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Hahn Wijsman Sequence Space

Ishfaq Ahmad Malik*, Jitika Negi and Shivani Thakur

ABSTRACT: In this paper, we introduce and examine the Hahn-Wijsman sequence space $h(W^u)$, a generalized sequence space defined for a metric space (X, d) and a positive sequence $\{u_k\}$, where a sequence $\{x_k\} \in h(W^u)$ if $\sum_{k=1} |x_k|u_k < \infty$. We investigate its structural characteristics by establishing that it forms a vector space and construct an appropriate norm under which the space $h(W^u)$ becomes a normed linear space. We further demonstrate that it is complete and hence a Banach space. The dual space is characterized by bounded linear functionals, and we explore isomorphic relationships between the Hahn-Wijsman space and classical sequence spaces such as $c_0(p), c(p)$ and $\ell(p)$. These results not only provide a deeper understanding of the topological and geometric aspects of the Hahn Wijsman space but also position it as a potential tool for further analysis in functional spaces and contribute to the ongoing development of sequence space theory.

Key Words: Hahn sequence space, Wijsman sequence space, norm, isomorphism.

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1. Introduction

1.1. Sequence Space

In functional analysis and related areas of mathematics, a sequence space is a vector space whose elements are infinite sequences of real or complex numbers. In other words, it is a function space whose elements are functions (or mapping) from the natural numbers to the field K of either real or complex numbers.

It is also called the linear subspace of $\mathbb{K}^{\mathbb{N}}$ where $\mathbb{K}^{\mathbb{N}}$ is the space of all sequences of elements of $\mathbb{K}$. It is equipped with a norm or a topological vector space structure. There are different types of sequence space as follows:

- c_0 (null sequences): This space contains sequences that converge to zero.

$$c_0 = \left\{x = \{x_n\} : \lim_{n\to\infty} x_n = 0\right\} \tag{1.1}$$

- c (convergent sequences): This space includes all sequences that converge to a finite value, which means that c_0 is a subset of c.

$$c = \left\{x = \{x_n\} : \lim_{n\to\infty} x_n \; exists\right\} \tag{1.2}$$

* Corresponding author.
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- ℓ_p (p- summable sequences): For any $1 \leq p < \infty$, this space contains sequences of real and complex numbers where the p^{th} power of the absolute values of the term is summable such that $\sum |x_n|^p$ is convergent.

$$\ell_p = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} \tag{1.3}$$

- ℓ_∞ (bounded sequences): This is the largest space, including all sequences in which the absolute value of each term is bounded by a finite constant.

$$c_0 \subseteq c \subseteq \ell_p(1 \leq p < \infty) \subseteq \ell_\infty \tag{1.4}$$

1.2. Hahn Sequence Space

The traditional sequence spaces approach sequence convergence, such as pointwise convergence, often prove to be restrictive, failing to capture the full spectrum of behaviors exhibited by sequences. Recognizing this limitation, Hans Hahn [11] introduced a Hahn Sequence Space 'h' defined by

$$h = \left\{ x = (x_k) \in \omega : \sum_k k\,|x_k - x_{k+1}| < \infty \text{ and } \lim_{k \to \infty} x_k = 0 \right\} \tag{1.5}$$

for all $k \in \mathbb{N}$.

1.3. Wijsman Convergence

Wijsman Convergence was defined by Robert Wijsman [1] in 1966. Let (X, d) be a metric space and let $Cl(X)$ denote the collection of all $d-$closed subsets of X. For a point $x \in X$ and a set $A \in Cl(X)$, set

$$\{d(x, A) = \inf_{a \in A} d(x, a)\} \tag{1.6}$$

A sequence of sets $A_n \to Cl(X)$ is said to be Wijsman Convergent to $A \in Cl(X)$ if, for each $x \in X$, $d(x, A_n) \to d(x, A)$. This means that the distance from any point x in the space X to the sets A_n converges to the distance from x to the set A.

Wijsman Convergence studies how the distance between points and sets behave rather than direct set to set convergence. Specifically, it deals with the convergence of sequences of closed sets in a metric space, having various applications in convex analysis, optimization, functional analysis and set-valued analysis, etc.

