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A Computationally Efficient Method for Analyzing Fractional Schrödinger Equation using Non-Singular Kernel

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ABSTRACT: This article investigates a weakly nonlocal Schrödinger equation characterized by parabolic law nonlinearity and an external potential, utilizing the Laplace-Adomian Decomposition Method (LADM) with a non-singular kernel. The LADM combines the Laplace transform method with the Adomian decomposition method, providing both approximate and exact solutions for three different cases: bright solitons, dark solitons, and exponential solutions. We present numerical and graphical solutions for these cases, demonstrating that accurate and reliable approximations can be achieved with only a few terms. We compared the obtained solutions using the proposed technique with the q-homotopy analysis transform method to validate the accuracy of our results. The physical properties of the LADM solutions are illustrated through plots for different fractional orders, complemented by numerical results.

Key Words: Caputo derivative, Adomian polynomial, Weakly nonlocal fractional Schrödinger equation, Laplace transform.

Contents

1. Introduction

The nonlinear Schrödinger equations (NLSEq) are essential nonlinear canonical evolution equations extensively applied in numerous fields of sciences. These types of equations are widely used to describe various physical processes, including signal transmission in optical fibers [1,2], the propagation of electromagnetic waves in plasma physics [3], the behavior of rogue waves [4], and wave motion in deep ocean waters [5]. Soliton theory plays a vital role in mathematical physics, as many models admit soliton-like solutions. These waveforms have a wide range of applications across mathematics and engineering due to their stability and unique propagation properties. In recent years, significant research attention has been directed towards analyzing complex optical soliton formations in nonlinear media ([6], [7], [8], [9], [10], [11], [12], [13]).
Fractional-order partial differential equations (PDEs) have gained increasing attention for their effectiveness in modeling nonlinear systems. Solutions to nonlinear PDEs of arbitrary order are essential for analyzing the dynamics and properties of complex phenomena in applied mathematics and various technological domains. However, obtaining exact analytical solutions for such equations often presents significant difficulties. Integral transforms are among the most powerful tools used to address integral and delay differential equations. These methods are highly adaptable and have proven valuable in both theoretical studies and practical applications. Their utility spans a wide array of disciplines, including

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biology ([14], [15]), electrodynamics [16], fluid dynamics ([17], [18]), mechanics ([19], [20]), nanotechnology [21], biotechnology [22], chaos theory [23], and many other fields.
Over the past decades, various fractional operators have been introduced to deepen the understanding of model dynamics. These include the Riemann-Liouville (RL), Hadamard ([24], [25]), Caputo ([26], [27], [28], [29]), generalized Caputo derivative ([30]), Caputo-Fabrizio (CF) ([31], [32], [33]), and Atangana-Baleanu ([34]) operators. Among these, Atangana and Baleanu significantly advanced fractional calculus by developing operators based on the Mittag-Leffler function to address problems involving fractional integrals and derivatives. Its ability to account for full memory effects in systems enables precise modeling of complex phenomena, leveraging its nonlocal properties. This innovative approach is particularly valuable in nonlinear models, where it enhances the accurate satisfaction of initial conditions, yielding more realistic representations of real-world problems. As such, the Atangana-Baleanu derivative is expected to play a crucial role in the future development of mathematical modeling across diverse applications.
The time-fractional weakly NLSEq ([35], [36]) by involving ABC derivative is given as

$$i{}_0^{AB}D_t^{\zeta}\Phi+\mu_1\frac{\partial^2\Phi}{\partial\kappa^2}+\mu_2\Phi\frac{\partial^2|\Phi|}{\partial\kappa^2}+\left(\mu_3|\Phi|^2+\mu_4|\Phi|^4\right)\Phi+\mu_5\Phi=0,\quad 0\prec\zeta\leq 1, \tag{1.1}$$

