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Existence and stability results for nonlinear fractional integrodifferential equations with nonlocal antiperiodic boundary conditions

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ABSTRACT: This work examines the existence and stability of solutions for nonlinear fractional integrodifferential equations with nonlocal antiperiodic boundary conditions, which include the Caputo derivative. The existence is established using the Banach contraction mapping theorem. Furthermore, two types of Ulam stability are being studied, namely Ulam-Hyers stability and generalized Ulam-Hyers stability. Finally, examples are given to demonstrate the applicability of the key findings.

Key Words: Fractional derivatives, antiperiodic boundary conditions, nonlocal boundary conditions, fixed point theorem.

Contents

1. Introduction

Fractional differential equations have emerged as powerful tools for modelling a broad spectrum of phenomena across diverse fields, including control theory, signal processing, rheology, fractals, chaotic dynamics, bioengineering and biomedical applications [1,4,5,13,20]. In recent years, the study of fractional differential and integrodifferential equations has gained considerable attention, yielding intriguing insights into their existence and uniqueness properties. For further exploration of these developments, refer to [1,13,15,18,19,25].

Fractional-order derivatives can be defined in multiple forms, such as Grunwald-Letnikov derivatives, Hadamard fractional derivatives, local fractional derivatives, Caputo-Fabrizio derivatives and Caputo fractional derivatives. Caputo fractional derivatives stand out by their distinctive characteristics, which include interpolation, linearity, the derivative of any constant equal to zero, and the ability to apply traditional initial and boundary conditions. These traits ensure that Caputo derivatives are compatible with classical derivative properties, making them particularly useful for boundary value problems.

In recent days several researchers have extensively investigated multi-point nonlocal boundary value problems [2,6,11,17]. Specifically, antiperiodic boundary conditions are increasingly being applied to new and intriguing scenarios. Antiperiodic boundary conditions, combined with fractional derivatives, are widely employed in diverse fields such as diffusion processes, signal processing, image processing, mechanical systems, bioengineering and economic systems [7,9,12,21,24].

Fractional derivatives play a key role in ensuring stability when modelling physical processes, especially

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at initial times [10,12,16,23]. Since stability with antiperiodic boundary conditions is complex, it offers significant opportunities to model systems with natural phase transitions or reversals effectively [8,14,22]. Based on the preceding considerations, this study intends to investigate the existence and stability of solutions for the following Nonlocal Fractional Integrodifferential Equations (NFIDEs) with nonlocal antiperiodic boundary conditions of order $p \in (2,3]$

$$^{C}D^{p}x(t) = f\left(t, x(t), \int_0^t g(t,s,x(s))ds\right), \quad 2 < p \leq 3, \quad t \in [0,c] = \mathfrak{J}, \tag{1.1}$$

$$x(\delta) = -x(c), \;\; x'(\delta) = -x'(c), \;\; x''(\delta) = -x''(c), \quad 0 \leq \delta < c, \tag{1.2}$$

where $^{C}D^{p}$ represents the Caputo fractional derivative of order p and $f : [0,c] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : [0,c] \times [0,c] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let us take

$$\mathcal{K}x(t) = \int_0^t g(t,s,x(s))ds.$$

Let $C(\mathfrak{J},\mathbb{R}) = S$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with norm

$$\|x\| = \sup\{|x(t)| : t \in \mathfrak{J}\}, \text{ for } \; x \in S.$$

The paper is structured as follows: Section 2 introduces the notations, basic concepts, and preliminary facts that will be utilized throughout the paper. In Section 3, classical fixed point theorem is employed to investigate the existence of solutions for the problem (1.1)-(1.2). Section 4 focuses on the stability of the proposed problem. Finally, Section 5 includes examples that corroborate the theory presented in this study.

