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A Study on the Stability and Solvability of Pantograph-Type Equations with (k,ψ)-Caputo Proportional Fractional Derivative Operator

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ABSTRACT: This study investigates a class of pantograph-type equations involving the (k,ψ)-Caputo proportional fractional derivative, subject to nonlocal fractional integral boundary conditions. The existence and uniqueness of solutions are established through the application of Banach's and Krasnoselskii's fixed point theorems. Furthermore, various forms of Ulam stability are analyzed. To illustrate the theoretical results and demonstrate their applicability, a numerical example is provided.

Key Words: Existence of solutions, Pantograph-type equations, (k,ψ)-Caputo proportional fractional derivative, Ulam–Hyers stability.

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1. Introduction

Differential equations are fundamental tools in modeling and analyzing natural phenomena, offering a rigorous mathematical framework for understanding dynamic processes and complex system behavior, see [8,9,10,17]. Over the past decades, they have enabled the prediction of various phenomena such as planetary motion and temperature fluctuations through diverse formulations. Readers may also refer to [13] for further details.

Among these, pantograph-type equations are particularly suited for systems with non-uniform time delays. Their applications span power transmission, automatic control, economic modeling, neuroscience, and heat transfer. By incorporating dependencies on past states over non-constant delays, they allow for more accurate representations of system dynamics. Within the domain of fractional calculus, considerable attention has been given to pantograph-type equations involving fractional derivatives, with particular focus on establishing the existence and uniqueness of solutions, typically using fixed point theorems such as those of Banach, Schauder, and Krasnoselskii. Stability analysis, especially in the context of Ulam-Hyers and Ulam-Hyers-Rassias frameworks, has also been extensively explored, see [2,3,12].

Several recent studies have addressed the existence, uniqueness, and Ulam-type stability of solutions to various fractional differential equations. For instance, [2] examined nonlinear fractional pantograph equations with proportional Caputo derivatives and mixed nonlocal conditions. Likewise, [11] and [1] considered ψ-Hilfer-type and neutral Caputo-Hadamard pantograph equations, respectively. In another contribution, [7] investigated boundary value problems involving nonlinear fractional differential equations with the ψ-Caputo derivative. In [16], the authors study existence and uniqueness results for a class

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of k-generalized ψ-Hilfer fractional differential equations with periodic conditions. For further details, see [14].

In [18], the existence, uniqueness, and Ulam–Hyers stability of solutions to the following Caputo – Hadamard - type pantograph fractional differential equation are investigated:

$$\begin{cases} {}^{C}_{H}D^{\alpha}u(t) = \varphi\left(t, u(t), u(\lambda t)\right), \quad t \in [1,T],\ 0<\alpha\leq 1,\ 0<\lambda<1, \\ \qquad u(1) = u_1 - \theta(u), \quad u_1 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^{C}_{H}D^{\alpha}$ denote the Caputo-Hadamard type fractional derivative of order α and $\varphi : [1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\theta : C([1,T],\mathbb{R})\to\mathbb{R}$ are given continuous functions.

Motivated by the aforementioned studies, the present work focuses on a class of pantograph-type equations with integral nonlocal boundary conditions. Our aim is to establish the existence and uniqueness of solutions, along with Ulam–Hyers stability for the following equation:

$$\begin{cases} {}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\left[\,{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(x) - \lambda\ g(x,\upsilon(x))\right] = f(x,\upsilon(x),\upsilon(\varrho x)), \quad x\in\Pi=[a,b], \\ \upsilon(b) = \sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\phi_i,\rho;\psi}\upsilon(\zeta_i), \qquad {}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(a) = \lambda\ g(a,\upsilon(a)). \end{cases} \tag{1.1}$$

where $0<\frac{\alpha}{k}\leq 1$, $0<\frac{\beta}{k}\leq 1$, with $k>0$, $0<\varrho<1$, and $\rho\in(0,1]$. Here, ${}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}(\cdot)$ denotes the (k,ψ)-Caputo proportional fractional derivative of order α, and ${}_{a,k}I^{\phi_i,\rho;\psi}(\cdot)$ represents the (k,ψ)-Riemann–Liouville proportional fractional integral operator of order ϕ_i. The parameters $\lambda,\sigma_i\in\mathbb{R}$, and $\zeta_i\in\Pi$, for $i=1,2,\ldots,m$, with $m\in\mathbb{N}$. The functions $f:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:[a,b]\times\mathbb{R}\to\mathbb{R}$ are continuous.

The proposed equation is characterized by its inclusiveness and generality, as it encompasses many well-known boundary value problems under suitable conditions.

By choosing

$$g(x,\upsilon(x)) = 1,$$

the problem (1.1) reduces to the following (k,ψ)-Caputo proportional fractional Langevin–pantograph-type equation:

$$\begin{cases} {}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\left[\,{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(x) - \lambda\right] = f(x,\upsilon(x),\upsilon(\varrho x)), \quad x\in\Pi, \\ \upsilon(b) = \sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\phi_i,\rho;\psi}\upsilon(\zeta_i), \qquad {}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(a) = \lambda. \end{cases}$$

Problem (1.1) represents a unified framework that captures a broad spectrum of cases involving different types of fractional derivatives. The diversity of problems covered by this formulation depends essentially on the specific choices of the function ψ and the parameters ρ and k

Under particular selections, the general problem reduces to several notable special cases, such as:

- If $\psi(x)=x$, Problem (1.1) reduces to a k-Caputo proportional fractional equation of pantograph type;
- If $\rho=1$, it takes the standard (k,ψ)-Caputo form;
- If $k=1$, it corresponds to the ψ-Caputo variant of the same equation.

This work introduces a fractional equation involving (k,ψ)-Caputo proportional derivatives with integral boundary conditions and presents a unified framework for various classes of fractional differential equations. It also establishes key results on existence, uniqueness, and Ulam–Hyers stability.

We outline the structure of this paper as follows: In Section 2, we present some essential definitions and lemmas. In Section 3, we study the existence and uniqueness of solutions for the proposed equation. Section 4 is devoted to the study of Ulam-Hyers stability and generalized Ulam-Hyers stability. In Section 5, an illustrative example is provided. Finally, Section 6 presents a summary of the main findings of this study.