Here, $i=\sqrt{-1}$, and μ_i $(1\leq j\leq 5)$ are real constants. In particular, μ_1 and μ_2 denote the coefficients corresponding to group velocity dispersion and weakly nonlocal nonlinearity, respectively. Meanwhile, μ_3 and μ_4 are associated with the parabolic law nonlinearity, and μ_5 corresponds to an external potential. Additionally, $\Phi=\Phi(\kappa,t)$ is a complex function.
This study focuses on analyzing the propagation of optical solitons in media governed by the combined effects of weakly nonlocal and parabolic law nonlinearities. To analyze the underlying dynamics, we utilize the Laplace transform decomposition method (LTDM) to derive both numerical and graphical solutions for various applications of equation (1.1). To address the challenges associated with computational complexity, the LTDM is introduced as a hybrid technique that integrates the Laplace transform with the Adomian decomposition method (ADM). This method is particularly suited for solving the time-fractional weakly nonlinear Schrödinger equation (NLSE), which constitutes the motivation of this research. The proposed approach yields a rapidly convergent series solution, offering both high accuracy and computational efficiency.

2. Preliminaries

This section provides key definitions, which are essential for understanding the subsequent results.

Definition 2.1 *The Caputo fractional derivative of $\Phi\in C_{-1}^{m}$ is defined as [37]*

$$^{C}D_t^{\zeta}\Phi(\kappa,t)=\begin{cases}\frac{\partial^{\gamma}\Phi(\kappa,t)}{\partial t^{\gamma}}, & \zeta=\rho\in\aleph,\\ \frac{1}{\Gamma(\gamma-1)}\int\limits_0^t(t-\rho)^{\gamma-1-\zeta}\frac{\partial^{\gamma}\Phi(\kappa,\rho)}{\partial\rho^{\rho}}d\rho, & \gamma-1<\zeta\leq\gamma.\end{cases} \tag{2.1}$$

Definition 2.2 *The Atangana-Baleanu-Caputo fractional derivative (ABC) [34] is defined as*

$${}_0^{AB}D_t^{\zeta}v(\kappa,t)=\frac{M(\zeta)}{1-\zeta}\int\limits_0^t v'(\kappa,\upsilon)E_{\zeta}\left(\frac{(t-\upsilon)^{\zeta}\zeta}{1-\zeta}\right)d\upsilon,\ m-1<\zeta\leq m, \tag{2.2}$$

where $\zeta\in\Re$, and $M(\zeta)>0$ is a normalization function satisfying $M(0)=1$ and $M(1)=1$.

The Laplace Transform (LT) of the ABC derivative operator is given by [34]:

$$L\left[{}_0^{AB}D_t^{\zeta}v(\kappa,t)\right]=\frac{M(\zeta)}{1-\zeta}\frac{\upsilon^{\zeta}L[v(\kappa,t)]-\upsilon^{\zeta-1}v(0)}{\upsilon^{\zeta}+\left(\frac{\zeta}{1-\zeta}\right)}. \tag{2.3}$$

3. Laplace transform decomposition method

Consider a nonlinear fractional partial differential equation as

$$^{AB}D_t^{\zeta}\Phi(\kappa,t) = \Re\Phi(\kappa,t) + \aleph\Phi(\kappa,t) + P(\kappa,t), \ m-1<\zeta\le m, \tag{3.1}$$

with initial condition

$$\Phi(\kappa,0) = \phi(\kappa), \tag{3.2}$$

Here, $^{AB}D_t^{\zeta} = \frac{\partial^{\zeta}}{\partial t^{\zeta}}$ denotes the Atangana–Baleanu fractional derivative of order ζ, $\Re$ is a linear function dependent on (κ,t), $\aleph$ represents the nonlinear function, and P stands for the source term.
Applying the LT to both sides of equation (3.1), we obtain

$$L\left[^{AB}D_t^{\zeta}\Phi(\kappa,t)\right] = L\left[\Re\Phi(\kappa,t)\right] + L\left[\aleph\Phi(\kappa,t)\right] + L\left[P(\kappa,t)\right]. \tag{3.3}$$

Utilizing the differentiation property of the LT, we obtain

$$L\left[\Phi(\kappa,t)\right] = \tfrac{1}{\upsilon}\left[\Phi(\kappa,0)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[P(\kappa,t)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\Re\Phi(\kappa,t)+\aleph\Phi(\kappa,t)\right]. \tag{3.4}$$

