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Spectral Properties of the Cartesian Product of $K_m \square K_g$ Graph

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ABSTRACT: This paper investigates the Laplacian spectral properties of the Cartesian product of graph $K_m \square K_g$, focusing on the trace, energy, and characteristic polynomial coefficients of Laplacian matrix of the graph. We derive general formulas for the trace of Laplacian matrix powers and provide recursive relations for the Laplacian coefficients of characteristic polynomial using trace identities for $K_m \square K_g$ graph, also found upper bounds of eigenvalue of Laplacian matrix of $K_m \square K_g$.

keywords : Cartesian product, Laplacian matrix, Trace.

Contents

1. Introduction.

For all terminology and concepts related to Graph theory, readers are referred to [7]. Here we consider simple and non-loop graphs including the restricted number of nodes and edges.

The Cartesian product of two simple graphs H and K is the graph $G = H \square K$ with $V(G) = V(H) \square V(K)$ in which vertices (a, b) and (c, d) are adjacent in K if and only if either $a = c$ and b, d are adjacent in K, or $b = d$ and a, c are adjacent in H.

Let $G = (V, E)$ be a connected graph, $L = D - A$ is the Kirchhoff matrix (Laplacian matrix) of G, where D and A are the diagonal degree matrix and the adjacency matrix of G respectively. It is recognized that L is positive semi-definite, symmetric and $|L| = 0$.

Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{(n-1)} \geq \alpha_n = 0$ are the eigenvalues of L. The Kirchhoff eigenvalues of a graph are important in the graph theory since they have a near relation to frequent graph invariants. Several new inequalities are obtained for the modules, the real part, and imaginary part of a linear arrangement of the ordered eigenvalues of a square matrix.

The spectrum of a graph G involves of its eigenvalues, with the leading eigenvalue α_1 mentioned to as the spectral radius or index of G. In spectral graph theory, spectral inequalities involve one or more spectral invariants, typically eigenvalues play a crucial role in describing a graph's structure. These inequalities have diverse applications and are derived using various methods, including the Rayleigh principle, eigenvalue interlacing, spectral moments, eigenvector analysis, and relationships between spectral and structural invariant. A specific type of spectral inequality is a bound on a spectral invariant, which may apply to entire graphs or a particular class of graphs. In both cases, extremal graphs for a given invariant are those that achieve the bound, if it exists. For instance, it is established that $\alpha_1 \geq 0$, with equality occurring only when G is absolutely disconnected. (Lower or greater)

In this article we gave a new greater bound for the Kirchhoff eigen values of a Cross product of graph.

Theorem 1.1 *The greatest Laplacian eigenvalue of $K_m \square K_g$ is $g + m$.*

Proof: Let $L(K_m)$ and $L(K_g)$ be the Laplacian matrices of K_m and K_g, respectively.

The Laplacian eigenvalues of K_m are:

$$\{0, m, m, \ldots, m\} \quad (m - 1 \text{ times } m)$$

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and of K_g are:

$$\{0, g, g, \ldots, g\} \quad (g-1 \text{ times } g)$$

Since the Laplacian of the Cartesian product satisfies:

$$\text{spec}(L(K_m \Box K_g)) = \{\lambda_i + \mu_j : \lambda_i \in \text{spec}(L(K_m)), \mu_j \in \text{spec}(L(K_g))\},$$

the maximum eigenvalue occurs at:

$$\alpha_1 = \max\{\lambda_i + \mu_j\} = m + g.$$

□

Proposition 1.1 *The trace of the Laplacian matrix of $K_m \Box K_g$ is given by*

$$\text{tr}(L) = (g-1)g + (m-1)m + (m-1)(g-1)(m+g).$$

Proof: The Laplacian eigenvalues of K_m are 0 (once) and m (multiplicity $m-1$), and of K_g are 0 (once) and g (multiplicity $g-1$). The eigenvalues of $K_m \Box K_g$ are given by $\lambda_i + \mu_j$ for eigenvalues λ_i of K_m and μ_j of K_g.

The trace sums over all combinations of eigenvalues:
$\text{tr}(L) = (g-1)g + (m-1)m + (m-1)(g-1)(m+g)$.

□

Proposition 1.2 *The trace of the k^{th} power of the Laplacian matrix of $K_m \Box K_g$ is*

$$\text{tr}(L^k) = (g-1)g^k + (m-1)m^k + (m-1)(g-1)(m+g)^k.$$

Proof: The eigenvalues of $L(K_m \Box K_g)$ consist of all sums $\lambda_i + \mu_j$ for eigenvalues $\lambda_i \in 0, m$ and $\mu_j \in 0, g$.