2. Main Results

2.1. Hahn Wijsman Sequence Space

Hahn Sequence Space [11] is a generalization of classical sequence space that consists of sequences whose weighted series a mathematical series where each term is multiplied by a weight sequence or weight function (a function that assigns a positive value to each element in a set often to control the influence of different elements in a mathematical structure given by

$$\sum_{k=1}^{\infty} x_k u_k \tag{2.1}$$

where $\{x_k\}$ is a sequence of real or complex numbers and $\{u_k\}$ is a weight sequence with $u_k > 0]$ converges absolutely, i.e. a sequence belongs to a Hahn type space if

$$\sum_{k=1}^{\infty} |x_k| u_k < \infty \tag{2.2}$$

By applying this summability [9,12] condition (summability generalizes the idea of convergence to sequence instead of requiring a sequence to converge in the usual sense, summability allows us to assign a meaningful limit to a sequence under certain transformations. A sequence $\{x_k\}$ is called summable if a transformation of its results in a finite sum or a well-defined limit. In the context of sequence spaces like Hahn Wijsman Sequence Space, summability is used to control the behavior of sequences by introducing weighted conditions) Hahn Sequence Space and using the Wijsman sequence convergence, we obtain the Hahn Wijsman Sequence Space $[h(W^u)]$, defined as

$$h(W^u) = \left\{ x = \{x_k\} : \sum_{(k=1)}^{\infty} |x_k| u_k < \infty \right\} \tag{2.3}$$

where, $x = \{x_k\}$ is a sequence of real or complex numbers.

$u = \{u_k\}$is a sequence of positive numbers, $u_k > 0 \; \forall \; k \in N$. (X, d) is a metric space [4,7].

That means this $h(W^u)$ consists of all sequence $\{x_k\}$ such that weighted series $\sum_{k=1}^{\infty} |x_k| u_k$ converges. Hence $h(W^u)$ is a more restrictive version where sequences must satisfy an absolute summability condition. Now, we will explore the various properties of Hahn- Wijsman Sequence Space which are as follows:

Theorem 2.1 *The space $h(W^u)$ is a vector space [8] over $\mathbb{R}$ (or $\mathbb{C}$).*

Proof: i) Closure under addition:
Let $x = \{x_k\}$ and $y = \{y_k\}$ be two elements of $h(W^u)$ meaning

$$\sum_{k=1}^{\infty} |x_k| u_k < \infty, \sum_{k=1}^{\infty} |y_k| u_k < \infty \tag{2.4}$$

,

We define a new sequence $z = x + y$, i.e. $z_k = x_k + y_k$ for each k. Now we check whether z belong to $h(W^u)$ i.e. whether the series

$$\sum_{k=1}^{\infty} |z_k| u_k \tag{2.5}$$

By triangle inequality,

$$|z_k| = |x_k + y_k| \leq |x_k| + |y_k| \tag{2.6}$$

Multiply both sides by u_k and summing

$$\sum |z_k| u_k \leq \sum (|x_k| u_k + |y_k| u_k) \tag{2.7}$$

Since both sums on the right are finite (because $x, y \in h(W^u)$ and their sum is also finite). Thus

$$\sum |z_k| u_k < \infty \tag{2.8}$$

proving $z \in (W^u)$. Hence $h(W^u)$ is closure under addition.
ii) Closure under scalar multiplication:
Let $\alpha \in \mathbb{R}(or\ \mathbb{C})$ and $x \in h(W^u)$. Define $y = \alpha x$, so $y_k = \alpha x_k$, then

$$\sum |y_k| u_k = \sum |\alpha x_k| u_k = |\alpha| \sum |x_k| u_k \tag{2.9}$$

Since

$$\sum |x_k| u_k < \infty \tag{2.10}$$

It follows

$$\sum |y_k| u_k < \infty \tag{2.11}$$

Thus, $y \in h(W^u)$, proving closure under scalar multiplication. Hence, $h(W^u)$ forms a linear space over $\mathbb{R}$ (*or* $\mathbb{C}$) [8]. □

Normed Space Property: We define the norm for the sequence $x = \{x_k\} \in h(W^u)$ as

$$\|x\|_{h(W^u)} = \sum_{k=1}^{\infty} |x_k| u_k \tag{2.12}$$

Theorem 2.2 *Under the norm defined in 2.12, the space $h(W^u)$ forms a normed linear space.*

Proof: i) Non-Negativity:
Since absolute value and positive weights are used, each term in the sum is non-negative.