Now, applying the inverse LT to both sides of the equation, we get (3.4), we get

$$L\left[\Phi(\kappa,t)\right] = \tfrac{1}{\upsilon}\left[\Phi(\kappa,0)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[P(\kappa,t)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\Re\Phi(\kappa,t)+\aleph\Phi(\kappa,t)\right]. \tag{3.5}$$

Next, we consider $\Phi(\kappa,t)$ has infinite series solution as

$$\Phi(\kappa,t) = \sum_{\tau=0}^{\infty}\Phi_{\tau}(\kappa,t), \tag{3.6}$$

and the nonlinear term $\aleph\Phi(\kappa,t)$ is expressed as

$$\aleph\Phi(\kappa,t) = \sum_{\tau=0}^{\infty}A_{\tau} \quad , \tag{3.7}$$

where A_{τ} is the Adomian polynomial, given by

$$A_{\tau} = \frac{1}{\Gamma(\tau+1)}\left[\frac{d^{\tau}}{d\varpi^{\tau}}\left\{\aleph\left(\sum_{\iota=0}^{\infty}\varpi^{\iota}\kappa_{\iota}, \sum_{\iota=0}^{\infty}\varpi^{\iota}t_{\iota}\right)\right\}\right]_{\varpi=0}. \tag{3.8}$$

Using equations (3.6) and (3.7) in equation (3.5), we get

$$\sum_{\tau=0}^{\infty}\Phi_{\tau}(\kappa,t) = L^{-1}\left[\tfrac{1}{\upsilon}\left[\upsilon(\kappa,0)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[P(\kappa,t)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\Re\Phi\left(\sum_{\tau=0}^{\infty}\kappa_{\tau}, \sum_{\tau=0}^{\infty}t_{\tau},\right) + \sum_{\tau=0}^{\infty}A_{\tau}\right]\right]. \tag{3.9}$$

From equation (3.9), we get

$$\begin{aligned}
\Phi_0(\kappa,t) &= L^{-1}\left[\tfrac{1}{\upsilon}\left[\upsilon(\kappa,0)\right] + \tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[P(\kappa,t)\right]\right],\\
\Phi_1(\kappa,t) &= L^{-1}\left[\tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\Re\Phi(\kappa_0,t_0)+A_0\right]\right],\\
\Phi_{\tau+1}(\kappa,t) &= L^{-1}\left[\tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\Re\Phi(\kappa_{\tau},t_{\tau})+A_{\tau}\right]\right], \ \tau\ge 1.
\end{aligned} \tag{3.10}$$

4. Solution of proposed problem using LADM

Consider the the Atangana–Baleanu fractional weakly nonlinear Schrödinger equation (NLSE) in the following form:

$$^{AB}D_t^{\zeta}\Phi(\kappa,t) = i\mu_1\frac{\partial^2\Phi}{\partial\kappa^2} + i\mu_2\Phi\frac{\partial^2|\Phi|}{\partial\kappa^2} + i\left(\mu_3|\Phi|^2+\mu_4|\Phi|^4\right)\Phi + i\mu_5\Phi, \quad 0\prec\zeta\le 1, \tag{4.1}$$

with IC:

$$\Phi(\kappa,0)=\phi(\kappa). \tag{4.2}$$

By applying the LT to both sides of equation (4.1) and simplifying, we obtain:

$$L\left[\Phi(\kappa,t)\right]=\tfrac{1}{\upsilon}\left[\Phi(\kappa,0)\right]+i\tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\mu_1\tfrac{\partial^2\Phi}{\partial\kappa^2}\ +\mu_2\Phi\tfrac{\partial^2|\Phi|}{\partial\kappa^2}+\left(\mu_3|\Phi|^2+\mu_4|\Phi|^4\right)\Phi+\mu_5\Phi\right]. \tag{4.3}$$

