Bol. Soc. Paran. Mat. (3s.) v. 2025 (43) : 1–6.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.77519

$Z-S-$ Coprime Modules

Farhan Dakhil Shyaa

ABSTRACT: In this essay, we present the idea $Z-S-$ coprime modules. M is called an $Z-S-$ coprime modules. if $ann_R M = ann_R \frac{M}{A} \forall A \ll_Z M$ and $A < M$. In this work examines the characteristics of the $Z-S-$ coprime modules as a expanding upon of the coprime. This paper provides various characterizations and properties Z- S- coprime modules.

Key Words: Coprime modules, $S-$ coprime modules, $Z-S-$ coprime modules.

Contents

1 Introduction 1

2 $Z-S-$Coprime Modules 1

2.1 Remarks and Examples 2

1. Introduction

Algebra has established it self as an important mathematical tool in wide variety subjects see [4,9]. During this article, all rings are assumed to be commute rings with unity $1 \neq 0$. Additionally, all modules considered as left module. If M is an R-module, we indicate $U \leq M (U < M)$ for U is a submodule (proper submodule) of M, $[U :_R M]$ stands for $\{r \in R : rM \subseteq U\}$, when $U = (0)$, then $[0 :_R M]$ means $ann_R M$. A submodule U of M is called small (denoted by $U \ll M$) if $U + K \neq M$, for any $K < M$ [6]. As a dual notion of small submodule is essential (or large) submodule, where a submodule is A of M is called essential denoted by $(A \leq_{ess} M)$ if $A \bigcap B \neq 0$ for any $0 \neq B \leq M$. Equivalently, if $A \bigcap B = 0, B \leq M$, then $B = 0$ [3].

Remember that if $ann_R M = ann_R U$ for every non-zero submodule A of M, then an R-module M is prime Equivalently M is prime if $ann_R M = ann_R (x)$, for each $0 \neq x \in R$ [8]. S. Annine in [2] presented a dual notion of prime module namely coprime module, an $R-$module M is called coprime if $ann_R M = ann_R \frac{M}{A}$ for each $A < M$. Equivalently M is coprime if for each $0 \neq r \in R$, eitherr $M = 0$ or $rM = M(M)$ is a second submodule of M. Note that second submodules introduced in 2001 by S. Yassmi [10].

Rasha in [1,7], introduced the concepts principally coprime, S-coprime modules by restricting the definition of coprime module on cyclic submodule, small submodules, respectively.

Within this work, we will be exploring $Z-S-$coprime using $Z-$small submodules which is introducedby in [5] where a submodule U of M is named $Z-$small (abbreviated $U \ll_Z M$) if whenever $U + W = M$ and $W \geq Z_2(M)$, then W=M. Note that $Z_2(M)$ is the second submodule of M(or Golde torsion) defined by $Z(\frac{M}{Z(M)} = \frac{Z_2(M)}{Z(M)}$ where $Z(M) = x \in R : ann(x) A \leq_{ess} R$. It is clear that every small submodule is $Z-$small, but not conversely, see [5].

Note that M is small submodule iff $M = 0$, however, M may be $Z-$small of M and $M \neq 0$. In this paper we introduce a new generalization of coprime namely $Z-S-$coprime satisfies $ann_R M = ann_R \frac{M}{A}$ for all $A \ll_Z M$ and $A < M$. In this work various properties for this class of module were discussed.

2. $Z-S-$Coprime Modules

Definition 2.1. *The name of an R-module M is $Z-S-$coprime module if $ann_R M = ann_R \frac{M}{A} \forall A \ll_Z M$ and $A < M$.*

2010 *Mathematics Subject Classification:* 16D10.