2. Preliminaries

Let $\Pi = [a,b]$ be a finite interval. For all $x \in \Pi$, let $\mathcal{G} = C(\Pi,\mathbb{R})$ denote the Banach space of continuous functions $\upsilon : \Pi \to \mathbb{R}$, equipped with the norm

$$\|\upsilon\| = \max_{x\in[a,b]} |\upsilon(x)|.$$

Let $\psi : \Pi \to \mathbb{R}$ be a strictly increasing and continuous function such that $\psi'(x) \neq 0$ for all $x \in \Pi$.

Moreover, define the space of functions that are n-times differentiable with $(n-1)$-th derivative absolutely continuous by

$$AC^n[a,b] = \left\{ \upsilon : \Pi \to \mathbb{R} \mid \upsilon^{(n-1)} \in AC[a,b] \right\}.$$

Definition 2.1 *[6] Assume that $u \in \mathbb{C}$ with $\mathrm{Re}(u) > 0$, and let $k > 0$. Then, the k-gamma function Γ_k is defined by*

$$\Gamma_k(u) = \int_0^\infty s^{u-1} e^{\frac{-s^k}{k}}\, ds. \tag{2.1}$$

Proposition 2.1 *[6] The k-gamma function satisfies the following properties:*

(i) $\Gamma_k(u+k) = u\Gamma_k(u)$

(ii) $\Gamma_k(k) = 1$

(iii) $\Gamma_k(u) = k^{\frac{u}{k}-1}\Gamma(\frac{u}{k})$.

Definition 2.2 *[5] Let $\upsilon \in L^1(\Pi,\mathbb{R})$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and $\rho \in (0,1]$. Then, the left-sided (k,ψ)-$\mathbb{RL}$ proportional fractional integral operator of order α is defined by*

$${}_{a,k}I^{\alpha,\rho;\psi}\upsilon(x) = \frac{1}{\rho^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_a^x {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(x,\delta)\psi'(\delta)\upsilon(\delta)\, d\delta,$$

where $\Gamma_k(\cdot)$ is defined by (2.1), and

$${}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(x,\delta) = e^{\frac{\rho-1}{k\rho}(\psi(x)-\psi(\delta))}(\psi(x)-\psi(\delta))^{\frac{\alpha}{k}-1}.$$

Lemma 2.1 *[5] Let $\alpha,\mu \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\mu) > 0$, $\rho \in (0,1]$, and $\frac{\mathrm{Re}(\mu)}{k} > -1$. Then, we obtain*

$${}_{a,k}I^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(x,a)\right] = \frac{\Gamma_k(\mu)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\mu+\alpha)}\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\mu+\alpha}{k}-1}(x,a).$$

Lemma 2.2 *[5] Let $\alpha_1,\alpha_2 \in \mathbb{C}$ with $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) > 0$, and $\rho \in (0,1]$. Then,*

$${}_{a,k}I^{\alpha_1,\rho;\psi}({}_{a,k}I^{\alpha_2,\rho;\psi}\upsilon(x)) = {}_{a,k}I^{\alpha_1+\alpha_2,\rho;\psi}\upsilon(x) = {}_{a,k}I^{\alpha_2,\rho;\psi}({}_{a,k}I^{\alpha_1,\rho;\psi}\upsilon(x)).$$

Definition 2.3 *[5] Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $0 < \rho \leq 1$, and let $\upsilon,\psi \in C^n(\Pi,\mathbb{R})$ with $\psi'(x) \neq 0$ for all $x \in \Pi$, where $n = \left\lfloor \frac{\mathrm{Re}(\alpha)}{k} \right\rfloor + 1$ and $n \in \mathbb{N}$. Then, the left-sided (k,ψ)-Caputo proportional fractional derivative operator of order α is defined by*

$${}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\upsilon(x) = {}_{a,k}I^{nk-\alpha,\rho;\psi}\left({}_{k}\mathbb{D}^{n,\rho;\psi}\upsilon(x)\right).$$

Lemma 2.3 *[5] Let $\alpha,\mu \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$, $\mathrm{Re}(\mu) > 0$, $k > 0$, $\rho \in (0,1]$, and $\frac{\mathrm{Re}(\mu)}{k} > -1$. Then, we obtain*

$${}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(x,a)\right] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\mu)}{\Gamma_k(\mu-\alpha)}\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\mu-\alpha}{k}-1}(x,a).$$

Lemma 2.4 *[5] Let $\alpha, \mu \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\mu) > nk$, $\rho \in (0,1]$, and $n = \left\lfloor \frac{\mathrm{Re}(\mu)}{k} \right\rfloor + 1$. Then,*

$$ {}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\big({}_{a,k}I^{\mu,\rho;\psi}\upsilon(x)\big) = {}_{a,k}I^{\mu-\alpha,\rho;\psi}\upsilon(x). $$

Lemma 2.5 *[5] Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $k > 0$, $\frac{\mathrm{Re}(\alpha)}{k} \in (n-1, n]$, $n \in \mathbb{N}$ and $\rho \in (0,1]$. Then,*

$$ {}_{a,k}I^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\upsilon(x)\right) = \upsilon(x) - \sum_{i=1}^{n} \frac{{}^{\rho}_{k}\Psi^{n-i}_{\psi}(x,a)}{(\rho k)^{n-i}(n-i)!}\ {}_{k}\mathbb{D}^{n-i,\rho;\psi}\upsilon(a). $$

In particular, for $n = 1$, we have

$$ {}_{a,k}I^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\upsilon(x)\right) = \upsilon(x) - {}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)\ {}_{k}\mathbb{D}^{0,\rho;\psi}\upsilon(a). $$

Lemma 2.6 *[4] (Banach's fixed point theorem) Let $\mathcal{X}$ be a non-empty closed subset of a Banach space $\mathcal{W}$. Then, every contraction mapping $\mathbb{T} : \mathcal{X} \to \mathcal{X}$ admits a unique fixed point.*

Lemma 2.7 *[19] (Krasnoselskii's fixed point theorem) Let $\mathcal{W}$ be a Banach space. Let $\mathcal{X}$ be a non-empty, closed, bounded, and convex subset of $\mathcal{W}$, and let $\mathcal{J}_1$, $\mathcal{J}_2$ be operators such that*

1. *$\mathcal{J}_1\upsilon_1 + \mathcal{J}_2\upsilon_2 \in \mathcal{X}$ whenever $\upsilon_1, \upsilon_2 \in \mathcal{X}$.*
2. *$\mathcal{J}_1$ is continuous and compact.*
3. *$\mathcal{J}_2$ is a contraction.*

Then, there exists $\upsilon_3 \in \mathcal{X}$ such that

$$ \upsilon_3 = \mathcal{J}_1\upsilon_3 + \mathcal{J}_2\upsilon_3. $$

3. Existence and Uniqueness Results

In this section, we prove the existence and uniqueness of the solution to equation (1.1) within the given domain.