2. Preliminaries

Definition 2.1 *[3] The Caputo fractional derivative, $^{C}D^{p}$ of order $p > 0$ of a continuous function $h : [0,\infty) \to \mathbb{R}$, is defined by*

$$^{C}D^{p}h(t) = \frac{1}{\Gamma(q-p)} \int_0^t (t-w)^{q-p-1} h^{q}(w)dw, \;\; q-1 < p < q.$$

Definition 2.2 *[3] For any order $p > 0$, the Riemann-Liouville fractional integral of a function $h(t)$, denoted I^{p}, is defined by*

$$I^{p}h(t) = \frac{1}{\Gamma(p)} \int_0^t (t-w)^{p-1} h(w)dw.$$

Definition 2.3 *[23] The problem (1.1)-(1.2) is Ulam-Hyers Stable (UHS) if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $x^* \in C(\mathfrak{J},\mathbb{R})$ of the inequality*

$$|^{C}D^{p}x^*(t) - f(t, x^*(t), \mathcal{K}x^*(t))| \leq \epsilon, \tag{2.1}$$

there exists a solution $x \in C(\mathfrak{J},\mathbb{R})$ of (1.1)-(1.2) with

$$|x^*(t) - x(t)| \leq c_f \epsilon, \quad t \in \mathfrak{J}.$$

Definition 2.4 *[23] The problem (1.1)-(1.2) is Generalized Ulam-Hyers Stable (GUHS) if there exists $\phi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\phi(0) = 0$ such that for each solution $x^* \in C(\mathfrak{J},\mathbb{R})$ of the inequality*

$$|^{C}D^{p}x^*(t) - f(t, x^*(t), \mathcal{K}x^*(t))| \leq \epsilon,$$

there exists a solution $x \in C(\mathfrak{J},\mathbb{R})$ of (1.1)-(1.2) with

$$|x^*(t) - x(t)| \leq \phi(\epsilon), \quad t \in \mathfrak{J}.$$

Lemma 2.1 *[3] The general solution of $^{C}D^{p}x(t) = 0$ where $p > 0$, is given by*

$$x(t) = K_1 + K_2 t + K_3 t^2 + ... + K_n t^{n-1},$$

where $K_m \in \mathbb{R}$, for $m = 1, 2, ..., n$ and $n = [p] + 1$.

3. Existence Results

To prove the main results, we need the following assumptions.

(A1) The function f is continuous and there exists a constant $\mathfrak{L}_1 > 0$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \mathfrak{L}_1\Big(|x_1-x_2| + |y_1-y_2|\Big), \text{ for all } t \in \mathfrak{J} \text{ and } x_1,y_1,x_2,y_2 \in \mathbb{R}.$$

(A2) The function g is continuous and there exists a constant $\mathfrak{L}_2 > 0$ such that

$$\Big|\int_0^t \Big(g(t,s,x) - g(t,s,y)\Big)ds\Big| \leq \mathfrak{L}_2(|x-y|), \text{ for all } t,s \in \mathfrak{J} \text{ and } x,y \in \mathbb{R}.$$

(A3) There exists a constant $\mathcal{G} > 0$ such that

$$\Big|\int_0^t g(t,s,x)ds\Big| \leq \mathcal{G}[1+|x|], \text{ for all } t,s \in \mathfrak{J} \text{ and } x \in \mathbb{R}.$$

(A4) The function f is bounded and there exists a constant $\mathcal{M} > 0$ such that

$$\mathcal{M} = \sup\{|f(t,x,y)| : t \in \mathfrak{J}, \ x,y \in \mathbb{R}\}.$$

Lemma 3.1 *[3] The solution of*

$$\begin{cases} {}^C D^p x(t) = f(t,x(t),\mathcal{K}x(t)), & 2 < p \leq 3, \quad 0 \leq \delta < c, \\ x(\delta) = -x(c), \ \ x'(\delta) = -x'(c), \ \ x''(\delta) = -x''(c), \end{cases}$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau - \frac{1}{2}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau \\ & + \int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau\Bigg] + \frac{\delta+c-2t}{4} \\ & \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau + \int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau\Bigg] \\ & + \frac{t[(\delta+c)-t]-\delta c}{4}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau \\ & + \int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau\Bigg]. \end{aligned}$$

Theorem 3.1 *Assume that the assumptions $(A1-A4)$ are satisfied. If*

$$\mathfrak{L}_1[1+\mathfrak{L}_2]\mathcal{Q} < 1, \tag{3.1}$$

then the antiperiodic boundary value problem (1.1)-(1.2) has a unique solution.