The terms are categorized by multiplicity:

- $0 + 0 = 0$ with multiplicity 1
- $0 + g = g$ with multiplicity $g - 1$
- $m + 0 = m$ with multiplicity $m - 1$
- $m + g = m + g$ with multiplicity $(m-1)(g-1)$

Then,

$$\text{tr}(L^k) = 1 \cdot 0^k + (g-1)g^k + (m-1)m^k + (m-1)(g-1)(m+g)^k$$
$$= (g-1)g^k + (m-1)m^k + (m-1)(g-1)(m+g)^k.$$

□

Example:

For k = 1, 2, 3 and 4 from the above expression we have the following.
$tr(L) = (g-1)g + (m-1)m + (m-1)(g-1)(m+g)$.
$tr(L^2) = (g-1)g^2 + (m-1)m^2 + (m-1)(g-1)(m+g)^2$.
$tr(L^3) = (g-1)g^3 + (m-1)m^3 + (m-1)(g-1)(m+g)^3$.
$tr(L^4) = (g-1)g^4 + (m-1)m^4 + (m-1)(g-1)(m+g)^4$.

2. Laplacian coefficients of Characteristic polynomial of $K_m \Box K_g$ graph.

In studying the chemical characteristics of molecules, coefficients of a characteristic polynomial are helpful. The characteristic polynomial of a graph can be evaluated using a variety of techniques ([1], [2], [10], [13], [14]). In [14], it is asserted that the Faddeev-Le Verrier- Frame approach [16,18] is the most effective technique for calculating the characteristic polynomial. Balasubramanian ([11], [12]) frequently used the computerized version technique.

The coefficients of the characteristic polynomial

$$p_g(\alpha) = \alpha^n + q_1\alpha^{n-1} + q_2\alpha^{n-2} + \cdots + q_n$$

of a graph G encode important structural information about the graph.

For instance, the number of edges in G is given by $-q_2$, and the number of triangles is given by $-\frac{q_3}{2}$. These relationships are derived from the combinatorial interpretation of the characteristic polynomial, as described in [15]. The following theorem from that work explicitly connects the polynomial's coefficients to the graph's topology.

Theorem 2.1 *[3] The Laplacian characteristic polynomial of a graph G is*

$$\psi(G, x) = x^n + q_1x^{n-1} + \cdots + q_{n-1}x + q_n,$$

where $q_i(i = 1, 2, 3...n)$ are the Laplacian coefficients, E_i $(1 \leq i \leq n)$ denotes the set of elementary sub graphs of G with i vertices, and $r(\wedge)$ and $s(\wedge)$ are the rank and co-rank of E_i, respectively. We then have

$$(-1)^i q_i = \sum_{\wedge \in E_i} (-1)^{r(\wedge)} 2^{s(\wedge)}.$$

Using the trace of the Laplacian matrix, Ivailo M. Mladenov et al. proposed an elegant method to determine the coefficients of the characteristic polynomial. In addition, Samuel Jurkiewicz et al. studied the computation of Laplacian coefficients using structural properties of the graph, as discussed in [3].

In this work, we introduce an alternative technique for identifying the Laplacian coefficients of the characteristic polynomial for the Cartesian product $K_m \Box K_g$. This approach is grounded in the use of the trace of the Laplacian matrix, combined with the classical Faddeev–LeVerrier algorithm and the theorem provided in [15].

Proposition 2.1 *Let the characteristic polynomial of the Laplacian matrix of $K_m \Box K_g$ be*

$$\phi(x) = \det(xI - L) = x^n + q_1x^{n-1} + q_2x^{n-2} + q_3x^{n-3} + q_4x^{n-4} + \cdots$$

then

$$q_1 = -\left[(g-1)g + (m-1)m + (m-1)(g-1)(m+g)\right].$$

$$q_2 = \tfrac{1}{2} \cdot mg \begin{pmatrix} g^3m + 2g^2m^2 - 4g^2m - g^2 + gm^3 - 4gm^2 \\ +2gm + 3g - m^2 + 3m - 2 \end{pmatrix}.$$

$$q_3 = -\frac{1}{6} \cdot gm \begin{pmatrix} g^5m^2 + 3g^4m^3 - 6g^4m^2 - 3g^4m + 3g^3m^4 - 12g^3m^3 + 3g^3m^2 + 15g^3m + 2g^3 \\ + g^2m^5 - 6g^2m^4 + 3g^2m^3 + 22g^2m^2 - 18g^2m - 8g^2 \\ - 3gm^4 + 15gm^3 - 18gm^2 + 6g + 2m^3 - 8m^2 + 6m \end{pmatrix}$$

.