$$|x_k|u_k \geq 0 \; for \; all \; k. \tag{2.13}$$

Since a sum of non-negative terms is also non-negative, we get

$$\|x\| = \sum_{k=1}^{\infty} |x_k|u_k \geq 0 \tag{2.14}$$

We need to show that $\|x\| = 0$ iff $x_k = 0$ for all k. If $x_k = 0$ for all k, then clearly

$$\sum_{k=1}^{\infty} |x_k|u_k = \sum_{k=1}^{\infty} 0 = 0 \tag{2.15}$$

Conversely, suppose $\|x\| = 0$,

$$\sum_{k=1}^{\infty} |x_k|u_k = 0 \tag{2.16}$$

Since $u_k > 0$ for all k . If $|x_k| = 0$ for every k, thus $x_k = 0$ for all k which implies $x = 0$. The function $\|x\|$ is non-negative and equal to zero if and only if $x = 0$.
ii) Triangle Inequality

$$\|x + y\| = \sum_{k=1}^{\infty} (|x_k + y_k|u_k) \tag{2.17}$$

Using the triangle inequality for real numbers, we have

$$|x_k + y_k| \leq |x_k| + |y_k| \tag{2.18}$$

for all k Multiplying both sides by u_k (which is positive),

$$|x_k + y_k|u_k \leq |x_k|u_k + |y_k|u_k \tag{2.19}$$

Summing over all k,

$$\sum_{k=1}^{\infty} |x_k + y_k|u_k \leq \sum_{k=1}^{\infty} |x_k|u_k + \sum_{k=1}^{\infty} |y_k|u_k \tag{2.20}$$

This simplifies to

$$\|x + y\| \leq \|x\| + \|y\| \tag{2.21}$$

iii) Homogeneity (Absolute Scalability)
By definition,

$$\|\alpha x\| = \sum_{k=1}^{\infty} |\alpha x_k|u_k \tag{2.22}$$

Using the absolute value property,

$$|\alpha x_k| = |\alpha||x_k| \tag{2.23}$$

Substituting this into the sum

$$\sum_{k=1}^{\infty} |\alpha||x_k|u_k \tag{2.24}$$

Since $|\alpha|$ is a constant, we factor it out

$$|\alpha| \sum_{k=1}^{\infty} |x_k|u_k = |\alpha|\|x\| \tag{2.25}$$

Hence, the function defined by 2.12 satisfies absolute homogeneity.
As the function defined in 2.12 satisfy all properties. So, the space $h(W^u)$ is a normed linear space. □

Theorem 2.3 *Prove that under the norm defined in Theorem 2.2, the space $h(W^u)$ is a Banach Space.*

Proof: Let $x^n = \{x_k^n\}$ be a Cauchy sequence A sequence x^n in $h(W^u)$ is Cauchy if

$$\forall \in > \ 0, \ \exists \, N \in \mathbb{N} \tag{2.26}$$

such that for all $m, n \geq N$

$$\|x^m - x^n\| = \sum_{k=1}^{\infty} |x_k^m - x_k^n| u_k < \in \tag{2.27}$$

This means that for large m, n the sequence x^m and x^n are closed in the norm and x_k^n is Cauchy for each fixed k. From the definition of the norm

$$\sum_{k=1}^{\infty} |x_k^m - x_k^n| u_k < \in \tag{2.28}$$

Since all terms are non-negative , each individual term must get arbitrarily small as $m, n \to \infty$ Thus, for each fixed k, the sequence $\{x_k^n\}$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$). Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, there exists a limit x_k, such that $x_k^n \to x_k$ as $n \to \infty$. This defines a new sequence $x = \{x_k\}$.
Now show that $x = \{x_k\}$ belongs to $h(W^u)$ and for that we need to prove that

$$\sum_{k=1}^{\infty} |x_k| u_k < \infty \tag{2.29}$$

From the triangle inequality,

$$|x_k| \leq |x_k^n| + |x_k - x_k^n| \tag{2.30}$$

Thus,

$$\sum_{k=1}^{\infty} |x_k| u_k \leq \sum_{k=1}^{\infty} |x_k^n| u_k + \sum_{k=1}^{\infty} |x_k - x_k^n| u_k \tag{2.31}$$

where, the first term $\sum_{k=1}^{\infty} |x_k^n| u_k$ is finite because $x^n \in h(W^u)$ and the second term $\sum_{k=1}^{\infty} |x_k - x_k^n| u_k$ can be made arbitrarily small because x^n is Cauchy. Thus, $x \in h(W^u)$, that is, $x = \{x_k\}$ is a valid element of $h(W^u)$.
Now we need to show that x^n converges to x in the norm. That is

$$\|x^n - x\| = \sum_{k=1}^{\infty} |x_k^n - x_k| u_k \to \infty \; as \; n \to \infty \tag{2.32}$$