Submitted June 26, 2025. Published December 05, 2025

2.1. Remarks and Examples

Remark 2.2 (1). *If M is a $Z-S-$coprime module, then M is $S-$coprime, but the converse may be not hold.*

Proof. Let $A \ll M$. Hence $A \ll_Z M$, as $MZ-S-$coprime, if $ann_R M = ann_R \frac{M}{A}$. Thus $MS-$coprime. □

Example 2.3 (2). *Consider Z_6 as $Z-$module. Z_6 is S-coprime since Z_6 is semi-simple see [[7], Example 3.2.2 (2)]. On other hand, every submodule of Z_6 is Z-small so $A =< \overline{2} > \ll_Z Z_6$, but $ann_Z Z_6 \neq ann \frac{Z_6}{<2>}$, so that Z_6 is not $Z-S-$coprime.*

Remark 2.4 (3). *A multiplication R-module is denoted by M. Then M is $Z-S-$coprime if and only if (0) is the only $Z-S-$small submodule in M.*

Proof. $\Rightarrow$ Let $A < M$, $A \ll_Z$. So that $ann_R M = [A :_R M], (if ann_R M)M = [A :_R M] M$ and so $(0) = A$.
$\Leftarrow$ It is clear since (0) is only $Z-$small in M and $ann_R M = ann_R \frac{M}{(0)}$. □

Remark 2.5 (4). *If M is hollow $R-$module. Next, the corresponding statement is equivalent:*

(i) M is $Z-S-$ coprime module.

(ii) M is S-coprime module.

(iii) M is coprime.

Proof. (i) $\Rightarrow$ (ii) by part (1). (ii) $\Rightarrow$ (iii) [[7], Proposition 3.2.7]. (iii) $\Rightarrow$ (i) Let $A \ll_Z M$ and $A \neq M$. Because of M is hollow, $A \ll M$. But M is S-coprime so, $ann_R M = ann_R \frac{M}{A}$.

Remark 2.6. *An $R-$module $M$with $Z_2(M) = 0$, then M is S-coprime if and only if M is $Z-S-$coprime.*

The following are characterizations of is $Z-S-$coprime module, but first recall that a homothety. An R-endomorphism r^* is called homothety if, for every x in M $r^*(x) = rx$ and M an R-module [10]. □

Theorem 2.7.

1. *M is Z- S-coprime.*
2. *$\forall r \in R, rM \ll_Z M$, then $rM = M$ or $rM = (0)$. For every ideal I of R, $IM \ll_Z M$, implies $IM = M$ or $IM = (0)$.*
3. *$\forall$ homothety $r^*(M) \ll_Z M$, implies $rM = M$ or $rM = (0)$.*

Proof. (1) $\Rightarrow$ (2) Let $rM \ll_Z M$. Then either $rM = M$ or $rM \neq M$. If $rM = M$, therefore nothing to prove. If $rM < M$, $rM \ll_Z M$, then $ann_R M = ann_R \frac{M}{rM} = [rM :_R M]$. As $r \in [rM :_R M]$, so that $r \in annM$; that is $rM = (0)$.

(2) $\Rightarrow$ (1) Let $A \ll_Z M, A < M$. Let $r \in [A :_R M], r \neq 0$; hence $rM \leq A$. Since $A \ll_Z M$, $rM \ll_Z M$. By condition (2) either $rM = M$ or $rM = 0$, but $rM = M$ implies contradiction. Thus $rM = 0$ and so $r \in annM$. Therefore, M is $Z-S-$coprime.

(2) $\Leftrightarrow$ (4) It is clear.

(1) $\Rightarrow$ (3) If $IM \ll_Z M$, then either $IM = M$or $IM < M$. If $IM = M$, therefore nothing to verify. If $IM < M$, then $I \leq [IM :_R M] \leq annM$. Hence, $IM = 0$.