Proof: Define a mapping $H : S \to S$ by

$$\begin{aligned} (Hx)(t) = & \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau - \frac{1}{2}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau \\ & + \int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau\Bigg] + \frac{\delta+c-2t}{4} \\ & \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau + \int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)} f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau\Bigg] \end{aligned}$$

$$+\frac{t[(\delta+c)-t]-\delta c}{4}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau$$
$$+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}f(\tau,x(\tau),\mathcal{K}x(\tau))d\tau\Bigg]$$

and we show that H has a fixed point. This fixed point is then a solution of (1.1)-(1.2). Choose

$$r \geq \frac{(\mathfrak{L}_1\mathcal{G}+\mathcal{M})\mathcal{Q}}{1-\mathcal{L}_1\mathcal{Q}(1+\mathcal{G})},$$

where

$$\mathcal{Q}=\max_{t\in[0,c]}\Bigg[\frac{t^p}{\Gamma(p+1)}-\frac{1}{2}\left(\frac{\delta^p}{\Gamma(p+1)}+\frac{c^p}{\Gamma(p+1)}\right)+\frac{|\delta+c-2t|}{4}\left(\frac{\delta^{p-1}}{\Gamma(p)}+\frac{c^{p-1}}{\Gamma(p)}\right)$$
$$+\frac{|t[(\delta+c)-t]-\delta c|}{4}\left(\frac{\delta^{p-2}}{\Gamma(p-1)}+\frac{c^{p-2}}{\Gamma(p-1)}\right)\Bigg]. \quad (3.2)$$

We can show that $H(\mathcal{B}_r)\subset\mathcal{B}_r$, where $\mathcal{B}_r=\{x\in S:\|x\|\leq r\}$. Let $x\in\mathcal{B}_r$. Then

$$|(Hx)(t)|\leq\int_0^t\frac{(t-\tau)^{p-1}}{\Gamma(p)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau+\int_0^t\frac{(t-\tau)^{p-1}}{\Gamma(p)}|f(\tau,0,0)|d\tau$$
$$-\frac{1}{2}\Bigg[\int_0^\delta\frac{(\delta-\tau)^{p-1}}{\Gamma(p)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau+\int_0^\delta\frac{(\delta-\tau)^{p-1}}{\Gamma(p)}$$
$$|f(\tau,0,0)|d\tau+\int_0^c\frac{(c-\tau)^{p-1}}{\Gamma(p)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau$$
$$+\int_0^c\frac{(c-\tau)^{p-1}}{\Gamma(p)}|f(\tau,0,0)|d\tau\Bigg]+\frac{|\delta+c-2t|}{4}$$
$$\Bigg[\int_0^\delta\frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau$$
$$+\int_0^\delta\frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}|f(\tau,0,0)|d\tau+\int_0^c\frac{(c-\tau)^{p-2}}{\Gamma(p-1)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau$$
$$+\int_0^c\frac{(c-\tau)^{p-2}}{\Gamma(p-1)}|f(\tau,0,0)|d\tau\Bigg]+\frac{|t[(\delta+c)-t]-\delta c|}{4}$$
$$\Bigg[\int_0^\delta\frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau+\int_0^\delta\frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}|f(\tau,0,0)|d\tau$$
$$+\int_0^c\frac{(c-\tau)^{p-3}}{\Gamma(p-2)}\Big|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,0,0)\Big|d\tau+\int_0^c\frac{(c-\tau)^{p-3}}{\Gamma(p-2)}|f(\tau,0,0)|d\tau\Bigg]$$
$$\leq\int_0^t\frac{(t-\tau)^{p-1}}{\Gamma(p)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^t\frac{(t-\tau)^{p-1}}{\Gamma(p)}\mathcal{M}d\tau$$

