n+3} & BhLe_{n+2} & BhLe_{n+1} \\ BhLe_{n+2} & BhLe_{n+1} & BhLe_n \\ BhLe_{n+1} & BhLe_n & BhLe_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} BhLe_3 & BhLe_2 & BhLe_1 \\ BhLe_2 & BhLe_1 & BhLe_0 \\ BhLe_1 & BhLe_0 & BhLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n,$$

$$b) \begin{bmatrix} BhLe_{-n+3} & BhLe_{-n+2} & BhLe_{-n+1} \\ BhLe_{-n+2} & BhLe_{-n+1} & BhLe_{-n} \\ BhLe_{-n+1} & BhLe_{-n} & BhLe_{-n-1} \end{bmatrix}$$

$$= \begin{bmatrix} BhLe_3 & BhLe_2 & BhLe_1 \\ BhLe_2 & BhLe_1 & BhLe_0 \\ BhLe_1 & BhLe_0 & BhLe_{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}^n.$$

Proof: a) For the proof, we use induction method on n. The equality holds for $n = 1$.

$$\begin{bmatrix} BhLe_3 & BhLe_2 & BhLe_1 \\ BhLe_2 & BhLe_1 & BhLe_0 \\ BhLe_1 & BhLe_0 & BhLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2BhLe_3 - BhLe_1 & BhLe_3 & BhLe_2 \\ 2BhLe_2 - BhLe_0 & BhLe_2 & BhLe_1 \\ 2BhLe_1 - BhLe_{-1} & BhLe_1 & BhLe_0 \end{bmatrix}$$

$$= \begin{bmatrix} BhLe_4 & BhLe_3 & BhLe_2 \\ BhLe_3 & BhLe_2 & BhLe_1 \\ BhLe_2 & BhLe_1 & BhLe_0 \end{bmatrix}.$$

Now suppose that the equality is true for $n > 1$. Then, we can verify for $n + 1$ as follows

$$\begin{bmatrix} BhLe_3 & BhLe_2 & BhLe_1 \\ BhLe_2 & BhLe_1 & BhLe_0 \\ BhLe_1 & BhLe_0 & BhLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{n+1}$$

$$= \begin{bmatrix} BhLe_3 & BhLe_2 & BhLe_1 \\ BhLe_2 & BhLe_1 & BhLe_0 \\ BhLe_1 & BhLe_0 & BhLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{n} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} BhLe_{n+3} & BhLe_{n+2} & BhLe_{n+1} \\ BhLe_{n+2} & BhLe_{n+1} & BhLe_{n} \\ BhLe_{n+1} & BhLe_{n} & BhLe_{n-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} BhLe_{n+4} & BhLe_{n+3} & BhLe_{n+2} \\ BhLe_{n+3} & BhLe_{n+2} & BhLe_{n+1} \\ BhLe_{n+2} & BhLe_{n+1} & BhLe_{n} \end{bmatrix}.$$

Thus, the theorem can be proved easily.

b) Similarly, the proof is seen by induction on n.

Lemma 1 [5] *Let Le_n be the nth Leonardo number. In this case*

$$\sum_{m=0}^{n} Le_m = Le_{n+2} - (n+2),$$

$$\sum_{m=0}^{n} Le_{2m} = Le_{2n+1} - n,$$

$$\sum_{m=0}^{n} Le_{2m+1} = Le_{2n+2} - (n+2).$$

In the next theorem, we can give the sum of the finite, finite odd and finite even terms of the bihyperbolic Leonardo numbers.

Theorem 6 *Let $BhLe_n$ be the nth bihyperbolic Leonardo number. In this case*

$$\sum_{m=0}^{n} BhLe_m = BhLe_{n+2} - \left(n \cdot \hat{\mathbf{1}} + BhLe_2 - BhLe_0\right),$$

$$\sum_{m=0}^{n} BhLe_{2m} = BhLe_{2n+1} - \left(n \cdot \hat{\mathbf{1}} + BhLe_1 - BhLe_0\right),$$

$$\sum_{m=0}^{n} BhLe_{2m+1} = BhLe_{2n+2} - \left(n \cdot \hat{\mathbf{1}} + BhLe_2 - BhLe_1\right).$$