We now present the following notations, which are crucial for the subsequent results.

$$ \Upsilon = {}^{\rho}_{k}\Psi^{0}_{\psi}(b,a) - \sum_{i=1}^{m} \frac{\sigma_i\ {}^{\rho}_{k}\Psi^{\frac{\phi_i}{k}}_{\psi}(\zeta_i,a)}{\rho^{\frac{\phi_i}{k}}\Gamma_k(\phi_i+k)} \neq 0. \tag{3.1} $$

$$ \Lambda = \frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\alpha+\beta+\phi_i}{k}}}{\rho^{\frac{\alpha+\beta+\phi_i}{k}}\Gamma_k(\alpha+\beta+\phi_i+k)} \tag{3.2} $$

$$ \begin{aligned} \Delta_1 = {} & \frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(b,a)}{|\Upsilon|}\left(\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)}\right. \\ & \left. + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\alpha+\beta+\phi_i}{k}}}{\rho^{\frac{\alpha+\beta+\phi_i}{k}}\Gamma_k(\alpha+\beta+\phi_i+k)}\right). \end{aligned} \tag{3.3} $$

$$ \Delta_2 = |\lambda|\left[\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)} + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(b,a)}{|\Upsilon|}\left(\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)} + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\beta+\phi_i}{k}}}{\rho^{\frac{\beta+\phi_i}{k}}\Gamma_k(\beta+\phi_i+k)}\right)\right]. \tag{3.4} $$

To lay the groundwork for the main result, we first establish the following lemma.

Lemma 3.1 *Let $h(x) \in C(\Pi, \mathbb{R})$ be a given function. The solution of the fractional differential equation*

$$\begin{cases} {}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi}\left[{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(x) - \lambda\, g(x,\upsilon(x))\right] = h(x), \quad x \in \Pi, \\ \upsilon(b) = \sum_{i=1}^{m} \sigma_i\ {}_{a,k}I^{\phi_i,\rho;\psi}\upsilon(\zeta_i), \quad {}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(a) = \lambda\, g(a,\upsilon(a)). \end{cases} \tag{3.5}$$

is given by

$$\upsilon(x) = {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(x) + \lambda\ {}_{a,k}I^{\beta,\rho;\psi}g(x,\upsilon(x)) + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{\Upsilon}\left[\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}h(\zeta_i) - {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(b) + \lambda\left(\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\beta+\phi_i,\rho;\psi}g(\zeta_i,\upsilon(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi}g(b,\upsilon(b))\right)\right].$$

Proof: Let $\upsilon \in C(\Pi, \mathbb{R})$ be a solution of the fractional differential equation (3.5). By applying the fractional integral operator ${}_{a,k}I^{\alpha,\rho;\psi}(\cdot)$ to both sides of equation (3.5), and invoking Lemma 2.5 with $n = 1$, we obtain:

$${}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(x) - \lambda\, g(x,\upsilon(x)) = {}_{a,k}I^{\alpha,\rho;\psi}h(x) + c_1\ {}^{\rho}_{k}\Psi^{0}_{\psi}(x,a), \tag{3.6}$$

where

$$c_1 = {}_{k}\mathbb{D}^{0,\rho;\psi}\left[{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi}\upsilon(a) - \lambda\, g(a,\upsilon(a))\right].$$

Taking into account the initial condition ${}^{C}_{a,k}\mathbb{D}^{\alpha_2,\rho;\psi}\upsilon(a) = \lambda\, g(a,\upsilon(a))$, we deduce that $c_1 = 0$. Applying once again the operator ${}_{a,k}I^{\beta,\rho;\psi}(\cdot)$ to both sides of equation (3.6), we obtain:

$$\upsilon(x) = {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(x) + \lambda\ {}_{a,k}I^{\beta,\rho;\psi}g(x,\upsilon(x)) + \overline{c}_1\ {}^{\rho}_{k}\Psi^{0}_{\psi}(x,a). \tag{3.7}$$

where

$$\overline{c}_1 = {}_{k}\mathbb{D}^{0,\rho;\psi}\upsilon(a),$$

Evaluating (3.7) at $x = b$ yields:

$$\upsilon(b) = {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(b) + \lambda\ {}_{a,k}I^{\beta,\rho;\psi}g(b,\upsilon(b)) + \overline{c}_1\ {}^{\rho}_{k}\Psi^{0}_{\psi}(b,a). \tag{3.8}$$

Furthermore, we have:

$$\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\phi_i,\rho;\psi}\upsilon(\zeta_i) = \sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}h(\zeta_i) + \lambda\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\beta+\phi_i,\rho;\psi}g(\zeta_i,\upsilon(\zeta_i)) + \overline{c}_1\sum_{i=1}^{m}\frac{\sigma_i\ {}^{\rho}_{k}\Psi^{\frac{\phi_i}{k}}_{\psi}(\zeta_i,a)}{\rho^{\frac{\phi_i}{k}}\Gamma_k(\phi_i+k)}. \tag{3.9}$$

Since it is known that $\upsilon(b) = \sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\phi_i,\rho;\psi}\upsilon(\zeta_i)$, it follows that:

$$\overline{c}_1 = \frac{1}{\Upsilon}\left[\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}h(\zeta_i) - {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(b) + \lambda\left(\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\beta+\phi_i,\rho;\psi}g(\zeta_i,\upsilon(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi}g(b,\upsilon(b))\right)\right]. \tag{3.10}$$