$$
\begin{aligned}
&-\frac{1}{2}\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)}\mathcal{M}d\tau \\
&+\int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)}\mathcal{M}d\tau\bigg]+\frac{|\delta+c-2t|}{4} \\
&\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}\mathcal{M}d\tau \\
&+\int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)}\mathcal{M}d\tau\bigg]+\frac{|t[(\delta+c)-t]-\delta c|}{4} \\
&\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}\mathcal{M}d\tau \\
&+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}\mathcal{M}d\tau\bigg] \\
\leq&\Big[\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]+\mathcal{M}\Big]\bigg[\frac{t^p}{\Gamma(p+1)}-\frac{1}{2}\bigg(\frac{\delta^p}{\Gamma(p+1)}+\frac{c^p}{\Gamma(p+1)}\bigg) \\
&+\frac{|\delta+c-2t|}{4}\bigg(\frac{\delta^{p-1}}{\Gamma(p)}+\frac{c^{p-1}}{\Gamma(p)}\bigg)+\frac{|t[(\delta+c)-t]-\delta c|}{4}\bigg(\frac{\delta^{p-2}}{\Gamma(p-1)}+\frac{c^{p-2}}{\Gamma(p-1)}\bigg)\bigg] \\
\|(Hx)\|\leq&\Big[\mathfrak{L}_1\Big[r+\mathcal{G}[1+r]\Big]+\mathcal{M}\Big]\Big(\mathcal{Q}\Big)\leq r.
\end{aligned}
$$

Now $x,y\in S$ and for each $t\in[0,c]$ we have

$$
\begin{aligned}
|(Hx)(t)-(Hy)(t)|\leq&\int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)}|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau \\
&-\frac{1}{2}\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)}|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau \\
&+\int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)}|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau\bigg] \\
&+\frac{\delta+c-2t}{4}\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau \\
&+\int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)}|f(\tau,x(\tau),\mathcal{K}x(\tau))-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau\bigg] \\
&+\frac{t[(\delta+c)-t]-\delta c}{4}\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}|f(\tau,x(\tau),\mathcal{K}x(\tau)) \\
&-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}|f(\tau,x(\tau),\mathcal{K}x(\tau)) \\
&-f(\tau,y(\tau),\mathcal{K}y(\tau))|d\tau\bigg] \\
\leq&\int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)}\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau \\
&-\frac{1}{2}\bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)}\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)}\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau\Bigg]+\frac{\delta+c-2t}{4}\\
&\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)}\\
&\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau\Bigg]+\frac{t[(\delta+c)-t]-\delta c}{4}\\
&\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}\\
&\mathfrak{L}_1\Big[\|x-y\|+\mathfrak{L}_2\|x-y\|\Big]d\tau\Bigg]\\
\leq&\mathfrak{L}_1\Big[1+\mathfrak{L}_2\Big]\|x-y\|\Bigg[\frac{t^p}{\Gamma(p+1)}-\frac{1}{2}\left(\frac{\delta^p}{\Gamma(p+1)}+\frac{c^p}{\Gamma(p+1)}\right)\\
&+\frac{|\delta+c-2t|}{4}\left(\frac{\delta^{p-1}}{\Gamma(p)}+\frac{c^{p-1}}{\Gamma(p)}\right)+\frac{|t[(\delta+c)-t]-\delta c|}{4}\\
&\left(\frac{\delta^{p-2}}{\Gamma(p-1)}+\frac{c^{p-2}}{\Gamma(p-1)}\right)\Bigg]\\
\leq&\mathfrak{L}_1\Big[1+\mathfrak{L}_2\Big](\mathcal{Q})\|x-y\|.
\end{aligned}
$$