$$q_4 = \tfrac{1}{24} \cdot mg \begin{pmatrix} g^7m^3 + 4g^6m^4 - 8g^6m^3 - 6g^6m^2 + 6g^5m^5 - 24g^5m^4 + 42g^5m^2 + 11g^5m \\ + 4g^4m^6 - 24g^4m^5 + 12g^4m^4 + 94g^4m^3 - 64g^4m^2 - 66g^4m - 6g^4 + g^3m^7 \\ - 8g^3m^6 + 94g^3m^4 - 134g^3m^3 - 62g^3m^2 + 103g^3m + 30g^3 - 6g^2m^6 + 42g^2m^5 \\ - 64g^2m^4 - 62g^2m^3 + 138g^2m^2 - 24g^2m - 24g^2 + 11gm^5 - 66gm^4 + 103gm^3 \\ - 24gm^2 - 24gm - 6m^4 + 30m^3 - 24m^2 \end{pmatrix}.$$

Proof: $q_1 = -tr(L)$

$$= -\left[(g-1)g + (m-1)m + (m-1)(g-1)(m+g)\right],$$

$$q_2 = \tfrac{-1}{2} tr\{B_1 \mathtt{L}\} \quad \text{where } B_1 = \mathtt{L} + q_1 I$$

$$= \tfrac{-1}{2} tr(L^2 + q_1 L)$$

from the above examples:

$$= \tfrac{1}{2} \cdot mg \begin{pmatrix} -g^4m - 2g^3m^2 + 3g^3m - g^2m^3 + 4g^2m^2 - 8g^2m \\ +3g^2 + 3gm^2 - 8gm + 6g - 2m + 2 \end{pmatrix},$$

$$q_3 = \tfrac{-1}{3} tr\{B_2 \mathtt{L}\} \quad \text{where } B_2 = B_1\mathtt{L} + q_2 I$$

$$= \tfrac{-1}{3} tr\{(\mathtt{L} + q_1 I)\mathtt{L}^2 + q_2 L\}$$

$$= \tfrac{-1}{3} tr\{\mathtt{L}^3 + q_1 \mathtt{L}^2 + q_2 \mathtt{L}\}$$

from the the above examples:

$$q_3 = -\tfrac{1}{6} \cdot gm \begin{pmatrix} g^5m^2 + 3g^4m^3 - 6g^4m^2 - 3g^4m + 3g^3m^4 - 12g^3m^3 + 3g^3m^2 + 15g^3m + 2g^3 \\ +g^2m^5 - 6g^2m^4 + 3g^2m^3 + 22g^2m^2 - 18g^2m - 8g^2 \\ -3gm^4 + 15gm^3 - 18gm^2 + 6g + 2m^3 - 8m^2 + 6m \end{pmatrix}$$

$$q_4 = \tfrac{-1}{4} tr\{B_3 \mathtt{L}\} \quad \text{where } B_3 = B_2\mathtt{L} + q_3 I$$

$$= \tfrac{-1}{4} tr\{(B_2\mathtt{L} + q_3 I)\mathtt{L}\}$$

$$= \tfrac{-1}{4} tr\{\mathtt{L}^4 + q_1\mathtt{L}^3 + q_2\mathtt{L}^2 + q_3\mathtt{L}\}$$

from the above examples:

$$q_4 = \frac{1}{24} \cdot mg \begin{pmatrix} g^7m^3 + 4g^6m^4 - 8g^6m^3 - 6g^6m^2 + 6g^5m^5 - 24g^5m^4 + 42g^5m^2 + 11g^5m \\ +4g^4m^6 - 24g^4m^5 + 12g^4m^4 + 94g^4m^3 - 64g^4m^2 - 66g^4m - 6g^4 + g^3m^7 \\ -8g^3m^6 + 94g^3m^4 - 134g^3m^3 - 62g^3m^2 + 103g^3m + 30g^3 - 6g^2m^6 + 42g^2m^5 \\ -64g^2m^4 - 62g^2m^3 + 138g^2m^2 - 24g^2m - 24g^2 + 11gm^5 - 66gm^4 + 103gm^3 \\ -24gm^2 - 24gm - 6m^4 + 30m^3 - 24m^2 \end{pmatrix}.$$

□

From the above Proposition we came to know that the following result:

$$\begin{aligned} q_1 &= -s_1 \\ q_2 &= \frac{1}{2}(s_1^2 - s_2) \\ q_3 &= \frac{1}{3}(-s_1^3 + 3s_1s_2 - 2s_3) \\ q_4 &= \frac{1}{4}(s_1^4 - 6s_1^2s_2 + 3s_2^2 + 8s_1s_3 - 6s_4)\cdots \end{aligned}$$

where

$$s_k = m^k(m-1) + g^k(g-1) + (m+g)^k(m-1)(g-1), \quad \text{for } k = 1,2,3,4\cdots$$

Theorem 2.2 *The Laplacian characteristic equation of the graph* $K_m\square K_g$ *is* $\psi(y,t) = y^n + t_1y^{n-1} + ... + t_{n-1}y + t_n$ *with* $t_0 = 1$ *then,*

$$t_\hbar = \frac{-1}{\hbar}\sum_{j=0}^{\hbar-1} t_j tr(L^{\hbar-j}).$$

where, $t_\hbar$ *are the coefficients of the characteristic polynomial and* $\hbar \neq 0$.