Hence,

$$\upsilon(x) = {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(x) + \lambda\, {}_{a,k}I^{\beta,\rho;\psi}g(x,\upsilon(x)) + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{\Upsilon}\Bigg[\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}h(\zeta_i) - {}_{a,k}I^{\alpha+\beta,\rho;\psi}h(b) + \lambda\Bigg(\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\beta+\phi_i,\rho;\psi}g(\zeta_i,\upsilon(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi}g(b,\upsilon(b))\Bigg)\Bigg].$$

□

Motivated by Lemma 3.1, we define the operator $\mathcal{J}:\mathcal{G}\to\mathcal{G}$ as follows:

$$(\mathcal{J}\upsilon)(x) = {}_{a,k}I^{\alpha+\beta,\rho;\psi}f\left(x,\upsilon(x),\upsilon(\varrho x)\right) + \lambda\, {}_{a,k}I^{\beta,\rho;\psi}g(x,\upsilon(x)) + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{\Upsilon} \times\Bigg[\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}f\left(\zeta_i,\upsilon(\zeta_i),\upsilon(\varrho\zeta_i)\right) - {}_{a,k}I^{\alpha+\beta,\rho;\psi}f\left(b,\upsilon(b),\upsilon(\varrho b)\right) + \lambda\Bigg(\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\beta+\phi_i,\rho;\psi}g(\zeta_i,\upsilon(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi}g(b,\upsilon(b))\Bigg)\Bigg].$$

We make the following assumptions:

($\mathcal{H}_1$) The functions $f:\Pi\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\Pi\times\mathbb{R}\to\mathbb{R}$ are continuous.

($\mathcal{H}_2$) There exists a constant $\mathfrak{L}_1>0$ such that

$$|f(x,u_1,v_1)-f(x,u_2,v_2)| \le \mathfrak{L}_1\left(|u_1-u_2|+|v_1-v_2|\right),$$

for all $x\in\Pi$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$.

($\mathcal{H}_3$) There exists a constant $\mathfrak{L}_2>0$ such that

$$|g(x,u)-g(x,v)| \le \mathfrak{L}_2|u-v|,$$

for all $x\in\Pi$ and $u,v\in\mathbb{R}$.

($\mathcal{H}_4$) There exists a positive function $\vartheta_f\in C(\Pi,\mathbb{R})$ such that

$$|f(x,u,v)| \le \vartheta_f(x), \quad \text{for all } x\in\Pi \text{ and } (u,v)\in\mathbb{R}^2.$$

($\mathcal{H}_5$) There exists a positive function $\vartheta_g\in C(\Pi,\mathbb{R})$ such that

$$|g(x,u)| \le \vartheta_g(x), \quad \text{for all } (x,u)\in\Pi\times\mathbb{R}.$$

Theorem 3.1 *Suppose that the assumptions ($\mathcal{H}_1$), ($\mathcal{H}_2$), and ($\mathcal{H}_3$) hold. If*

$$2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2<1,$$

then the fractional differential equation (1.1) admits a unique solution on Π.

Proof: First, let $\varpi=\sup_{x\in\Pi}|f(x,0,0)|<\infty$ and $\varsigma=\sup_{x\in\Pi}|g(x,0)|<\infty$. We prove that $\mathcal{J}\mathcal{B}_r\subset\mathcal{B}_r$, where

$$\mathcal{B}_r=\{\upsilon\in\mathcal{G}:\|\upsilon\|\le r\} \quad\text{and}\quad r>\frac{\varpi\Delta_1+\varsigma\Delta_2}{1-(2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2)}.$$

For all $\upsilon \in \mathcal{B}_r$, we have

$$\begin{aligned}
|f(x,\upsilon(x),\upsilon(\varrho x))| &= |f(x,\upsilon(x),\upsilon(\varrho x)) - f(x,0,0) + f(x,0,0)| \\
&\leq |f(x,\upsilon(x),\upsilon(\varrho x)) - f(x,0,0)| + |f(x,0,0)| \\
&\leq 2\mathfrak{L}_1|\upsilon(x)| + \varpi \\
&\leq 2\mathfrak{L}_1\|\upsilon\| + \varpi \\
&\leq 2\mathfrak{L}_1 r + \varpi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|g(x,\upsilon(x))| &= |g(x,\upsilon(x)) - g(x,0) + g(x,0)| \\
&\leq |g(x,\upsilon(x)) - g(x,0)| + |g(x,0)| \\
&\leq \mathfrak{L}_2|\upsilon(x)| + \varsigma \\
&\leq \mathfrak{L}_2\|\upsilon\| + \varsigma \\
&\leq \mathfrak{L}_2 r + \varsigma.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
|(\mathcal{J}\upsilon)(x)| \leq\ & {}_{a,k}I^{\alpha+\beta,\rho;\psi}|f\left(x,\upsilon(x),\upsilon(\varrho x)\right)| + |\lambda|\ {}_{a,k}I^{\beta,\rho;\psi}|g(x,\upsilon(x))| \\
&+ \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{|\Upsilon|}\left[\sum_{i=1}^{m}|\sigma_i|\ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}|f\left(\zeta_i,\upsilon(\zeta_i),\upsilon(\varrho\zeta_i)\right)|\right. \\
&+ {}_{a,k}I^{\alpha+\beta,\rho;\psi}|f\left(b,\upsilon(b),\upsilon(\varrho b)\right)| \\
&\left.+ |\lambda|\left(\sum_{i=1}^{m}|\sigma_i|\ {}_{a,k}I^{\beta+\phi_i,\rho;\psi}|g(\zeta_i,\upsilon(\zeta_i))| + {}_{a,k}I^{\beta,\rho;\psi}|g(b,\upsilon(b))|\right)\right] \\
&\leq \big(2\mathfrak{L}_1 r + \varpi\big)\left[\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(b,a)}{|\Upsilon|}\right. \\
&\left.\times\left(\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\alpha+\beta+\phi_i}{k}}}{\rho^{\frac{\alpha+\beta+\phi_i}{k}}\Gamma_k(\alpha+\beta+\phi_i+k)}\right)\right] \\
&+ |\lambda|\big(\mathfrak{L}_2 r + \varsigma\big)\left[\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)} + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(b,a)}{|\Upsilon|}\right. \\
&\left.\times\left(\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)} + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\beta+\phi_i}{k}}}{\rho^{\frac{\beta+\phi_i}{k}}\Gamma_k(\beta+\phi_i+k)}\right)\right] \\
&\leq \big(2\mathfrak{L}_1 r + \varpi\big)\Delta_1 + \big(\mathfrak{L}_2 r + \varsigma\big)\Delta_2.
\end{aligned}$$

Which implies that

$$\|(\mathcal{J}\upsilon)(x)\| \leq r,$$

thus,

$$\mathcal{J}\mathcal{B}_r \subset \mathcal{B}_r$$

.