Thus,

$$\|(Hx)-(Hy)\|\leq \mathfrak{L}_1\Big[1+\mathfrak{L}_2\Big](\mathcal{Q})\|x-y\|.$$

Given the assumption $\mathfrak{L}_1\Big[1+\mathfrak{L}_2\Big]\mathcal{Q}<1$, the operator $H:S\to S$ becomes a contraction. Therefore, by the Banach fixed point theorem, the mapping H has a unique fixed point. As a result, the antiperiodic boundary value problem (1.1)-(1.2) has a unique solution in $\mathfrak{J}$. □

4. Stability Results

We discuss two types of Ulam Stability for the solutions of problem (1.1)-(1.2)

Remark 4.1 *A function $x^*\in C(\mathfrak{J},\mathbb{R})$ satisfies the inequality*

$$|{}^C D^p x^*(t)-f(t,x^*(t),\mathcal{K}x^*(t))|\leq\epsilon,\ t\in\mathfrak{J},$$

if and only if there exists a function $h\in C(\mathfrak{J},\mathbb{R})$ such that,

(i) $|h(t)|\leq\epsilon,\quad\forall\ t\in\mathfrak{J}$,

(ii) ${}^C D^p x^*(t)=f(t,x^*(t),\mathcal{K}x^*(t))+h(t),\ \ \forall\ t\in\mathfrak{J}$.

Theorem 4.1 *Suppose that the hypotheses* $(A1-A2)$ *and condition* (3.1) *are satisfied. Then, the problem* (1.1)-(1.2) *is UHS. Moreover, it is GUHS.*

Proof: Assume $\epsilon>0$. Let $x^*\in C([0,c],\mathbb{R})$ be a function that satisfies the inequality (2.1). It follows from Remark (4.1) that

$$
\begin{aligned}
x^*(t) = & \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau - \frac{1}{2}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau \\
& + \int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau \Bigg] + \frac{\delta + c - 2t}{4} \\
& \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau + \int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau \Bigg] \\
& + \frac{t[(\delta+c)-t] - \delta c}{4} \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau \\
& + \int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)} f(\tau, x^*(\tau), \mathcal{K}x^*(\tau))d\tau \Bigg] + I^p h(\tau). \qquad (4.1)
\end{aligned}
$$

On the other hand, consider $x \in C(\mathfrak{J}, \mathbb{R})$ as a unique solution of NFIDEs (1.1)-(1.2). In Lemma (3.1), we have

$$
\begin{aligned}
x(t) = & \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau - \frac{1}{2}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau \\
& + \int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau \Bigg] + \frac{\delta + c - 2t}{4} \\
& \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau + \int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau \Bigg] \\
& + \frac{t[(\delta+c)-t] - \delta c}{4} \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau \\
& + \int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)} f(\tau, x(\tau), \mathcal{K}x(\tau))d\tau \Bigg]. \qquad (4.2)
\end{aligned}
$$