Proof:

Here $t_1 = -tr(\mathtt{L})$.

$$\begin{aligned} t_2 &= \tfrac{-1}{2}tr(t_1\mathtt{L}) \\ &= \tfrac{-1}{2}tr((\mathtt{L}+t_1)\mathtt{L}) \\ &= \tfrac{-1}{2}\{tr(\mathtt{L}^2) + t_1tr(\mathtt{L})\} \\ &= \tfrac{-1}{2}\{t_0tr(\mathtt{L}^2) + t_1tr(\mathtt{L})\}. \end{aligned}$$

Similarly for an integer k,

$$t_k = \tfrac{-1}{k}\textstyle\sum_{j=0}^{k-1} t_j\ tr(\mathtt{L}^{k-j}).$$

we have,

$$\begin{aligned} t_{k+1} &= \tfrac{-1}{k+1}tr(t_k\mathtt{L}) \\ &= \tfrac{-1}{k+1}tr((t_{k-1}L + t_k)\mathtt{L}) \\ &= \tfrac{-1}{k+1}tr((\mathtt{L}t_{k-2} + t_{k-1})\mathtt{L}^2 + t_k\mathtt{L}) \\ &= \tfrac{-1}{k+1}(tr(\mathtt{L}^3t_{k-2}) + t_{k-1}tr(\mathtt{L}^2) + t_ktr(\mathtt{L})) \\ &= \tfrac{-1}{k+1}(tr(\mathtt{L}^4t_{k-3}) + tr(\mathtt{L}^3)t_{k-1} + tr(\mathtt{L}^2)t_k + tr(\mathtt{L})) \end{aligned}$$

. . .

$$t_{k+1} = \tfrac{-1}{k+1}\{t_0tr(\mathtt{L}^{k+1}) + t_1tr(\mathtt{L}^k) + t_2tr(\mathtt{L}^{k-1}) + ... + t_ktr(\mathtt{L})\}$$

i.e., $t_{k+1} = \frac{-1}{k+1}\sum_{j=0}^{k} t_j\ tr(\mathtt{L}^{k+1-j})$.

Hence by induction,

$$t_\hbar = \tfrac{-1}{\hbar}\textstyle\sum_{j=0}^{\hbar-1} t_jtr(\mathtt{L}^{\hbar-j}) \qquad \textit{where} \quad \hbar \neq 0.$$

□

Theorem 2.3 *Let* $G = K_m\square K_g$*, the* k^{th} *Laplacian coefficient of a graph* G *is given by*

$$t_k = \frac{-1}{k}\sum_{j=0}^{k-1} t_j\left[(m-1)(g-1)(m+g)^{k-j} + (m-1)m^{k-j} + (g-1)g^{k-j}\right].$$

Proof: We substitute the trace formula from the previous Theorem 2.2 & Proposition 1.2 into Newton's identities to obtain the recursive form of t_k. □

The energy $E(G)$ of a graph G is defined to be the sum of the absolute values of its eigen values. Hence if $A(G)$ is the adjacency matrix of G and $\rho_1, \rho_2, \rho_3 \dots \rho_n$ are eigenvalues of $A(G)$, then $E(G) = \sum_{n=1}^{n} |\rho_i|$.

Theorem 2.4 *The energy of the Cartesian product of graph $K_m \square K_g$ is given by:*

$$E(K_m \square K_g) = (m + g - 2) + |m - 2|(g - 1) + |g - 2|(m - 1) + 2(m - 1)(g - 1).$$

Proof:
The adjacency spectrum of a complete graph K_n is:

$$\text{Spec}(K_n) = \{n - 1, (-1)^{(n-1)}\}$$

That is:

- One eigenvalue $n - 1$
- $n - 1$ eigenvalues of -1

Now consider the Cartesian product $K_m \square K_g$. The spectrum is the sum of eigenvalues of K_m and K_g:

$$\text{Spec}(K_m \square K_g) = \{\lambda_i + \mu_j \mid \lambda_i \in \text{Spec}(K_m), \mu_j \in \text{Spec}(K_g)\}$$

So the eigenvalues of $K_m \square K_g$ are:

$$\begin{aligned} &(m - 1) + (g - 1) = m + g - 2 && \text{(1 time)} \\ &(m - 1) + (-1) = m - 2 && \text{(g - 1 times)} \\ &(-1) + (g - 1) = g - 2 && \text{(m - 1 times)} \\ &(-1) + (-1) = -2 && \text{((m - 1)(g - 1) times)} \end{aligned}$$

The energy of a graph is the sum of the absolute values of its eigenvalues:

$$\begin{aligned} E(K_m \square K_g) &= |m + g - 2| \cdot 1 + |m - 2| \cdot (g - 1) + |g - 2| \cdot (m - 1) + |-2| \cdot (m - 1)(g - 1) \\ &= (m + g - 2) + |m - 2|(g - 1) + |g - 2|(m - 1) + 2(m - 1)(g - 1) \end{aligned}$$

. □

3. Upper bounds for α_1 of the graph $K_m \square K_g$.

:

Bounds on eigenvalues of a graph provide valuable insights into the graph's structure and properties. These bounds can be used to estimate the connectivity, degree distribution, and other characteristics of the graph. Various theorems and techniques exist to derive these bounds, depending on the type of graph and the specific eigenvalue being considered.