(3) $\Rightarrow$ (1) Let $A \ll_Z M, A \neq M$. Assume $r \in [A :_R M]$, $r \neq 0$. Then $rM \leq A$ and also, $rM \ll_Z M, rM \neq M$. Thus, $(r)M \ll_Z M$. By part (3), $(r)M = 0$; that is $rM = (0)$, then $r \in annM$. Therefore, M is $Z-S-$coprime module

□

Lemma 2.8. *Suppose M represent a finitely generated faithful multiplication of an R-module. Let I be an ideal of R. Then $I \ll_Z R$ if and only if $IM \ll_Z M$.*

Proof. $\Rightarrow$ Asume $I \ll_Z R$ and $IM + B = M$ with $B \geq Z_2(M)$. As M is faithful finitely generated multiplication R-module, so $Z_2(M) = Z_2(R) M$, also $B = JM$ for some ideal J of R. Hence $IM + JM = M$; that is $(I + J) M = RM$. But Mis a faithful finitely generated multiplication R-module implies that $I + J = R$ and $J \geq Z_2(R)$. Since $I \ll_Z R$, we conclude that $J = R$ and so $B = JM = RM = M$.

$\Leftarrow$ Assume $IM \ll_Z M$ and $I + K = R$ with $J \geq Z_2(R)$.Then $IM + JM = M$ and $JM \geq Z_2(R) M = Z_2(M)$. As $IM \ll_Z M$, we get $JM = M$ and since M s a faithful multiplication with finite generation, $J = R$. □

Corollary 2.9. *Considering that M is a finitely generated faithful multiplication module, $A \leq M$, then A is Z-small submodule of $M \iff [A :_R M] \ll_Z R$.*

Proposition 2.10. *Presume M be a faithful multiplication $R-$module that is finitely generated. R is $Z-S-$coprime, If and only if M is $Z-S-$coprime.*

Proof. $\Rightarrow LetI \ll_Z R$. Then via Lemma 2.8 $IM \ll_Z M$. But M is $Z-S-$coprime, so either $IM = M$ or $IM = 0$, if $IM = M$, then $I = R$ if $IM = 0$ implies $I \leq annM = 0$.

$\Leftarrow LetA \ll_Z M$. Then $A = IM$ for some $I \leq R$. Hence $IM \ll_Z M$ which implies $I \ll_Z R$ via Lemma 2.8. But R is $Z-S-$coprime, so $I = 0$ (since R is $Z-S-$coprime). Thus $A = IM = 0$. □

Lemma 2.11. *If $\frac{W}{N} \ll_Z \frac{M}{N}, N \ll_Z M$ and $Z_2\left(\frac{M}{N}\right) = \frac{Z_2(M)+N}{N}$. Then $W \ll_Z M$.*

Proof. Let $W + V = M$ such that $V \geq Z_2(M).\frac{W}{N} + \frac{V+N}{N} = \frac{M}{N}, \frac{V+N}{N} \geq \frac{Z_2(M)+N}{N} = Z_2\left(\frac{M}{N}\right)$. Since $\frac{W}{N} \ll_Z \frac{M}{N}$, we get $\frac{V+N}{N} = \frac{M}{N}$, implies that $V + N = M$. But $N \ll_Z M$, hence $V = M$. Thus $W \ll_Z M$. □

Proposition 2.12. *Let M be $Z-S-$coprime, $N \ll_Z M$. Then $\frac{M}{N}$ is $Z-S-$coprime, provided $Z_2\left(\frac{M}{N}\right) = \frac{Z_2(M)+N}{N}$.*

Proof. Let $\frac{W}{N} \ll_Z \frac{M}{N}$. But $N \ll_Z M$, then via Lemma 2.11 $W \ll_Z M$. As M is an $Z-S-$coprime, then $ann_R M = ann_R \frac{M}{N}$. Since $ann_R \frac{M}{N}/\frac{W}{N} = ann_R \frac{M}{W}$. Thus $ann_R \frac{M}{N}/\frac{W}{N} = ann_R M$. Therefore, $ann_R \frac{M}{N}/\frac{W}{N} = ann_R \frac{M}{N}$. Thus $\frac{M}{N}$ is $Z-S-$coprime.