From the equations (4.1)-(4.2) and the assumptions $(A1 - A2)$, we obtain

$$
\begin{aligned}
|x^*(t) - x(t)| = & \Big| \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)} \Big(f(\tau, x^*(\tau), \mathcal{K}x^*(\tau)) - f(\tau, x(\tau), \mathcal{K}x(\tau)) \Big) d\tau \\
& - \frac{1}{2}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)} \Big(f(\tau, x^*(\tau), \mathcal{K}x^*(\tau)) - f(\tau, x(\tau), \mathcal{K}x(\tau)) \Big) d\tau \\
& + \int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)} \Big(f(\tau, x^*(\tau), \mathcal{K}x^*(\tau)) - f(\tau, x(\tau), \mathcal{K}x(\tau)) \Big) d\tau \Bigg] + \frac{\delta + c - 2t}{4} \\
& \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)} \Big(f(\tau, x^*(\tau), \mathcal{K}x^*(\tau)) - f(\tau, x(\tau), \mathcal{K}x(\tau)) \Big) d\tau \\
& + \int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)} \Big(f(\tau, x^*(\tau), \mathcal{K}x^*(\tau)) - f(\tau, x(\tau), \mathcal{K}x(\tau)) \Big) d\tau \Bigg] \\
& + \frac{t[(\delta+c)-t] - \delta c}{4} \Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)} \Big(f(\tau, x^*(\tau), \mathcal{K}x^*(\tau)) - f(\tau, x(\tau), \mathcal{K}x(\tau)) \Big) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}\Big(f(\tau,x^*(\tau),\mathcal{K}x^*(\tau))-f(\tau,x(\tau),\mathcal{K}x(\tau))\Big)d\tau\Bigg]+I^p h(t)\Bigg|\\
&\leq \int_0^t \frac{(t-\tau)^{p-1}}{\Gamma(p)}\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau-\frac{1}{2}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-1}}{\Gamma(p)}\\
&\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-1}}{\Gamma(p)}\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau\Bigg]\\
&+\frac{\delta+c-2t}{4}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-2}}{\Gamma(p-1)}\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-2}}{\Gamma(p-1)}\\
&\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau\Bigg]+\frac{t[(\delta+c)-t]-\delta c}{4}\Bigg[\int_0^\delta \frac{(\delta-\tau)^{p-3}}{\Gamma(p-2)}\\
&\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau+\int_0^c \frac{(c-\tau)^{p-3}}{\Gamma(p-2)}\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]d\tau\Bigg]\\
&+\frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}|h(\tau)|d\tau\\
&\leq\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]\Bigg[\frac{t^p}{\Gamma(p+1)}-\frac{1}{2}\left(\frac{\delta^p}{\Gamma(p+1)}+\frac{c^p}{\Gamma(p+1)}\right)\\
&+\frac{|\delta+c-2t|}{4}\left(\frac{\delta^{p-1}}{\Gamma(p)}+\frac{c^{p-1}}{\Gamma(p)}\right)+\frac{|t[(\delta+c)-t]-\delta c|}{4}\left(\frac{\delta^{p-2}}{\Gamma(p-1)}+\frac{c^{p-2}}{\Gamma(p-1)}\right)\Bigg]\\
&+\epsilon\Big(\frac{t^p}{\Gamma(p+1)}\Big)\\
&\leq\mathcal{L}_1\Big[\|x^*-x\|+\mathcal{L}_2\|x^*-x\|\Big]\Big(\mathcal{Q}\Big)+\epsilon\Big(\frac{t^p}{\Gamma(p+1)}\Big),
\end{aligned}
$$

where $\mathcal{Q}$ is defined in (3.2), consequently

$$|x^*-x|\leq \frac{c^p}{\Gamma(p+1)\Big[1-\mathfrak{L}_1(1+\mathfrak{L}_2)\mathcal{Q}\Big]}(\epsilon),$$

provided that $\mathfrak{L}_1(1+\mathfrak{L}_2)\mathcal{Q}<1$. We conclude that the problem (1.1)-(1.2) is UHS. By setting,

$$\phi(\epsilon)=\frac{c^p}{\Gamma(p+1)\Big[1-\mathfrak{L}_1(1+\mathfrak{L}_2)\mathcal{Q}\Big]}(\epsilon),$$

we obtain,

$$|x^*-x|\leq\phi(\epsilon). \tag{4.3}$$

Clearly $\phi(0)=0$. Therefore, the problem (1.1)-(1.2) is also GUHS. □

5. Examples

Here are some examples that demonstrate the results we have obtained throughout the paper.