We specifically found some upper bounds of α_1:

Proposition 3.1 *Let $G = K_m \square K_g$, then the largest Laplacian eigenvalue α_1 of G satisfies:*

$$\alpha_1 \leq \tfrac{5}{m(m+g-2)} \left[\tfrac{(g-1)^{\sqrt{\frac{2\pi}{5}}} g^{\sqrt{\frac{2\pi}{5}}} + (m-1)^{\sqrt{\frac{2\pi}{5}}} m^{\sqrt{\frac{2\pi}{5}}} + (m-1)^{\sqrt{\frac{2\pi}{5}}} (g-1)^{\sqrt{\frac{2\pi}{5}}} (m+g)^{\sqrt{\frac{2\pi}{5}}}}{5} \right]^{\frac{1}{\sqrt{\frac{2\pi}{5}}}} + m \tag{1}$$

Proof:

We have by the power mean inequality,

$$\left[\frac{(g-1)g+(m-1)m+(m-1)(g-1)(m+g)}{5}\right] \leq \left[\frac{(g-1)^{\sqrt{\frac{2\pi}{5}}}g^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}m^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}(g-1)^{\sqrt{\frac{2\pi}{5}}}(m+g)^{\sqrt{\frac{2\pi}{5}}}}{5}\right]^{\frac{1}{\sqrt{\frac{2\pi}{5}}}}$$

$$\frac{tr\{\mathrm{L}\}}{5} \leq \left[\frac{(g-1)^{\sqrt{\frac{2\pi}{5}}}g^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}m^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}(g-1)^{\sqrt{\frac{2\pi}{5}}}(m+g)^{\sqrt{\frac{2\pi}{5}}}}{5}\right]^{\frac{1}{\sqrt{\frac{2\pi}{5}}}}$$

$$\frac{mg(m+g-2)}{5} \leq \left[\frac{(g-1)^{\sqrt{\frac{2\pi}{5}}}g^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}m^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}(g-1)^{\sqrt{\frac{2\pi}{5}}}(m+g)^{\sqrt{\frac{2\pi}{5}}}}{5}\right]^{\frac{1}{\sqrt{\frac{2\pi}{5}}}}$$

$$g \leq \frac{5}{m(m+g-2)}\left[\frac{(g-1)^{\sqrt{\frac{2\pi}{5}}}g^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}m^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}(g-1)^{\sqrt{\frac{2\pi}{5}}}(m+g)^{\sqrt{\frac{2\pi}{5}}}}{5}\right]^{\frac{1}{\sqrt{\frac{2\pi}{5}}}}$$

From Theorem 1.1,

$$\alpha_1 \leq \frac{5}{m(m+g-2)}\left[\frac{(g-1)^{\sqrt{\frac{2\pi}{5}}}g^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}m^{\sqrt{\frac{2\pi}{5}}}+(m-1)^{\sqrt{\frac{2\pi}{5}}}(g-1)^{\sqrt{\frac{2\pi}{5}}}(m+g)^{\sqrt{\frac{2\pi}{5}}}}{5}\right]^{\frac{1}{\sqrt{\frac{2\pi}{5}}}}+m$$

□

The following theorem establishes an upper bound that improves upon inequality (1).

Theorem 3.1 *Let α_1 be the maximum Laplacian eigenvalue of $K_m \square K_g$. Then*

$$\alpha_1 \leq h + \frac{1}{g}, \qquad \text{where } h = m + g. \tag{2}$$

Proof: By Theorem 1.1, we know that the largest Laplacian eigenvalue of $K_m \square K_g$ is:

$$\alpha_1(K_m \square K_g) = h = m + g.$$

Since $g > 1$, define:

$$u(g) = g + m + \frac{1}{g} > g + m = h.$$

As $g \to \infty$, we observe:

$$\lim_{g\to\infty} u(g) = g + m = h.$$

Therefore,

$$\alpha_1 \leq h + \frac{1}{g}.$$

□

For a vertex v, we denote its degree by d_v, the set of its adjacent vertices by N_v, and the mean degree of the neighbors of v by m_v [15].