For instance, we see that the opposite of this **assertion 2.8** is generally untrue. □

Example 2.13. $\frac{Z_{12}}{(\mathbf{2})} \cong Z_2$ *as $Z-$module is $Z-S-$coprime and Z_{12} is not $Z-S-$coprime.*

However, the converse of Proposition 2.12 is true under specific circumstances, according to the following assertion

Proposition 2.14. *Consider M be an R-module and $N \ll M$ such that $ann_R \frac{M}{N} = ann_R M$ and $Z_2\left(\frac{M}{N}\right) = \frac{Z_2(M)+N}{N}$. If $\frac{M}{N}$ is $Z-S-$coprime, then M is $Z-S-$coprime.*

Proof. To show M is $Z-S-$coprime, we must prove $ann_R \frac{M}{W} = ann_R M$ for each $W \ll_Z M$ and $W < M$. Since N and W are $Z-$small in M, then by [[8], Remarks and examples 2.2 (8)] $W + N \ll_Z M$ and $\frac{W+N}{N} \ll_Z \frac{M}{N}$. But $\frac{M}{N}$ is $Z-S-$coprime, hence $W + N \neq M$, so $ann_R \frac{M}{N} = ann_R \frac{\frac{M}{N}}{\frac{W+N}{N}}$. Thus $ann_R \frac{M}{N} = ann_R \frac{M}{W+N}$.

Now, let $r \in ann_R \frac{M}{W}$, then $rM \leq W$. Therefore, $rM \leq W + N$; that is $\in [W + N :_R M]$. Thus $r \in ann_R \frac{M}{W+N} = ann_R \frac{M}{N}$. But $ann_R M = ann_R \frac{M}{N}$ by hypothesis. Therefore, $r \in ann_R M$ and $ann_R \frac{M}{W} \leq ann_R M$. Thus $ann_R \frac{M}{W} = ann_R M$. Hence, M is $Z-S-$coprime. □

Corollary 2.15. *Presume N be a small submodule of M and $ann_R \frac{M}{N} = ann_R M$. Then $\frac{M}{N}$ $Z-S-$coprime iff M $Z-S-$coprime.*

Proposition 2.16. *If $M \cong M'$, then M is $Z-S-$coprime if and only if M' $Z-S-$coprime.*

Proof. Let M be $Z-S-$coprime. Since $M \cong M'$, then there exists $f : M' \longmapsto M$ such that f is an isomorphism. Let $rM' \ll_Z M', r \neq 0$, then $f\left(rM'\right) = rf\left(M'\right) = rM$. By [[5], Proposition 2.3] $rM \ll_Z M$ and since M is $Z-S-$coprime $rM = 0$, then $f\left(rM'\right) = 0$. Hence, $rM' = 0$. Thus M' $Z-S-$coprime.

The opposite is similarly proven.

Since the zero submodule is only $Z-$small in M but $N = (0) \oplus Z_4$ is not $Z-S-$coprime in M because it is not $S-$coprime [3].

Nevertheless, we possess the subsequent. □

Proposition 2.17. *Consider M as an R-module that is $Z-S-$coprime and N be a submodule of M such that $[U :_R N] = [U :_R M]$ for every $Z-$small submodule U of N, then N is $Z-S-$coprime.*

Proof. We must prove $ann_R N = [U :_R N]$ for every $U \ll_Z N$, and $U \neq N$ then we have $U \ll_Z M$. Since M is $Z-S-$coprime, then $[U :_R M] = ann_R M \leq ann_R N$. But by hypothesis $[U :_R N] = [U :_R M]$. Thus $[U :_R N] \leq ann_R N$.Therefore, N is $Z-S-$coprime.