Applying the inverse LT to both sides of equation (4.3), we obtain:

$$\Phi(\kappa,t)=\phi(\kappa)+iL^{-1}\left[\tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\mu_1\tfrac{\partial^2\Phi}{\partial\kappa^2}\ +\mu_2\Phi\tfrac{\partial^2|\Phi|}{\partial\kappa^2}+\left(\mu_3|\Phi|^2+\mu_4|\Phi|^4\right)\Phi+\mu_5\Phi\right]\right]. \tag{4.4}$$

Let, the function $\Phi(\kappa,t)$ has infinite series solution as

$$\Phi(\kappa,t)=\sum_{\tau=0}^{\infty}\Phi_{\tau}(\kappa,t), \tag{4.5}$$

and nonlinear term $N\Phi(\kappa,t)=\mu_2\left(\Phi^2\left(\overline{\Phi}\right)_{\kappa\kappa}+\Phi\overline{\Phi}(\Phi)_{\kappa\kappa}+2\Phi\left(\overline{\Phi}\right)_{\kappa}(\Phi)_{\kappa}\right)+\mu_3\Phi^2\overline{\Phi}+\mu_4\Phi^3\overline{\Phi}^{\,2}$, so the adomain polynomial is calculated as:

$$N\Phi(\kappa,t)=\sum_{\tau=0}^{\infty}A_{\tau}(\Phi_0,\Phi_1,........,\Phi_{\tau}), \tag{4.6}$$

here,

$$\begin{aligned}
A_0 &= \mu_2{\Phi_0}^2\left(\overline{\Phi}_0\right)_{\kappa\kappa}+\mu_2\Phi_0\overline{\Phi}_0(\Phi_0)_{\kappa\kappa}+2\mu_2\Phi_0(\Phi_0)_{\kappa}\left(\overline{\Phi}_0\right)_{\kappa}+\mu_3{\Phi_0}^2\overline{\Phi}_0+\mu_4{\Phi_0}^3{\overline{\Phi}_0}^{\,2},\\
A_1 &= 2\mu_2\Phi_0\Phi_1\left(\overline{\Phi}_0\right)_{\kappa\kappa}+\mu_2{\Phi_0}^2\left(\overline{\Phi}_1\right)_{\kappa\kappa}+\mu_2\Phi_1\overline{\Phi}_0(\Phi_0)_{\kappa\kappa}+\mu_2\Phi_0\overline{\Phi}_1(\Phi_0)_{\kappa\kappa}\\
&\quad+\mu_2\Phi_0\overline{\Phi}_0(\Phi_1)_{\kappa\kappa}+2\mu_2\Phi_1(\Phi_0)_{\kappa}\left(\overline{\Phi}_0\right)_{\kappa}+2\mu_2\Phi_0(\Phi_1)_{\kappa}\left(\overline{\Phi}_0\right)_{\kappa}+2\mu_2\Phi_0(\Phi_0)_{\kappa}\left(\overline{\Phi}_1\right)_{\kappa}\\
&\quad+2\mu_3\Phi_0\Phi_1\overline{\Phi}_0+\mu_3{\Phi_0}^2\overline{\Phi}_1+3\mu_4{\Phi_0}^2\Phi_1{\overline{\Phi}_0}^{\,2}+2\mu_4{\Phi_0}^3\overline{\Phi}_0\overline{\Phi}_1,\\
A_2 &= 2\mu_2\Phi_0\Phi_2\left(\overline{\Phi}_0\right)_{\kappa\kappa}+\mu_2{\Phi_1}^2\left(\overline{\Phi}_0\right)_{\kappa\kappa}+2\mu_2\Phi_0\Phi_1\left(\overline{\Phi}_1\right)_{\kappa\kappa}+\mu_2{\Phi_0}^2\left(\overline{\Phi}_2\right)_{\kappa\kappa}+\mu_2\Phi_2\overline{\Phi}_0(\Phi_0)_{\kappa\kappa}\\
&\quad+\mu_2\Phi_0\overline{\Phi}_2(\Phi_0)_{\kappa\kappa}+\mu_2\Phi_0\overline{\Phi}_0(\Phi_2)_{\kappa\kappa}+\mu_2\Phi_1\overline{\Phi}_1(\Phi_0)_{\kappa\kappa}+\mu_2\Phi_1\overline{\Phi}_0(\Phi_1)_{\kappa\kappa}+\mu_2\Phi_0\overline{\Phi}_1(\Phi_1)_{\kappa\kappa}\\
&\quad+2\mu_2\Phi_2(\Phi_0)_{\kappa}\left(\overline{\Phi}_0\right)_{\kappa}+2\mu_2\Phi_1(\Phi_1)_{\kappa}\left(\overline{\Phi}_0\right)_{\kappa}+2\mu_2\Phi_1(\Phi_0)_{\kappa}\left(\overline{\Phi}_1\right)_{\kappa}+2\mu_2\Phi_0(\Phi_2)_{\kappa}\left(\overline{\Phi}_0\right)_{\kappa}\\
&\quad+2\mu_2\Phi_0(\Phi_1)_{\kappa}\left(\overline{\Phi}_1\right)_{\kappa}+2\mu_2\Phi_0(\Phi_0)_{\kappa}\left(\overline{\Phi}_2\right)_{\kappa}\\
&\quad+2\mu_3\Phi_0\Phi_2\overline{\Phi}_0+\mu_3{\Phi_1}^2\overline{\Phi}_0+2\mu_3\Phi_1\Phi_0\overline{\Phi}_1+\mu_3{\Phi_0}^2\overline{\Phi}_2\\
&\quad+3\mu_4{\Phi_0}^2\Phi_2{\overline{\Phi}_0}^{\,2}+6\mu_4{\Phi_0}^2\overline{\Phi}_0\overline{\Phi}_1\Phi_1+3\mu_4{\Phi_1}^2\Phi_0{\overline{\Phi}_0}^{\,2}+2\mu_4{\Phi_0}^3\overline{\Phi}_0\overline{\Phi}_2+\mu_4{\Phi_0}^3{\overline{\Phi}_1}^{\,2},
\end{aligned}$$