Since $x_k^n \to x_k$ for each k and x^n is Cauchy, the sum can be made arbitrarily small, therefore,

$$\sum_{k=1}^{\infty} |x_k^n - x_k| u_k \to 0 \tag{2.33}$$

It implies $x^n \to x$ in $h(W^u)$ meaning x^n converges in norm. Since every Cauchy sequence in $h(W^u)$ converges in norm to an element of $h(W^u)$, it follows that the space is complete. Thus, $h(W^u)$ forms a Banach space [6]. □

Dual Space: The dual space [10] consists of all bounded, linear functions of $h(W^u)$. A function f acting on $x \in h(W^u)$ has the form

$$f(x) = \sum_{k=1}^{\infty} x_k y_k \tag{2.34}$$

for some sequence $\{y_k\}$. For f to be a bounded function, we need the supremum condition

$$\sup_{x\neq 0} \frac{|f(x)|}{\|x\|} < \infty \tag{2.35}$$

Applying Holder's inequality, this means that y_k should satisfy

$$\sup_k \left| \frac{y_k}{u_k} \right| < \infty \tag{2.36}$$

Thus, y_k must belong to the weighted $l^\infty(\frac{1}{u})$ space, hence its dual space is given by:

$$(h(W^u))^* = l^\infty(\frac{1}{u}) \tag{2.37}$$

i) α- Dual Space: The α-dual [2] of $h(W^u)$ denoted by $(h(W^u))^\alpha$, consists of all sequence $y = y_k$ such that the series

$$\sum_{k=1}^{\infty} x_k y_k \tag{2.38}$$

converges for all $x \in h(W^u)$. This is equivalent to requiring $y = y_k$ to be in $\ell^\infty(\frac{1}{u})$, which means $\sup_k |y_k|| \frac{y_k}{u_k}| < \infty$. Thus, the α-dual space of $h(W^u)$ is given by:

$$(h(W^u))^\alpha = \ell^\infty(\frac{1}{u}) \tag{2.39}$$

ii) β- Dual Space: The β-dual [2] of $h(W^u)$, denoted by $(h(W^u))^\beta$, consists of all sequences $y = y_k$ such that

$$\sum_{k=1}^{\infty} x_k y_k < \infty, \ \forall\, x \in h(W^u) \tag{2.40}$$

This means $y = \{y_k\}$ must belong to the weighted ℓ^1 space

$$\sum_{k=1}^{\infty} \frac{|y_k|}{u_k} < \infty \tag{2.41}$$

Thus, the β- dual space is given by:

$$(h(W^u))^\beta = \ell^1(\frac{1}{u}) \tag{2.42}$$

iii) γ- Dual Space: The γ- dual [2] of $h(W^u)$ denoted by $(h(W^u))^\gamma$ consists of all sequences $y = y_k$ such that

$$\sum_{k=1}^{\infty} x_k y_k \tag{2.43}$$

converges for all $x \in h(W^u)$ This requires that y belong to $c_0(\frac{1}{u})$ meaning that $\frac{y_k}{u_k} \to 0$ as$k \to \infty$. Thus the γ- dual space is given by:

$$(h(W^u))^\gamma = c_0(\frac{1}{u}) \tag{2.44}$$

Theorem 2.4 *Isomorphism: Prove that the Hahn Wijsman Sequence Space $h(W^u)$ is isomorphic [2,3] to the classical sequence spaces $c(p), c_0(p)$ and ℓ_p*

Proof: The Hahn Wijsman Sequence Space is defined as:

$$h(W^u) = \{x = \{x_k\} : \sum_{k=1}^{\infty} |x_k| u_k < \infty\} \tag{2.45}$$

where u_k is a positive sequence.
I. Checking isomorphic with ℓ_p: The classical weighted ℓ_p space [3] is

$$\ell_p = \left\{ x = \{x_k\} : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\} \tag{2.46}$$

Define the mapping

$$T(x) = \{x_k u_k\} \tag{2.47}$$

i) Linearity: If $x, y \in h(W^u)$, then

$$T(ax + by) = \{(ax + by)u_k\} = \{a\{x_k.u_k\} + b\{y_k.u_k\}\} = a\,T(x) + b\,T(y) \tag{2.48}$$