Now, let $u, \upsilon \in \mathcal{G}$. Then

$$
\begin{aligned}
|(\mathcal{J}u)(x)-(\mathcal{J}v)(x)| \leq\ & {}_{a,k}I^{\alpha+\beta,\rho;\psi}|f\left(x,u(x),u(\varrho x)\right)-f\left(x,v(x),v(\varrho x)\right|\\
&+|\lambda|\ {}_{a,k}I^{\beta,\rho;\psi}|g(x,u(x))-g(x,v(x))|+\frac{{}_{k}^{\rho}\Psi_{\psi}^{0}(x,a)}{|\Upsilon|}\\
&\times\left[\sum_{i=1}^{m}|\sigma_i|\ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}|f\left(\zeta_i,u(\zeta_i),u(\varrho\zeta_i)\right)-f\left(\zeta_i,v(\zeta_i),v(\varrho\zeta_i)\right)|\right.\\
&+\ {}_{a,k}I^{\alpha+\beta,\rho;\psi}|f\left(b,u(b),u(\varrho b)\right)-f\left(b,v(b),v(\varrho b)\right)|\\
&+|\lambda|\Bigg(\sum_{i=1}^{m}|\sigma_i|\ {}_{a,k}I^{\beta+\phi_i,\rho;\psi}|g(\zeta_i,u(\zeta_i))-g(\zeta_i,v(\zeta_i))|\\
&\left.+\ {}_{a,k}I^{\beta,\rho;\psi}|g(b,u(b))-g(b,v(b))|\Bigg)\right]\\
&\leq\left(2\mathfrak{L}_1\|u-v\|\right)\left[\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)}+\frac{{}_{k}^{\rho}\Psi_{\psi}^{0}(b,a)}{|\Upsilon|}\right.\\
&\left.\times\left(\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)}+\sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\alpha+\beta+\phi_i}{k}}}{\rho^{\frac{\alpha+\beta+\phi_i}{k}}\Gamma_k(\alpha+\beta+\phi_i+k)}\right)\right],\\
&+|\lambda|\left(\mathfrak{L}_2\|u-v\|\right)\left[\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)}+\frac{{}_{k}^{\rho}\Psi_{\psi}^{0}(b,a)}{|\Upsilon|},\right.\\
&\left.\times\left(\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)}+\sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\beta+\phi_i}{k}}}{\rho^{\frac{\beta+\phi_i}{k}}\Gamma_k(\beta+\phi_i+k)}\right)\right],\\
&\leq 2\mathfrak{L}_1\|u-v\|\Delta_1+\mathfrak{L}_2\|u-v\|\Delta_2,\\
&\leq\left(2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2\right)\|u-v\|.
\end{aligned}
$$

Which implies that

$$
\|(\mathcal{J}u)(x)-(\mathcal{J}v)(x)\|\leq\left(2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2\right)\|u-v\|,
$$

for all $u,v\in\mathcal{G}$.

Since $2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2<1$, the operator $\mathcal{J}$ is a contraction.

Therefore, by applying Theorem 3.1, we conclude that $\mathcal{J}$ has a unique fixed point, which corresponds to the unique solution of equation (1.1). This completes the proof. □

Theorem 3.2 *Suppose that conditions $(\mathcal{H}_1)$, $(\mathcal{H}_4)$, and $(\mathcal{H}_5)$ hold. Then, there exists at least one solution to the fractional differential equation (1.1) on the interval Π.*

Proof: Based on the conditions $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$, we select $\wp>\|\vartheta_f\|\Delta_1+\|\vartheta_g\|\Delta_2$, where

$$
\mathcal{B}_{\wp}=\{u\in\mathcal{G}:\|u\|\leq\wp\},
$$

and

$$
\|\vartheta_f\|=\sup_{x\in\Pi}\vartheta_f(x),\quad \|\vartheta_g\|=\sup_{x\in\Pi}\vartheta_g(x).
$$

.

Now, we define two operators $\mathcal{J}_1$ and $\mathcal{J}_2$ as follows:

$$
\begin{aligned}
(\mathcal{J}_1 v)(x)=\ & {}_{a,k}I^{\alpha+\beta,\rho;\psi}f\big(x,v(x),v(\varrho x)\big)\\
&+\frac{{}_{k}^{\rho}\Psi_{\psi}^{0}(x,a)}{\Upsilon}\left(\sum_{i=1}^{m}\sigma_i\ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}f\big(\zeta_i,v(\zeta_i),v(\varrho\zeta_i)\big)-\ {}_{a,k}I^{\alpha+\beta,\rho;\psi}f\big(b,v(b),v(\varrho b)\big)\right).
\end{aligned}
$$

and

$$(\mathcal{J}_2 \upsilon)(x) = \lambda\, {}_{a,k}I^{\beta,\rho;\psi} g(x,\upsilon(x)) + \frac{\lambda\, {}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{\Upsilon}\left(\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\beta+\phi_i,\rho;\psi} g(\zeta_i,\upsilon(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi} g(b,\upsilon(b))\right).$$

We will now demonstrate that the conditions of Theorem 3.2 are satisfied. To this end, we proceed to verify the hypotheses through a detailed three-step process.