Known Upper Bounds for α_1 of the graph G

Among the known upper bounds for the largest Laplacian eigenvalue α_1 of a graph, the following are noteworthy:

- **Anderson and Morley's bound** [19]:

$$\alpha_1 \leq \max\{d_u + d_v \mid uv \in E\} \tag{3}$$

- **Li and Zhang's bound** [9]:

$$\alpha_1 \le 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \tag{4}$$

- **Another Li and Zhang's bound** [9]:

$$\alpha_1 \le 2 + \sqrt{(r-2)(s-2)} \tag{5}$$

where $r = d_x + d_y$ for all $xy \in E$, and $s = \max\{d_u + d_v \mid uv \in E \setminus \{xy\}\}$.

- **Merris's bound** [17]:

$$\alpha_1 \le \max\{d_v + m_v \mid v \in V\} \tag{6}$$

Upper bounds of α_1 of Laplacian matrix of $K_m \square K_g$ as follows:

Table1:Upper bounds of α_1

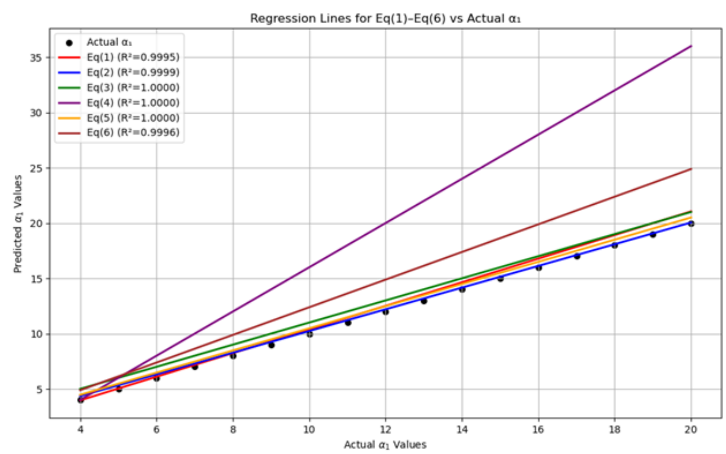
m	g	α_1	Eq(1)	Eq(2)	Eq(3)	Eq(4)	Eq(5)	Eq(6)
2	2	4	4.1285	4.5000	5	4.0000	4.4495	5.0000
2	3	5	5.2310	5.3333	6	6.0000	5.4641	6.0000
3	3	6	6.2590	6.3333	7	8.0000	6.4721	7.5000
3	4	7	7.3780	7.2500	8	10.0000	7.4772	8.5000
4	4	8	8.4123	8.2500	9	12.0000	8.4807	10.0000
4	5	9	9.5444	9.2000	10	14.0000	9.4833	11.0000
5	5	10	10.5760	10.2000	11	16.0000	10.4853	12.5000
5	6	11	11.7172	11.1667	12	18.0000	11.4868	13.5000
6	6	12	12.7456	12.1667	13	20.0000	12.4881	15.0000
6	7	13	13.8934	13.1429	14	22.0000	13.4891	16.0000
7	7	14	14.9190	14.1429	15	24.0000	14.4900	17.5000
7	8	15	16.0718	15.1250	16	26.0000	15.4907	18.5000
8	8	16	17.0951	16.1250	17	28.0000	16.4914	20.0000
8	9	17	18.2518	17.1111	18	30.0000	17.4919	21.0000
9	9	18	19.2731	18.1111	19	32.0000	18.4924	22.5000
9	10	19	20.4329	19.1000	20	34.0000	19.4929	23.5000
10	10	20	21.4525	20.1000	21	36.0000	20.4932	25.0000
10	11	21	22.6149	21.0909	22	38.0000	21.4936	26.0000
11	11	22	23.6331	22.0909	23	40.0000	22.4939	27.5000
11	12	23	24.7977	23.0833	24	42.0000	23.4942	28.5000
12	12	24	25.8147	24.0833	25	44.0000	24.4944	30.0000
12	13	25	26.9810	25.0769	26	46.0000	25.4947	31.0000
13	13	26	27.9970	26.0769	27	48.0000	26.4949	32.5000
13	14	27	29.1649	27.0714	28	50.0000	27.4951	33.5000