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Furthermore, equation (4.4) can be rewritten as:

$$\sum_{\tau=0}^{\infty}\Phi_{\tau}(\kappa,t)=\Phi(\kappa)+L^{-1}\left[\frac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\mu_1\sum_{\tau=0}^{\infty}\frac{\partial^2\Phi_{\tau}}{\partial\kappa^2}+\mu_5\sum_{\tau=0}^{\infty}\Phi_{\tau}+\sum_{\tau=0}^{\infty}A_{\tau}\right]\right]. \tag{4.7}$$

Finally, We have recurrence relations as:

$$\begin{aligned}
&\Phi_0(\kappa,t)=\Phi(\kappa),\\
&\Phi_1(\kappa,t)=L^{-1}\left[\tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\mu_1\tfrac{\partial^2\Phi_0}{\partial\kappa^2}+\mu_5\Phi_0+A_0\right]\right],\\
&\Phi_{\tau+1}(\kappa,t)=L^{-1}\left[\tfrac{\upsilon^{\zeta}(1-\zeta)+\zeta}{\upsilon^{\zeta}M(\zeta)}L\left[\mu_1\tfrac{\partial^2\Phi_{\tau}}{\partial\kappa^2}+\mu_5\Phi_{\tau}+A_{\tau}\right]\right],\ \tau\geq 1.
\end{aligned} \tag{4.8}$$

Example 1: [35] Consider IC as:

$$\Phi(\kappa,0)=-q\sqrt{\frac{6\mu_2}{\mu_4}}\sec h(q\kappa), \tag{4.9}$$

and

$$\mu_3 = \frac{\mu_1\mu_4 - 12q^2{\mu_2}^2}{3\mu_2}. \tag{4.10}$$

By applying equations (4.8), (4.9), and (4.10) for $\tau = 1, 2, 3, \ldots$, we obtain the following result:

$\Phi_0(\kappa, t) = -q\sqrt{\frac{6\mu_2}{\mu_4}}\sec h(q\kappa)$,

$\Phi_1(\kappa, t) = -iq\sqrt{\frac{6\mu_2}{\mu_4}}\left(\mu_5 + \mu_1 n^2\right)\sec h(q\kappa)\frac{1}{M(\zeta)}\left(1 - \zeta + \zeta\left(\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right)$,