step 1. Let $u, w \in \mathcal{B}_{\wp}$. Then,

$$\begin{aligned}
|(\mathcal{J}_1 u)(x) + (\mathcal{J}_2 w)(x)| &= \Bigg|\, {}_{a,k}I^{\alpha+\beta,\rho;\psi} f\left(x,u(x),u(\varrho x)\right) + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{\Upsilon} \\
&\times \left(\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi} f\left(\zeta_i,u(\zeta_i),u(\varrho\zeta_i)\right)\right. \\
&\left. -\ {}_{a,k}I^{\alpha+\beta,\rho;\psi} f\left(b,u(b),u(\varrho b)\right)\right) + \lambda\, {}_{a,k}I^{\beta,\rho;\psi} g(x,w(x)) \\
&+ \frac{\lambda\, {}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{|\Upsilon|}\left(\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\beta+\phi_i,\rho;\psi} g(\zeta_i,w(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi} g(b,w(b))\right)\Bigg| \\
&\le {}_{a,k}I^{\alpha+\beta,\rho;\psi}|f\left(x,u(x),u(\varrho x)\right)| + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{|\Upsilon|} \\
&\times \left(\sum_{i=1}^{m}|\sigma_i|\, {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}|f\left(\zeta_i,u(\zeta_i),u(\varrho\zeta_i)\right)|\right. \\
&\left. +\ {}_{a,k}I^{\alpha+\beta,\rho;\psi}|f\left(b,u(b),u(\varrho b)\right)|\right) + |\lambda|\, {}_{a,k}I^{\beta,\rho;\psi}|g(x,w(x))| \\
&+ \frac{|\lambda|\, {}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{|\Upsilon|}\left(\sum_{i=1}^{m}|\sigma_i|\, {}_{a,k}I^{\beta+\phi_i,\rho;\psi}|g(\zeta_i,w(\zeta_i))| + {}_{a,k}I^{\beta,\rho;\psi}|g(b,w(b))|\right), \\
&\le \|\vartheta_f\|\Bigg[\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(b,a)}{|\Upsilon|}\Bigg(\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} \\
&+ \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\alpha+\beta+\phi_i}{k}}}{\rho^{\frac{\alpha+\beta+\phi_i}{k}}\Gamma_k(\alpha+\beta+\phi_i+k)}\Bigg)\Bigg], \\
&+ \|\vartheta_g\||\lambda|\Bigg[\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)} + \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(b,a)}{|\Upsilon|}, \\
&\times \Bigg(\frac{(\psi(b)-\psi(a))^{\frac{\beta}{k}}}{\rho^{\frac{\beta}{k}}\Gamma_k(\beta+k)} + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\beta+\phi_i}{k}}}{\rho^{\frac{\beta+\phi_i}{k}}\Gamma_k(\beta+\phi_i+k)}\Bigg)\Bigg] \\
&\le \|\vartheta_f\|\Delta_1 + \|\vartheta_g\|\Delta_2.
\end{aligned}$$

This implies that

$$\|(\mathcal{J}_1 u)(x) + (\mathcal{J}_2 w)(x)\| \le \wp.$$

Therefore,

$$(\mathcal{J}_1 u)(x) + (\mathcal{J}_2 w)(x) \in \mathcal{B}_{\wp}.$$

Step 2. Since the operator $\mathcal{J}$ is a contraction, it follows that

$$\|(\mathcal{J}_2 u)(x) - (\mathcal{J}_2 v)(x)\| \le \mathfrak{L}_2\Delta_2\,\|u-v\|.$$

Therefore, $\mathcal{J}_2$ is also a contraction mapping.

Step 3. Now, we prove that the operator $\mathcal{J}_1$ is compact. Let $x_1, x_2 \in \Pi$ with $x_1 < x_2$, and set

$$\overline{f} = \sup_{(x,u,v)\in\Pi\times\mathcal{B}_\wp\times\mathcal{B}_\wp} |f(x, \upsilon(x), \upsilon(\varrho x))|$$

Then, it holds that

$$|(\mathcal{J}_1\upsilon)(x_2) - (\mathcal{J}_1\upsilon)(x_1)| \leq \frac{\overline{f}}{\rho^{\frac{\alpha+\beta}{k}} k\Gamma_k(\alpha+\beta)} \Bigg| \int_a^{x_1} \left({}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha+\beta}{k}-1}(x_2,\delta) - {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha+\beta}{k}-1}(x_1,\delta) \right) \psi'(\delta)\, d\delta + \int_{x_2}^{x_1} {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha+\beta}{k}-1}(x_2,\delta)\psi'(\delta)\, d\delta \Bigg| + \frac{|\,{}^{\rho}_{k}\Psi^{0}_{\psi}(x_2,a) - {}^{\rho}_{k}\Psi^{0}_{\psi}(x_1,a)|\, \overline{f}\, \Lambda}{|\Upsilon|}.$$

The right-hand side asymptotically approaches zero as $(x_2 - x_1) \to 0$, independently of $\upsilon \in \mathbb{B}_\wp$. Hence, by the Arzelà–Ascoli theorem, the operator $\mathcal{J}_1$ is compact on $\mathcal{B}_\wp$.

Applying Krasnoselskii's fixed point theorem, we conclude that problem (1.1) admits at least one solution on $\mathcal{G}$.

□

4. Stability

The definitions below establish the concepts of Ulam-Hyers and generalized Ulam-Hyers stability for problem (1.1). For a given $\epsilon > 0$, consider the inequality:

$$\left| {}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi} \left[{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi} \kappa(x) - \lambda\, g(x,\kappa(x)) \right] - f(x,\kappa(x),\kappa(\varrho x)) \right| \leq \epsilon, \quad x \in \Pi. \tag{4.1}$$

Definition 4.1 *[15] The proposed problem (1.1) is said to be Ulam-Hyers (U-H) stable if there exists a constant $\Omega > 0$ such that, for every $\epsilon > 0$ and for every function $\kappa \in \mathcal{G}^1 = C^1(\Pi,\mathbb{R})$ satisfying inequality (4.1), there exists a solution $\upsilon \in C^1(\Pi,\mathbb{R})$ of problem (1.1) such that*

$$|\kappa(x) - \upsilon(x)| \leq \Omega\epsilon, \quad x \in \Pi. \tag{4.2}$$

Definition 4.2 *[15] The proposed problem (1.1) is said to be generalized Ulam-Hyers stable if there exists a continuous function $\Omega_f \in C([0,\infty),[0,\infty))$ with $\Omega_f(0) = 0$, such that for each $\epsilon > 0$ and for each function $\kappa \in C^1(\Pi,\mathbb{R})$ satisfying inequality (4.1), there exists a solution $\upsilon \in C^1(\Pi,\mathbb{R})$ of problem (1.1) such that*

$$|\kappa(x) - \upsilon(x)| \leq \Omega_f(\epsilon), \quad x \in \Pi. \tag{4.3}$$

From the previous definitions, it follows that Definition 4.1 implies Definition 4.2.