5.1. Example

Consider the following fractional differential equation:

$$
\begin{gathered}
{}^{C}\mathcal{D}^{2.5}x(t) = \frac{1}{10} + \frac{e^{-\lambda t}}{2}x(t) + \frac{1}{2}\int_0^t e^{-(t-s)}x(s)ds, \quad t \in [0,1] = \mathfrak{J}, \\
x(0) = -x(1), \quad x'(0) = -x'(1), \quad x''(0) = -x''(1),
\end{gathered}
$$

here, we take

$$
\begin{aligned}
p = 2.5, \quad \delta = 0, \quad c = 1, \quad \lambda > 0 \ \text{ and} \\
f(t, x(t), \mathcal{K}x(t)) = \frac{1}{10} + \frac{e^{-\lambda t}}{2}x(t) + \frac{1}{2}\int_0^t e^{-(t-s)}x(s)ds, \\
\mathcal{K}x(t) = \int_0^t e^{-(t-s)}x(s)ds.
\end{aligned}
$$

Let $x, y \in \mathbb{R}$ and $t \in \mathfrak{J}$, then we have

$$
\begin{aligned}
|f(t, x, \mathcal{K}x) - f(t, y, \mathcal{K}y)| \leq & \frac{1}{2}\Big(|x-y| + |\mathcal{K}x - \mathcal{K}y|\Big), \\
\Big|\int_0^t \Big(g(t,s,x) - g(t,s,y)\Big)ds\Big| \leq & |x-y|.
\end{aligned}
$$

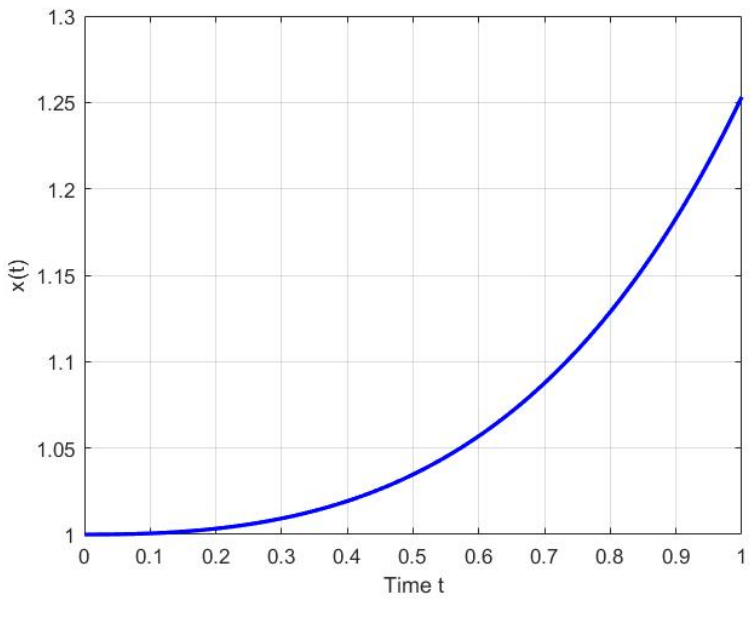

Figure 1: Graph of $x(t)$ for $p = 2.5$

Thus, the condition $(A1)-(A2)$ are satisfied with $\mathfrak{L}_1 = \frac{1}{2}$ and $\mathfrak{L}_2 = 1$. By simple calculation, we obtain

$$
\begin{aligned}
\mathfrak{L}_1[1+\mathfrak{L}_2]\mathcal{Q} < & \frac{1}{2}\Big[1+1\Big]\max_{t\in[0,1]}\left(\frac{t^{2.5}}{3.32335} - \frac{1}{2}(0.3009) + \frac{|1-2t|}{4}(0.7522) + \frac{|t[1-t]|}{4}(1.1283)\right) \\
< & \, 0.1286 < 1.
\end{aligned}
$$

Therefore, by Theorem 3.1, the problem (1.1)-(1.2) has a unique solution on $\mathfrak{J}$. Moreover, according to Theorem 4.1, the problem (1.1)-(1.2) exhibits UHS and GUHS.