$\Phi_2(\kappa, t) = q\sqrt{\frac{6\mu_2}{\mu_4}}\left(\mu_5 + \mu_1 n^2\right)^2\sec h(q\kappa)\frac{1}{[M(\zeta)]^2}\left(1 + 2\zeta\left(1 + \frac{t^\zeta}{\Gamma(\zeta+1)}\right) + \zeta^2\left(1 + \frac{t^{2\zeta}}{\Gamma(2\zeta+1)} - 2\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right)$,

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If we let $\zeta = 1$, then the solution of (4.1) is given as

$\Phi(\kappa, t) = -q\sqrt{\frac{6\mu_2}{\mu_4}}\sec h(q\kappa)\left(1 + i\Upsilon t + \frac{1}{2!}(i\Upsilon)^2 t^2 + \frac{1}{3!}(i\Upsilon)^3 t^3 + ...\right)$,

where, $\Upsilon = \mu_5 + n^2\mu_1$, and $\frac{\mu_2}{\mu_4} > 0$.

Finally, the exact solution (bright soliton) of equation (4.1) is given as

$$\Phi(\kappa, t) = -q\sqrt{\frac{6\mu_2}{\mu_4}}\sec h(q\kappa)e^{i\Upsilon t}. \tag{4.11}$$

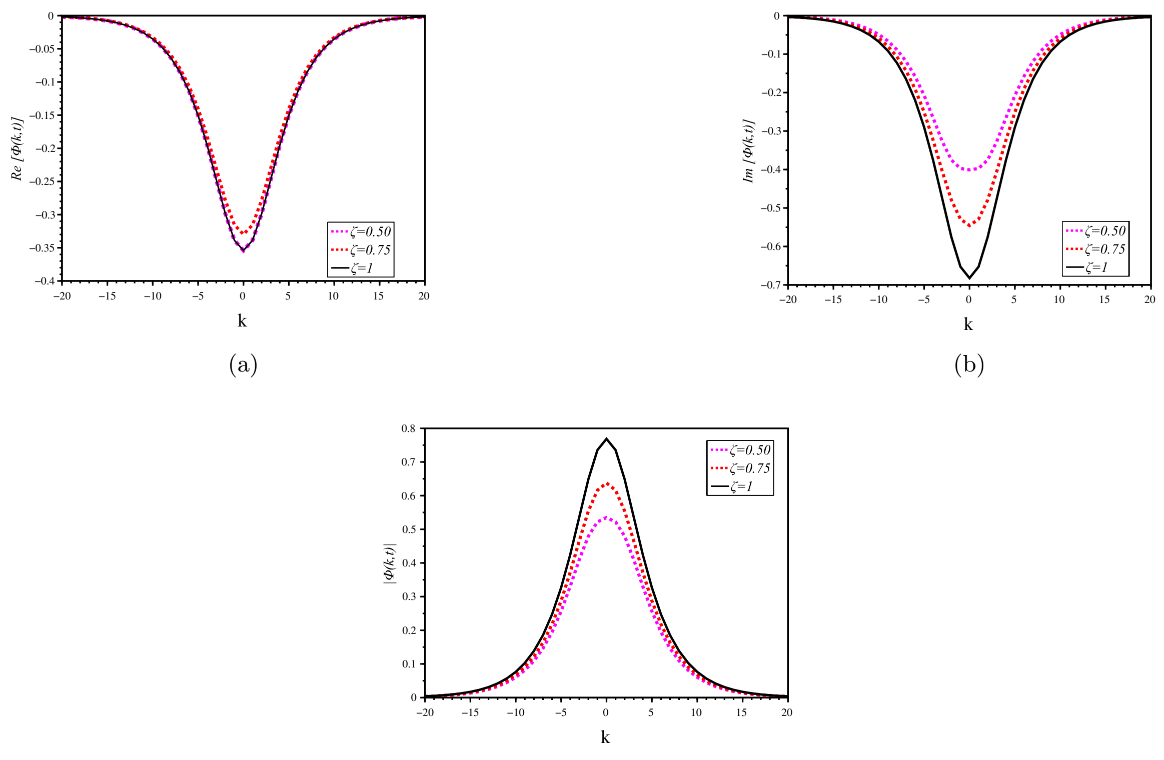

Figure 1: 2D plots for different values of ζ with parameters $q = 0.3$, $t = 2$, $\mu_1 = 0.1$, $\mu_2 = \mu_4 = 1$, and $\mu_5 = 0.5$ for Example 1.

Example 2: [35] Consider IC as:

$$\Phi(\kappa, 0) = -q\sqrt{-\frac{6\mu_2}{\mu_4}}\tan h(q\kappa), \tag{4.12}$$

Table 1: Comparison analysis with $q-HATM$ [36] at
$q=0.3,\ t=0.5,\ \mu_1=0.1,\ \mu_2=\mu_4=1,\ \mu_5=0.5$ and $\zeta=1$ for example 1.

	LTDM			$q-HATM$ [36]		
κ	$\mathrm{Re}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Im}\left[\Phi_{(3)}\right]^{Abs}$	$\lvert\Phi_{(3)}\rvert^{Abs}$	$\mathrm{Re}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Im}\left[\Phi_{(3)}\right]^{Abs}$	$\lvert\Phi_{(3)}\rvert^{Abs}$
−20	6.3541×10^{-07}	3.2362×10^{-08}	6.2309×10^{-07}	6.3541×10^{-07}	3.2362×10^{-08}	6.2309×10^{-07}