Remark 4.1 *A function $\kappa \in C^1(\Pi,\mathbb{R})$ is a solution of inequality (4.1) if and only if there exists a function $\mathcal{D} \in C^1(\Pi,\mathbb{R})$ such that:*

(i) $|\mathcal{D}(x)| \leq \epsilon, \quad x \in \Pi;$

(ii) ${}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi} \left[{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi} \kappa(x) - \lambda\, g(x,\kappa(x)) \right] = f(x,\kappa(x),\kappa(\varrho x)) + \mathcal{D}(x), \quad x \in \Pi.$

In view of part *(ii)* of Remark 4.1, it follows that

$${}^{C}_{a,k}\mathbb{D}^{\alpha,\rho;\psi} \left[{}^{C}_{a,k}\mathbb{D}^{\beta,\rho;\psi} \kappa(x) - \lambda\, g(x,\kappa(x)) \right] = f(x,\kappa(x),\kappa(\varrho x)) + \mathcal{D}(x), \quad x \in \Pi. \tag{4.4}$$

By applying Lemma 3.1, the solution of problem (4.4) is given by

$$
\begin{aligned}
\kappa(x) = & \ {}_{a,k}I^{\alpha+\beta,\rho;\psi} f\left(x, \kappa(x), \kappa(\varrho x)\right) + {}_{a,k}I^{\alpha+\beta,\rho;\psi}\mathcal{D}(x) + \lambda \ {}_{a,k}I^{\beta,\rho;\psi} g(x, \kappa(x)) \\
& + \frac{{}_k^\rho\Psi_\psi^0(x,a)}{\Upsilon}\Bigg[\sum_{i=1}^{m}\sigma_i \ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi} f\left(\zeta_i, \kappa(\zeta_i), \kappa(\varrho\zeta_i)\right) \\
& - {}_{a,k}I^{\alpha+\beta,\rho;\psi} f\left(b, \kappa(b), \kappa(\varrho b)\right) + \sum_{i=1}^{m}\sigma_i \ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}\mathcal{D}(\zeta_i) - {}_{a,k}I^{\alpha+\beta,\rho;\psi}\mathcal{D}(b) \\
& + \lambda\Bigg(\sum_{i=1}^{m}\sigma_i \ {}_{a,k}I^{\beta+\phi_i,\rho;\psi} g(\zeta_i, \kappa(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi} g(b, \kappa(b))\Bigg)\Bigg].
\end{aligned}
\tag{4.5}
$$

Firstly, we present an important lemma that will be instrumental in the proofs of Ulam-Hyers stability and generalized Ulam-Hyers stability.

Lemma 4.1 *Let $\kappa \in C^1(\Pi, \mathbb{R})$ be a function that satisfies the inequality (4.1). Then, κ also satisfies the following inequality:*

$$|\kappa(x) - (\mathcal{J}\kappa)(x)| \leq \Delta_1\epsilon, \quad \epsilon \in (0, 1], \tag{4.6}$$

where Δ_1 is the constant defined in (3.3).

Proof:

From Remark 4.1 and equation (4.5), we obtain

$$
\begin{aligned}
|\kappa(x) - (\mathcal{J}\kappa)(x)| = & \Bigg| \ {}_{a,k}I^{\alpha+\beta,\rho;\psi}\mathcal{D}(x) \\
& + \frac{{}_k^\rho\Psi_\psi^0(x,a)}{\Upsilon}\Bigg[\sum_{i=1}^{m}\sigma_i \ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}\mathcal{D}(\zeta_i) - {}_{a,k}I^{\alpha+\beta,\rho;\psi}\mathcal{D}(b)\Bigg]\Bigg| \\
\leq & \ {}_{a,k}I^{\alpha+\beta,\rho;\psi}|\mathcal{D}(x)| \\
& + \frac{{}_k^\rho\Psi_\psi^0(b,a)}{|\Upsilon|}\Bigg[\sum_{i=1}^{m}|\sigma_i| \ {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi}|\mathcal{D}(\zeta_i)| + {}_{a,k}I^{\alpha+\beta,\rho;\psi}|\mathcal{D}(b)|\Bigg] \\
\leq & \ \epsilon\Bigg[\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \frac{{}_k^\rho\Psi_\psi^0(b,a)}{|\Upsilon|} \\
& \times\Bigg(\frac{(\psi(b)-\psi(a))^{\frac{\alpha+\beta}{k}}}{\rho^{\frac{\alpha+\beta}{k}}\Gamma_k(\alpha+\beta+k)} + \sum_{i=1}^{m}|\sigma_i|\frac{(\psi(\zeta_i)-\psi(a))^{\frac{\alpha+\beta+\phi_i}{k}}}{\rho^{\frac{\alpha+\beta+\phi_i}{k}}\Gamma_k(\alpha+\beta+\phi_i+k)}\Bigg)\Bigg] \\
\leq & \ \Delta_1\epsilon.
\end{aligned}
$$

□

Now, we present the results concerning the Ulam-Hyers stability and the generalized Ulam-Hyers stability of the proposed problem.

Theorem 4.1 *Assume that the conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied. If $2\mathfrak{L}_1\Delta_1 + \mathfrak{L}_2\Delta_2 < 1$, then problem (1.1) is both Ulam-Hyers stable and generalized Ulam-Hyers stable on the interval Π.*

Proof: Let $\kappa \in C^1(\Pi, \mathbb{R})$ be a solution of the inequality (4.1), and let υ be the unique solution of the problem (1.1). By applying the triangle inequality, namely $|u - v| \leq |u| + |v|$, and using Lemma 4.1, we obtain