Proof:

$$\sum_{m=0}^{n} BhLe_m = \sum_{m=0}^{n}(Le_m + Le_{m+1}j_1 + Le_{m+2}j_2 + Le_{m+3}j_3)$$
$$= \sum_{m=0}^{n} Le_m + j_1\sum_{m=0}^{n} Le_{m+1} + j_2\sum_{m=0}^{n} Le_{m+2} + j_3\sum_{m=0}^{n} Le_{m+3}$$
$$= (Le_{n+2} - (n+2)) + (Le_{n+3} - (n+4))j_1$$
$$+ (Le_{n+4} - (n+6))j_2 + (Le_{n+5} - (n+10))j_3$$
$$= BhLe_{n+2} - (n(1 + j_1 + j_2 + j_3) + 2 + 4j_1 + 6j_2 + 10j_3)$$
$$= BhLe_{n+2} - \left(n \cdot \hat{\mathbf{1}} + BhLe_2 - BhLe_0\right).$$

Other sum formulas are proven using through the same method.

3. One-parameter generalization of bihyperbolic Leonardo numbers

One-parameter generalization of Leonardo numbers, i.e. generalized Fibonacci–Leonardo numbers were introduced in [3] quite recently.

Let $n \geq 0$, $t \geq 1$ be integers. The generalized Fibonacci-Leonardo numbers $Le(t,n)$ are given by the recurrence relation

$$Le(t,n) = Le(t,n-1) + Le(t,n-2) + (t-1) \text{ for } n \geq 2, \tag{3.1}$$

with initial terms $Le(t,0) = Le(t,1) = 1$.

As you can see, for $t = 1$ we obtain $Le(1,n) = F_{n+1}$ and for $t = 2$, we have $Le(2,n) = Le_n$. Recall that F_n denotes the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = 0$, $F_1 = 1$.

Theorem 7 [3] (*Binet formula for generalized Fibonacci–Leonardo numbers*) *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$Le(t,n) = t\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - (t-1),$$

where α, β are given by (2.1).

Let $n \geq 0, t \geq 1$ be integers. The nth bihyperbolic generalized Fibonacci-Leonardo number $BhLe(t,n)$ is given by

$$BhLe(t,n) = Le(t,n) + Le(t,n+1)j_1 + (t,n+2)j_2 + Le(t,n+3)j_3, \tag{3.2}$$

where $Le(t,n)$ is the nth generalized Fibonacci-Leonardo number and j_1, j_2, j_3 are units which satisfy (1.1).

The generalized Fibonacci-Leonardo numbers starting from $n = 0$, for $t = 2$, can be written as

$$BhLe(2,0) = 1 + 1j_1 + 3j_2 + 5j_3,$$
$$BhLe(2,1) = 1 + 3j_1 + 5j_2 + 9j_3,$$
$$BhLe(2,2) = 3 + 5j_1 + 9j_2 + 15j_3,$$
$$BhLe(2,3) = 5 + 9j_1 + 15j_2 + 25j_3,$$
$$BhLe(2,4) = 15 + 25j_1 + 41j_2 + 67j_3,$$
$$BhLe(2,5) = 25 + 41j_1 + 67j_2 + 109j_3.$$

Theorem 8 (*Binet formula for bihyperbolic generalized Fibonacci–Leonardo numbers*) *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$BhLe(t,n) = \frac{t\alpha^{n+1}}{\alpha - \beta}\hat{\alpha} - \frac{t\beta^{n+1}}{\alpha - \beta}\hat{\beta} - (t-1)\hat{\mathbf{1}},$$

where α, β and $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mathbf{1}}$ are given by (2.1) *and* (2.3), *respectively.*

Proof: The proof of Theorem 8 is analogous to the proof of Theorem 2.

Using (3.2), we get a general bilinear index-reduction formula for bihyperbolic generalized Fibonacci–Leonardo numbers and other identities (Catalan, Cassini, etc.)

Theorem 9 *(General bilinear index-reduction formula for bihyperbolic generalized Fibonacci–Leonardo numbers) Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$\begin{aligned} & BhLe(t,a)\cdot BhLe(t,b) - BhLe(t,c)\cdot BhLe(t,d) = \\ & = \frac{t^2}{5}\left(\alpha^a\beta^b + \beta^a\alpha^b - \alpha^c\beta^d - \beta^c\alpha^d\right)\hat{\alpha}\hat{\beta} \\ & + \frac{t(t-1)}{\sqrt{5}}\left(-\alpha^{a+1} - \alpha^{b+1} + \alpha^{c+1} + \alpha^{d+1}\right)\hat{\alpha}\hat{\mathbf{1}} \\ & + \frac{t(t-1)}{\sqrt{5}}\left(\beta^{a+1} + \beta^{b+1} - \beta^{c+1} - \beta^{d+1}\right)\hat{\beta}\hat{\mathbf{1}}, \end{aligned}$$

where α, β and $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mathbf{1}}$ are given by (2.1) and (2.3), respectively.