5.2. Example

Consider the following fractional differential equation:

$$ {}^{C}\mathcal{D}^{2.8}x(t) = \frac{1}{20} + \frac{e^{-\lambda t}}{2}x(t) + \frac{1}{2}\int_0^t e^{-(t-s)}x(s)ds, \quad t \in [0,1] = \mathfrak{J}, $$
$$ x(0) = -x(1), \quad x'(0) = -x'(1), \quad x''(0) = -x''(1) $$

here, we take

$$ p = 2.8, \quad \delta = 0, \quad c = 1, \quad \lambda > 0 \text{ and} $$

$$ f(t, x(t), \mathcal{K}x(t)) = \frac{1}{20} + \frac{e^{-\lambda t}}{2}x(t) + \frac{1}{2}\int_0^t e^{-(t-s)}x(s)ds, $$
$$ \mathcal{K}x(t) = \int_0^t e^{-(t-s)}x(s)ds. $$

Let $x, y \in \mathbb{R}$ and $t \in \mathfrak{J}$, then we have

$$ |f(t,x,\mathcal{K}x) - f(t,y,\mathcal{K}y)| \leq \frac{1}{2}\Big(|x-y| + |\mathcal{K}x - \mathcal{K}y|\Big), $$
$$ \Big|\int_0^t \Big(g(t,s,x) - g(t,s,y)\Big)ds\Big| \leq |x-y|. $$

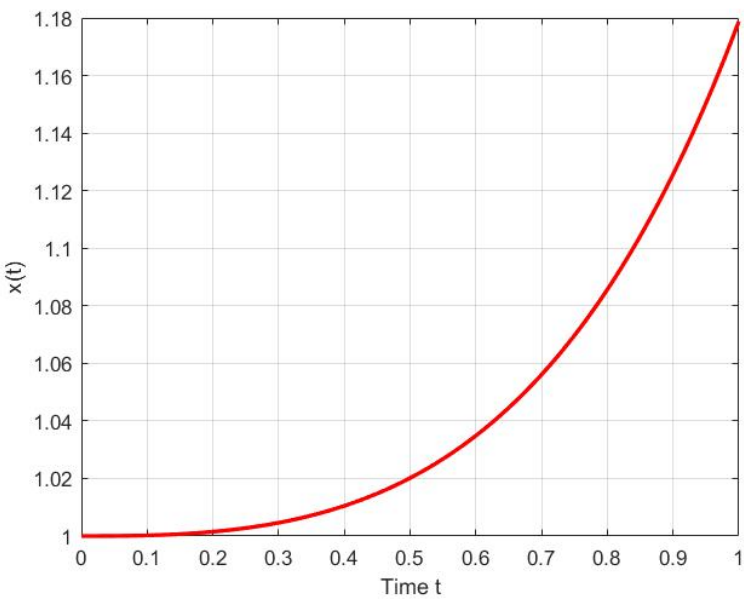

Figure 2: Graph of $x(t)$ for $p = 2.8$

Thus, the conditions $(A1)-(A2)$ are satisfied with $\mathfrak{L}_1 = \frac{1}{2}$ and $\mathfrak{L}_2 = 1$. By simple calculation, we obtain

$$ \mathfrak{L}_1[1+\mathfrak{L}_2]\mathcal{Q} < \frac{1}{2}\Big[1+1\Big]\max_{t\in[0,1]}\left(\frac{t^{2.8}}{2.9530} - \frac{1}{2}(0.3386) + \frac{|1-2t|}{4}(0.6157) + \frac{|t[1-t]|}{4}(1.0570)\right) $$
$$ < 0.1047 < 1. $$

Therefore, by Theorem 3.1, the problem (1.1)-(1.2) has a unique solution on $\mathfrak{J}$. Moreover, according to Theorem 4.1, the problem (1.1)-(1.2) exhibits UHS and GUHS.

6. Conclusion

This study confirms the existence and Ulam-type stability of solutions for nonlinear fractional integrodifferential equations with nonlocal antiperiodic boundary conditions using the Banach fixed point theorem. Illustrative examples are provided to support the theoretical results. For further research, stability results can be expanded to include models with delay or impulsive effects as well as systems incorporating various kinds of fractional derivatives.

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