m	g	α_1	Eq(1)	Eq(2)	Eq(3)	Eq(4)	Eq(5)	Eq(6)
14	14	28	29.8623	28.0714	29	52.0000	28.4953	35.0000
14	15	29	30.1800	29.0667	30	54.0000	29.4955	36.0000
15	15	30	31.3492	30.0667	31	56.0000	30.4956	37.5000
15	16	31	32.3635	31.0625	32	58.0000	31.4958	38.5000
16	16	32	33.5339	32.0625	33	60.0000	32.4959	40.0000
16	17	33	35.7190	33.0588	34	62.0000	33.4960	41.0000
17	17	34	36.7318	34.0588	35	64.0000	34.4962	42.5000
17	18	35	37.9043	35.0556	36	66.0000	35.4963	43.5000
18	18	36	38.9165	36.0556	37	68.0000	36.4964	45.0000
18	19	37	40.0899	37.0526	38	70.0000	37.4965	46.0000
19	19	38	41.1016	38.0526	39	72.0000	38.4966	47.5000
19	20	39	42.2757	39.0500	40	74.0000	39.4967	48.5000
20	20	40	43.2869	40.0500	41	76.0000	40.4968	50.0000
20	21	41	44.4617	41.0476	42	78.0000	41.4968	51.0000
21	21	42	45.4725	42.0476	43	80.0000	42.4969	52.5000
21	22	43	46.6479	43.0455	44	82.0000	43.4970	53.5000
22	22	44	47.6583	44.0455	45	84.0000	44.4971	55.0000
22	23	45	48.8342	45.0435	46	86.0000	45.4971	56.0000
23	23	46	49.8443	46.0435	47	88.0000	46.4972	57.5000
23	24	47	51.0208	47.0417	48	90.0000	47.4973	58.5000
24	24	48	52.0304	48.0417	49	92.0000	48.4973	60.0000
24	25	49	53.2074	49.0400	50	94.0000	49.4974	61.0000
25	25	50	54.2168	50.0400	51	96.0000	50.4974	62.5000
25	26	51	55.3942	51.0385	52	98.0000	51.4975	63.5000
26	26	52	56.4033	52.0385	53	100.0000	52.4975	65.0000
26	27	53	57.5811	53.0370	54	102.0000	53.4976	66.0000
27	27	54	58.5899	54.0370	55	104.0000	54.4976	67.5000
27	28	55	59.7682	55.0357	56	106.0000	55.4977	68.5000
28	28	56	60.7767	56.0357	57	108.0000	56.4977	70.0000
28	29	57	61.9553	57.0345	58	110.0000	57.4977	71.0000
29	29	58	62.9635	58.0345	59	112.0000	58.4978	72.5000
29	30	59	64.1425	59.0333	60	114.0000	59.4978	73.5000
30	30	60	65.1505	60.0333	61	116.0000	60.4979	75.0000
30	31	61	66.3298	61.0323	62	118.0000	61.4979	76.0000
31	31	62	67.3376	62.0323	63	120.0000	62.4979	77.5000
31	32	63	68.5172	63.0312	64	122.0000	63.4980	78.5000
32	32	64	69.5248	64.0312	65	124.0000	64.4980	80.0000
32	33	65	70.7047	65.0303	66	126.0000	65.4980	81.0000
33	33	66	71.7121	66.0303	67	128.0000	66.4981	82.5000
33	34	67	72.8922	67.0294	68	130.0000	67.4981	83.5000
34	34	68	73.8995	68.0294	69	132.0000	68.4981	85.0000
34	35	69	75.0799	69.0286	70	134.0000	69.4981	86.0000
35	35	70	76.0869	70.0286	71	136.0000	70.4982	87.5000
35	36	71	77.2675	71.0278	72	138.0000	71.4982	88.5000
36	36	72	78.2744	72.0278	73	140.0000	72.4982	90.0000
36	37	73	79.4553	73.0270	74	142.0000	73.4983	91.0000
37	37	74	80.4620	74.0270	75	144.0000	74.4983	92.5000
37	38	75	81.6431	75.0263	76	146.0000	75.4983	93.5000
38	38	76	82.6496	76.0263	77	148.0000	76.4983	95.0000
38	39	77	83.8309	77.0256	78	150.0000	77.4983	96.0000
39	39	78	84.8373	78.0256	79	152.0000	78.4984	97.5000
39	40	79	86.0188	79.0250	80	154.0000	79.4984	98.5000
40	40	80	87.0251	80.0250	81	156.0000	80.4984	100.0000
40	41	81	88.2068	81.0244	82	158.0000	81.4984	101.0000

m	g	α_1	Eq(1)	Eq(2)	Eq(3)	Eq(4)	Eq(5)	Eq(6)
41	41	82	89.2129	82.0244	83	160.0000	82.4984	102.5000
41	42	83	90.3948	83.0238	84	162.0000	83.4985	103.5000
42	42	84	91.4008	84.0238	85	164.0000	84.4985	105.0000
42	43	85	92.5828	85.0233	86	166.0000	85.4985	106.0000
43	43	86	93.5887	86.0233	87	168.0000	86.4985	107.5000
43	44	87	94.7709	87.0227	88	170.0000	87.4985	108.5000
44	44	88	95.7767	88.0227	89	172.0000	88.4986	110.0000
44	45	89	96.9590	89.0222	90	174.0000	89.4986	111.0000
45	45	90	97.9647	90.0222	91	176.0000	90.4986	112.5000
45	46	91	99.1472	91.0217	92	178.0000	91.4986	113.5000
46	46	92	100.1528	92.0217	93	180.0000	92.4986	115.0000
46	47	93	101.3354	93.0213	94	182.0000	93.4986	116.0000
47	47	94	102.3409	94.0213	95	184.0000	94.4986	117.5000
47	48	95	103.5236	95.0208	96	186.0000	95.4987	118.5000
48	48	96	104.5290	96.0208	97	188.0000	96.4987	120.0000
48	49	97	105.7119	97.0204	98	190.0000	97.4987	121.0000
49	49	98	106.7172	98.0204	99	192.0000	98.4987	122.5000
49	50	99	107.9002	99.0200	100	194.0000	99.4987	123.5000