□

Proposition 2.18. *Let M_1, M_2 be two faithful's $Z-S-$coprime modules, then $M = M_1 \oplus M_2$ is $Z-S-$coprime.*

Proof. Let $r \in R, r \neq 0$ such that $r\left(M_1 \oplus M_2\right) \ll_Z M$. $r\left(M_1 \oplus M_2\right) = rM_1 \oplus rM_2$. Hence by [8, proposition 2.7] $rM_1 \ll_Z M_1$ and $rM_2 \ll_Z M_2$. Since M_1 and M_2 are $Z-S-$coprime, then by Theorem 2.7 $rM_1 = M_1$, and $rM_2 = M_2$. This implies that $r\left(M_1 \oplus M_2\right) = M_1 \oplus M_2$. Thus, via Theorem 2.7 $M_1 \oplus M_2$ is $Z-S-$coprime. □

Theorem 2.19. *Consider two R -modules, denoted as M_1, M_2 such that $ann_R M_1$ and $ann_R M_2$ are non-comparable prime ideals of R. If $M = M_1 \bigoplus M_2$ is an $Z-S-$coprime, then M_1 and M_2 are $Z-S-$coprime.*

Proof. $N_1 \ll_Z M_1$ and $N_2 \ll_Z M_2$ and $N_1 \neq M_1$, $N_2 \neq M_2$. To prove $ann_R M_1 = [N_1 :_R M_1]$ and $ann_R M_2 = [N_2 :_R M_2]$. Since $N_1 \ll_Z M_1$ and $N_2 \ll_Z M_2$, then by [8, proposition 2.7] $N_1 \oplus N_2 \ll_Z M_1 \oplus M_2$, moreover, $N_1 \oplus N_2 \neq M_1 \oplus M_2$ But M is $Z-S-$coprime, hence $ann_R M_1 \oplus M_2 = [N_1 \oplus N_2 :_R M_1 \oplus M_2]$, so $ann_R M_1 \cap ann_R M_2 = [N_1 :_R M_1] \cap [N_2 :_R M_2]$. Thus, $= [N_1 :_R M_1] \cap [N_2 :_R M_2] \leq ann_R M_1$ and $= [N_1 : {}_R M_1] \cap [N_2 :_R M_2] \leq ann_R M_2$. Since, $ann_R\ M_1$ and $ann_R M_2$ are prime ideals, then $[N_1 :_R M_1] \leq ann_R M_1$ or $[N_2 :_R M_2] \leq ann_R M_1$ and $[N_1 :_R M_1] \leq ann_R M_2$ or $[N_2 :_R M_2] \leq ann_R M_2$. Hence, we get $[N_1 :_R M_1] = ann_R M_1$ or $ann_R M_1 \leq ann_R M_2$ and $ann_R M_1 \leq ann_R M_2$ or $[N_2 :_R M_2] = ann_R M_2$. But $ann_R M_1$ and $ann_R M_2$ are non-comparable, then $[N_1 :_R M_1] = ann_R M_1$ and $[N_2 :_R M_2] = ann_R M_2$. Therefor, M_1 and M_2 are $Z-S-$coprime. □

Lemma 2.20. *Consider two $R-$modules, M_1, M_2, such that $ann_R M_1 + ann_R M_2 = R$. Then $ann_R M_1$ and $ann_R M_2$ are non-comparable.*

Hance, We get the following result.

Corollary 2.21. *Presume M_1, M_2 are two R-modules, $M = M_1 \oplus M_2$ such that $ann_R M_1 + ann_R M_2 = R$ and $ann_R M_1$, $ann_R M_2$ are prime ideals. If M is $Z-S-$coprime, then M_1, M_2 are $Z-S-$coprime.*

Corollary 2.22. *Let M_1, M_2 be two R-modules such that $ann_R\ M_1$, $ann_R M_2$ are prime ideals, then M_1, M_2 are $Z-S-$coprime if and only if $M_1 \oplus M_2$ is $Z-S-$coprime.*

Proposition 2.23. *Let M be an R-module such that $Hom_R\left(M, N\right) = 0$ for every $N \ll_Z M$ and $N \neq M$. Then M is an Z-S-coprime module.*