−15	2.8474×10^{-06}	1.4502×10^{-07}	2.7922×10^{-06}	2.8474×10^{-06}	1.4502×10^{-07}	2.7922×10^{-06}
−10	1.2731×10^{-05}	6.4842×10^{-07}	1.2484×10^{-05}	1.2731×10^{-05}	6.4842×10^{-07}	1.2484×10^{-05}
−5	5.4486×10^{-05}	2.7750×10^{-06}	5.3429×10^{-05}	5.4486×10^{-05}	2.7750×10^{-06}	5.3429×10^{-05}
0	1.2817×10^{-04}	6.5280×10^{-06}	1.2568×10^{-04}	1.2817×10^{-04}	6.5280×10^{-06}	1.2568×10^{-04}
5	5.4486×10^{-05}	2.7750×10^{-06}	5.3429×10^{-05}	5.4486×10^{-05}	2.7750×10^{-06}	5.3429×10^{-05}
10	1.2731×10^{-05}	6.4842×10^{-07}	1.2484×10^{-05}	1.2731×10^{-05}	6.4842×10^{-07}	1.2484×10^{-05}
15	2.8474×10^{-06}	1.4502×10^{-07}	2.7922×10^{-06}	2.8474×10^{-06}	1.4502×10^{-07}	2.7922×10^{-06}
20	6.3541×10^{-07}	3.2362×10^{-08}	6.2309×10^{-07}	6.3541×10^{-07}	3.2362×10^{-08}	6.2309×10^{-07}

and

$$\mu_3=\frac{\mu_1\mu_4+24q^2{\mu_2}^2}{3\mu_2}. \tag{4.13}$$

By applying equations (4.8), (4.12) and (4.13), for $\tau=1,2,3,\ldots$, we obtain the following result:

$$\Phi_0(\kappa,t)=-q\sqrt{-\tfrac{6\mu_2}{\mu_4}}\tan h(q\kappa),$$

$$\Phi_1(\kappa,t)=-i\frac{\sqrt{6}q\left(-\frac{\mu_2}{\mu_4}\right)^{\frac{3}{2}}\left(12n^4{\mu_2}^2+\mu_4\left(2n^2\mu_1-\mu_5\right)\right)}{\mu_2}\tan h(q\kappa)\frac{1}{M(\zeta)}\left(1-\zeta+\zeta\left(\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right),$$

$$\Phi_2(\kappa,t)=\frac{\sqrt{6}q\left(-\frac{\mu_2}{\mu_4}\right)^{\frac{1}{2}}\left(12n^4{\mu_2}^2+\mu_4\left(2n^2\mu_1-\mu_5\right)\right)^2}{{\mu_4}^2}\tan h(q\kappa)\frac{1}{[M(\zeta)]^2}\left(1+2\zeta\left(1+\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right.$$
$$\left.+\zeta^2\left(1+\frac{t^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right),$$