Theorem 10 *The generating formula for the bihyperbolic generalized Fibonacci-Leonardo numbers is*

$$\begin{aligned} & \sum_{n=0}^{\infty} BhLe(t,n)x^n = \\ & = \frac{\left(1-x+tx^2-x^2\right)+\left(1-x+tx-x^2\right)j_1+\left(1+t-x-x^2\right)j_2}{1-2t-t^3} \\ & + \frac{\left(1+2t-x-x^2-tx^2\right)j_3}{1-2t-t^3}. \end{aligned}$$

Proof: The proof of Theorem 10 is analogous to the proof of Theorem 4.

Lemma 2 [3] *Let $Le(t,n)$ be the nth generalized Fibonacci-Leonardo number. In this case*

$$\sum_{m=0}^{n} Le(t,m) = Le(t,n+2) - (t-1)n - t,$$

$$\sum_{m=0}^{n} Le(t,2m) = Le(t,2n+1) - (t-1)n,$$

$$\sum_{m=0}^{n} Le(t,2m+1) = Le(t,2n+2) - (t-1)n - t.$$

In the next theorem, we can give the sum of the finite, finite odd and finite even terms of the bihyperbolic generalized Fibonacci-Leonardo numbers.

Theorem 11 *Let $BhLe(t,n)$ be the nth bihyperbolic generalized Fibonacci-Leonardo number. In this case*

$$\begin{aligned} & \sum_{m=0}^{n} BhLe(t,m) = \\ & = BhLe(t,n+2) - \left(n(t-1)\cdot\hat{\mathbf{1}} + t(BhLe(t,2) - BhLe(t,0))\right), \end{aligned}$$

$$\begin{aligned} & \sum_{m=0}^{n} BhLe(t,2m) = \\ & = BhLe(t,2n+1) - \left(n(t-1)\cdot\hat{\mathbf{1}} + t(BhLe(t,1) - BhLe(t,0))\right), \end{aligned}$$

$$\sum_{m=0}^{n} BhLe(t,2m+1) = \\ = BhLe(t,2n+2) - \left(n(t-1)\cdot\hat{\mathbf{1}} + t(BhLe(t,2) - BhLe(t,1))\right).$$

Proof:

$$\begin{aligned}
&\sum_{m=0}^{n} BhLe(t,m) = \\
&= \sum_{m=0}^{n} (Le(t,m) + Le(t,m+1)j_1 + Le(t,m+2)j_2 + Le(t,m+3)j_3) \\
&= \sum_{m=0}^{n} Le(t,m) + j_1 \sum_{m=0}^{n} Le(t,m+1) + j_2 \sum_{m=0}^{n} Le(t,m+2) \\
&\ + j_3 \sum_{m=0}^{n} Le(t,m+3) \\
&= (Le(t,n+2) - (t-1)n - t) + (Le(t,n+3) - (t-1)(n+1) - t - 1)j_1 \\
&\ + (Le(t,n+4) - (t-1)(n+2) - t - 2)j_2 \\
&\ + (Le(t,n+5) - (t-1)(n+3) - 2t - 3)j_3 \\
&= BhLe(t,n+2) - (n(t-1)(1+j_1+j_2+j_3) + t(1+2j_1+3j_2+5j_3)) \\
&= BhLe(t,n+2) - \left(n(t-1)\cdot\hat{\mathbf{1}} + t(BhLe(t,2) - BhLe(t,0))\right).
\end{aligned}$$

Other sum formulas are proven using through the same method.

4. Concluding Remarks

In this paper, we introduced and studied bihyperbolic Leonardo numbers and their one-parameter generalization bihyperbolic generalized Fibonacci–Leonardo numbers. The Leonardo sequence can also be generalized using Leonardo polynomials. It will be interesting to continue this research by defining and then examining Leonardo bihypernomials. Recently, the sequences of numbers and polynomials have been intensively studied, and the sequences of numbers have been widely used in many research areas, such as architecture, nature, art, statistics, biology, finance, physics, engineering, cryptography and algebraic topology.

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