From the above table it is easily seen that (2) is better than (1), (3), (4), (5) and (6) for large values of $m\&g$.

To show that equation number (2) is the best among mentioned inequalities we used linear regression model as follows:



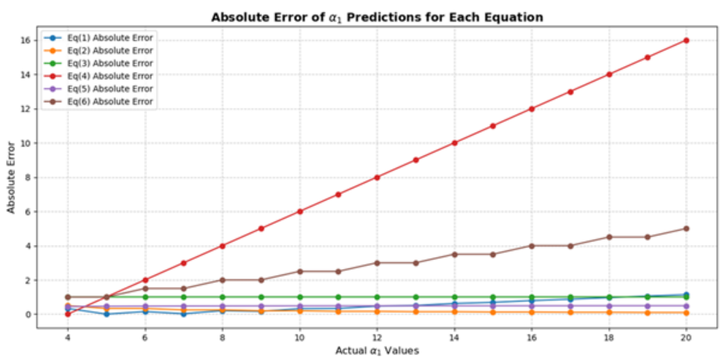
Conclusion

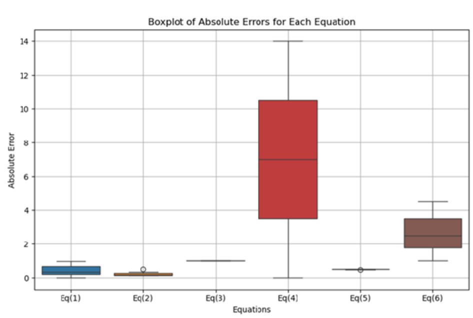
In this work, we have computed the coefficients of the characteristic polynomial of the Laplacian matrix for the Cartesian product graph $K_m \Box K_g$. Furthermore, we have derived an upper bound for the largest Laplacian eigenvalue of the graph $K_m \Box K_g$, and calculated its graph energy based on the eigenvalues of the corresponding adjacency matrix.

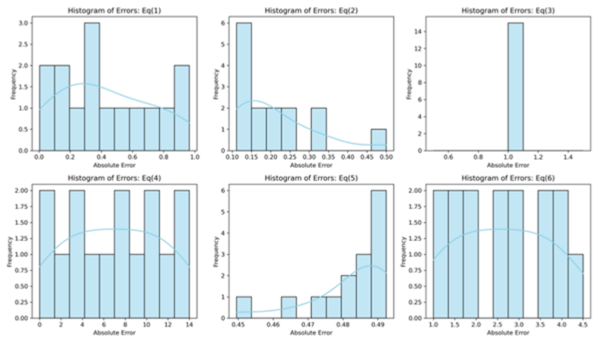
As discussed earlier, Equations (1) and (2) yield more accurate or tighter results compared to the other considered expressions. This conclusion is supported by the data presented in the accompanying tables and graphical illustrations.

Table2:Error Metrics for Different Equations

Equation	**RMSE**	**MAE**	R^2
Eq(1)	0.619	0.5099	0.984
Eq(2)	0.2235	0.1975	0.9979
Eq(3)	1.0	1.0	0.9583
Eq(4)	9.3808	8.0	-2.6667
Eq(5)	0.4836	0.4835	0.9903
Eq(6)	3.1343	2.8824	0.5907







To check how well the six equations, work in predicting our data, we used three common error-checking methods: Root Mean Square Error (RMSE), Mean Absolute Error (MAE), and the Coefficient of Determination (R^2). These tools help us measure how close the predicted values are to the actual values. The lower the RMSE and MAE, the better the prediction. A higher R^2 (closer to 1) means the model explains the data well. Out of all the equations, Equation (2) gave the best results. It had a very low RMSE of 0.2235, MAE of 0.1975, and an almost perfect R^2 of 0.9979. This means Equation (2) predicts the data extremely well and makes only very small errors. Equation (1) also performed well, with an RMSE of 0.619, MAE of 0.5099, and R^2 of 0.984. While not as accurate as Equation (2), it still provides good predictions and fits the data closely. These results make it clear that Equations (1) and (2) are far better than the others, and among them, Equation (2) is the most accurate and reliable.

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