$$\begin{aligned}|\kappa(x)-\upsilon(x)| = \Bigg| & \kappa(x) - {}_{a,k}I^{\alpha+\beta,\rho;\psi} f\left(x,\upsilon(x),\upsilon(\varrho x)\right) - \lambda\, {}_{a,k}I^{\beta,\rho;\psi} g(x,\upsilon(x)) \\ & - \frac{{}^{\rho}_{k}\Psi^{0}_{\psi}(x,a)}{\Upsilon}\Bigg[\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\alpha+\beta+\phi_i,\rho;\psi} f\left(\zeta_i,\upsilon(\zeta_i),\upsilon(\varrho\zeta_i)\right) - {}_{a,k}I^{\alpha+\beta,\rho;\psi} f\left(b,\upsilon(b),\upsilon(\varrho b)\right) \\ & + \lambda\Bigg(\sum_{i=1}^{m}\sigma_i\, {}_{a,k}I^{\beta+\phi_i,\rho;\psi} g(\zeta_i,\upsilon(\zeta_i)) - {}_{a,k}I^{\beta,\rho;\psi} g(b,\upsilon(b))\Bigg)\Bigg]\Bigg| \\ & = |\kappa(x) - (\mathcal{J}\kappa)(x) + (\mathcal{J}\kappa)(x) - (\mathcal{J}\upsilon)(x)| \\ & \le |\kappa(x) - (\mathcal{J}\kappa)(x)| + |(\mathcal{J}\kappa)(x) - (\mathcal{J}\upsilon)(x)| \\ & \le \Delta_1\epsilon + (2\mathfrak{L}_1\Delta_1 + \mathfrak{L}_2\Delta_2)|\kappa(x)-\upsilon(x)|.\end{aligned}$$

This implies that

$$|\kappa(x)-\upsilon(x)| \le \frac{\Delta_1\epsilon}{1-(2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2)}.$$

Consequently, we obtain

$$|\kappa(x)-\upsilon(x)| \le \Omega\epsilon,$$

where

$$\Omega = \frac{\Delta_1}{1-(2\mathfrak{L}_1\Delta_1+\mathfrak{L}_2\Delta_2)}.$$

Hence, the problem (1.1) is Ulam-Hyers stable. Furthermore, by defining

$$\Omega_f(\epsilon) = \sqrt{\Omega}\epsilon,$$

which satisfies $\Omega_f(0)=0$, we conclude that the problem (1.1) is also generalized Ulam-Hyers stable. This completes the proof.

□

5. Example

We consider the following nonlinear (k,ψ)-Caputo proportional fractional Pantograph-type differential equation subject to nonlocal boundary conditions:

$$\begin{cases} {}^{C}_{0,2}\mathbb{D}^{\frac{1}{2},\frac{1}{2};x^2+x}\left[{}^{C}_{0,2}\mathbb{D}^{\frac{1}{3},\frac{1}{2};x^2+x}\upsilon(x) - \frac{1}{55}\left(\frac{\sin(x)|\upsilon(x)|}{30(x^2+1)(1+|\upsilon(x)|)}\right)\right] = \frac{|\upsilon(x)|+|\upsilon(\frac{x}{3})|}{20\sqrt{x^2+1}(1+|\upsilon(x)|+|\upsilon(\frac{x}{3})|)}, \\ \upsilon(1) = \frac{1}{5}\,{}_{0,2}I^{\frac{1}{3},\frac{1}{2};x^2+x}\upsilon\left(\frac{3}{2}\right), \quad x\in\Pi=[0,1]. \end{cases} \tag{5.1}$$

For this example, we choose the parameters as follows: $a=0$, $b=1$, $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $k=2$, $n=1$, $\rho=\frac{1}{2}$, $\psi(x)=x^2+x$, $\lambda=\frac{1}{55}$, $\varrho=\frac{1}{3}$, $m=1$, $\sigma_1=\frac{1}{5}$, $\phi_1=\frac{1}{3}$ and $\zeta_1=\frac{3}{2}$.
Numerical computations yield:

$$\Upsilon\approx 0.3222,\quad \Lambda\approx 1.9530,\quad \Delta_1\approx 3.6474,\quad \Delta_2\approx 0.0544\le 1.$$

The nonlinear functions involved in the problem are defined as follows:

$$g(x,\upsilon) = \frac{\sin(x)}{30(x^2+1)}\left(\frac{|\upsilon|}{1+|\upsilon|}\right),\quad f(x,\upsilon,w) = \frac{1}{20\sqrt{x^2+1}}\left(\frac{|\upsilon|+|w|}{1+|\upsilon|+|w|}\right).$$

For all $x\in\Pi$ and $\upsilon_i, w_i\in\mathbb{R}$, $i=1,2$, we have

$$|f(x,\upsilon_1,w_1)-f(x,\upsilon_2,w_2)| \le \frac{1}{20}\left(|\upsilon_1-\upsilon_2|+|w_1-w_2|\right),$$

therefore, the Lipschitz condition $(\mathcal{H}_2)$ is satisfied.

Similarly, for all $x \in \Pi$ and $\upsilon, w \in \mathbb{R}$, we have

$$|g(x,\upsilon) - g(x,w)| \leq \frac{1}{30}\,|\upsilon - w|,$$

hence, the condition $(\mathcal{H}_3)$ is satisfied.

Moreover, since

$$2\mathfrak{L}_1\Delta_1 + \mathfrak{L}_2\Delta_2 \approx 0.36655 < 1,$$

by invoking Theorem 3.1, it follows that problem (5.1) admits a unique solution.

In addition, we observe that

$$|f(x,\upsilon,w)| \leq \vartheta_f(x), \quad \text{where} \quad \vartheta_f(x) = \frac{1}{\sqrt{x^2+1}},$$

which ensures that condition $(\mathcal{H}_4)$ is satisfied.

Moreover, we find that

$$|g(x,\upsilon)| \leq \vartheta_g(x), \quad \text{where} \quad \vartheta_g(x) = \frac{1}{x^2+1},$$

so condition $(\mathcal{H}_5)$ holds.

Finally, since

$$\mathfrak{L}_2\Delta_2 \approx 0.001813 < 1,$$

all the assumptions of Theorem 3.2 are satisfied. Hence, by applying Theorem 3.2, equation (5.1) admits at least one solution on Π.

Finally, after straightforward calculations, we find that $\Omega = 5.758$. Since all the assumptions of Theorem 4.1 are met, it follows that the problem (5.1) is both Ulam-Hyers stable and generalized Ulam-Hyers stable.

6. Conclusion

This paper addresses the existence, uniqueness, and Ulam-type stability of solutions for a class of pantograph-type equations involving the (k,ψ)-Caputo fractional derivative under nonlocal integral boundary conditions. The analysis is based on Banach and Krasnoselskii fixed point theorems. A numerical example illustrates the theoretical results. As a direction for future work, we aim to perform an in-depth numerical study of the considered equation.

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