Proof. Let $N \ll_Z M$ and $N \neq M$ clearly $ann_R M \subseteq [N :_R M]$. Let $r \in [N :_R M]$. Then $rM \subseteq N$. Define $f : M \longmapsto N$ by $f(m) = rm$ for each $m \in M$ f is well defined R-homomorphism; that is $f \in Hom(M, N) = 0$. Thus $f(M) = rM = 0$ and so $r \in ann_R M$. □

Proposition 2.24. *Let M be scalar R-module, then M is an $Z-S-$coprime module over $E = End(M)$ if and only if $Hom(M, N) = 0$ for each $Z-$small N of M.*

Proof. To prove $Hom_R(N, M) = 0$ for every $N \ll_Z M, N \neq M$. Let $f \in Hom_R(N, M)$, then $f(M) \subseteq N$. Since M is a scalar $R-$module every submodule is $E-$submodule. Also, every $Z-$small submodule is $Z-$small $E-$submodule. Thus, N is $Z-$small E-submodule and so $f(M) \subseteq N$ implies $f \in [N :_R M] = ann_E M = 0$. Hence, $f = 0$. Thus $f = 0$ for every $N \ll_Z M,\ N \neq M$. Then, by Proposition 2.23 M is an $Z-S-$coprime.

The proof of the converse is similarly. □

Proposition 2.25. *If M is an $Z-S-$coprime $E-$module, then M is $-S-$coprime $R-$module.*

Proof. Let $rM \ll_Z M$, $rM \neq M, r \neq 0$. To prove rM is $Z-$small $E-$submodule. Let $rM + B = M$, $B \supseteq Z_2(M)$, since $rM \ll_Z M$, so $B = M$. But M $Z-S-$coprime $E-$module, so $rM = 0$. Thus, M is an $Z-S-$coprime $R-$module. □

Proposition 2.26. *Let M be a chained module over regular ring, then the following are equivalent.*

1. *M is an S-coprime R-module.*
2. *M is an $Z-S-$coprime R-module.*
3. *M is a coprime R-module.*
4. *M is a prime M-module.*
5. *M is a quasi-dedkind R-module.*

Proof.
(1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3) by Remark 2.2 (3)
(3)$\Longleftrightarrow$(4)$\Longleftrightarrow$ (5). It follows that by [[7], Proposition 3.2.27]

□

References

1. A. M. Ali and I. K. Rasha, *S-coprime modules*, Journal of Basrah Researches: Sciences **34** (2011), no. 4.
2. S. Annin, *Associated and attached primes over noncommutative rings*, Phd thesis, University of California, Berkeley, Berkeley, CA, USA, 2002.
3. Kenneth Goodearl, *Ring theory: Nonsingular rings and modules*, vol. 33, CRC Press, 1976.
4. I. M. A. Hadi and F. D. Shyaa, *T-stable-extending modules and strongly t-stable extending modules*, Iraqi Journal of Science **61** (2020), no. 2, 401–408.
5. Amina T Hamad and Alaa A Elewi, *Z-small submodules and z-hollow modules*, Iraqi journal of science **62** (2021), no. 8, 2708–2713.
6. F. Kasch, *Modules and rings*, London Mathematical Society Monograph, Academic Press, London, UK, 1989.
7. RI Khalaf, *Dual notions of prime submodules and prime modules*, Msc. thesis, University of Baghdad, Baghdad, Iraq, 2009.
8. SA Saymach, *On prime submodules*, University Noc. Tucumare Ser. A **29** (1979), 121–136.
9. F. D. Shyaa, *Approximately 2-absorbing and weakly approximately 2-absorbing sub-modules*, Journal of Discrete Mathematical Sciences & Cryptography **28** (2025), no. 2, 303–309.
10. Siamak Yassemi, *The dual notion of prime submodules*, Archivum Mathematicum **37** (2001), no. 4, 273–278.

Farhan Dakhil Shyaa,
Department of Mathematics,
College of Education, University of Al-Qadisiyah,
Al-Qadisiya, Iraq.
E-mail address: farhan.shyaa@qu.edu.iq