Thus, T is linear.
ii) Injectivity: If $T(x) = 0$, then $x_k.u_k = 0 \; \forall \; k$

$$\Rightarrow x_k = 0 \tag{2.49}$$

So, $x = 0$, proving injectivity.
iii) Boundedness of T: If $x \in h(W^u)$, then

$$\sum_{k=1}^{\infty} |x_k| u_k < \infty \tag{2.50}$$

But this is exactly the norm definition of ℓ_p

$$\Rightarrow T(x) \in \ell_p \tag{2.51}$$

iv) Inverse Mapping:

$$\ell_p \to h(W^u) \tag{2.52}$$

Define $S(y) = \left\{ \frac{y_k}{u_k} \right\}$.
If $y_k \in \ell_p$, then

$$\sum_{k=1}^{\infty} \left| \frac{y_k}{u_k} \right| u_k = \sum_{k=1}^{\infty} |y_k| < \infty \tag{2.53}$$

So $S(y) \in h(W^u)$

$$\Rightarrow S = T^{-1} \tag{2.54}$$

Thus, T is a bounded bijective linear operator with a bounded inverse.

$$\Rightarrow h(W^u) \simeq \ell_p \tag{2.55}$$

II. Checking isomorphism with $c(p)$: Here we have,

$$c(p) = \left\{ x = \{x_k\} \in \ell_\infty : \lim_{k\to\infty} x_k \; exists \right\} \tag{2.56}$$

Define the mapping:

$$T(x) = \{x_k p_k\} \tag{2.57}$$

i) Linearity: It holds since multiplication by p_k preserves addition.
ii) Injectivity: $T(x) = 0$ implies $x_k p_k = 0$ meaning $x_k = 0$.

iii) Boundedness: Since $\sum_{k=1}^{\infty} |x_k| u_k < \infty$, this ensures weighted convergence.

iv) Inverse Mapping: Define $S(y) = \left\{ \frac{y_k}{p_k} \right\}$, which maps back to $h(W^u)$

$$\Rightarrow h(W^u) \simeq c(p) \tag{2.58}$$

III. Checking isomorphism with $c_0(p)$: It consists of sequences $x = \{x_k\}$ such that $\lim_{k\to\infty} x_k = 0$. That is,

$$c_0 = \left\{ x = \{x_k\} : \lim_{k\to\infty} x_k = 0 \right\} \tag{2.59}$$

Define the mapping :

$$T(x) = \{x_k p_k\} \tag{2.60}$$

Similarly, linearity holds, injectivity follows from the same argument and boundedness is also satisfied.

$$\Rightarrow h(W^u) \simeq c_0(p) \tag{2.61}$$

Hence proved. □

3. Conclusion

In this paper, we introduce and study the Hahn- Wijsman sequence space $h(W^u)$, motivated by the convergence concepts introduced by Wijsman [1,13] and developed through the structural framework of Hahn sequence spaces [11]. We thoroughly investigated its algebraic and topological properties, demonstrating that it forms a vector space [8] under the standard operations of vector addition and scalar multiplication. By constructing an appropriate norm, we proved that this space satisfies all the conditions required to be classified as a normed linear space [5]. We further established its completeness, thus confirming that it is a Banach space [6]. A key contribution of this work lies in the detailed characterization of the space's duals [2]. Specifically, we have determined the α-dual, β-dual and γ-dual of $h(W^u)$ offering deeper insights into its structure, transformation behavior, and functional interactions with other sequence spaces [2,10]. These duals play an essential role in understanding the continuity of linear operators and in the study of matrix transformations.

Furthermore, we have shown that $h(W^u)$ is isomorphic [2,3] to classical sequence spaces such as $c_0(p), c(p)$ *and* $l(p)$; thereby reinforcing its place in the wider landscape of sequence space theory. These results place the Hahn-Wijsman sequence space within the broader framework of summability theory [9] and functional analysis. The properties derived in this paper not only enrich the existing body of knowledge on sequence spaces but also open new directions for further mathematical research in operator theory, summability theory [9,12], topology of functional spaces, and applied analysis.

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Ishfaq Ahmad Malik,
Department of Mathematics,
Yogananda School of AI Computers and Data Sciences
Shoolini University,
India.
E-mail address: ishfaqmalik@shooliniuniversity.com