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If we let $\zeta=1$, then the solution of (4.1) is given as

$$\Phi(\kappa,t)=-n\sqrt{-\tfrac{6\mu_2}{\mu_4}}\tan h(n\kappa)\left(1+i\Theta t+\tfrac{1}{2!}(i\Theta)^2t^2+\tfrac{1}{3!}(i\Theta)^3t^3+.....\right),$$

where, $\Theta=\left(12n^4{\mu_2}^2+\mu_4\left(2n^2\mu_1-\mu_5\right)\right)$, and $\frac{\mu_2}{\mu_4}<0$.

Finally, the exact solution (dark soliton) of equation (4.1) is given as

$$\Phi(\kappa,t)=-n\sqrt{-\frac{6\mu_2}{\mu_4}}\tan h(n\kappa)e^{i\Theta t}. \tag{4.14}$$

Example 3: Consider IC as:

$$\Phi(\kappa,0)=\eta e^{iq\kappa}. \tag{4.15}$$

By applying equations (4.8) and (4.15), for $\tau=1,2,3,\ldots$, we obtain the following result:

$$\Phi_0(\kappa,t)=\eta e^{iq\kappa},$$

$$\Phi_1(\kappa,t)=i\eta\left(\mu_4\eta^4+\mu_3\eta^2+\mu_5-\mu_1q^2\right)e^{iq\kappa}\frac{1}{M(\zeta)}\left(1-\zeta+\zeta\left(\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right),$$

$$\Phi_2(\kappa,t)=-\eta\left(\mu_4\eta^4+\mu_3\eta^2+\mu_5-\mu_1q^2\right)^2e^{iq\kappa}\frac{1}{[M(\zeta)]^2}\left(1+2\zeta\left(1+\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right.$$
$$\left.+\zeta^2\left(1+\frac{t^{2\zeta}}{\Gamma(2\zeta+1)}-2\frac{t^\zeta}{\Gamma(\zeta+1)}\right)\right),$$

.
.

If we let $\zeta=1$, then the solution of (4.1) is given as

$$\Phi(\kappa,t)=\Xi e^{iqx}\left(1+i\eta t+\tfrac{1}{2!}(i\eta)^2t^2+\tfrac{1}{3!}(i\eta)^3t^3+....\right),$$

where, $\Xi=\mu_4\eta^4+\mu_3\eta^2+\mu_5-\mu_1n^2$

Table 2: Comparison analysis with $q-HATM$ [36] at $q=0.2$, $t=0.5$, $\mu_2=-1$, $\mu_4=1$, $\mu_5=0.5$ and $\zeta=1$ for example 2.

	$LTDM$			$q-HATM$ [36]		
κ	$\mathrm{Re}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Im}\left[\Phi_{(3)}\right]^{Abs}$	$\lvert\Phi_{(3)}\rvert^{Abs}$	$\mathrm{Re}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Im}\left[\Phi_{(3)}\right]^{Abs}$	$\lvert\Phi_{(3)}\rvert^{Abs}$
-20	6.3589×10^{-05}	3.0081×10^{-06}	6.2525×10^{-05}	6.3589×10^{-05}	3.0081×10^{-06}	6.2525×10^{-05}
-15	6.3317×10^{-05}	2.9952×10^{-06}	6.2257×10^{-05}	6.3317×10^{-05}	2.9952×10^{-06}	6.2257×10^{-05}
-10	6.1343×10^{-05}	2.9018×10^{-06}	6.0316×10^{-05}	6.1343×10^{-05}	2.9018×10^{-06}	6.0316×10^{-05}
-5	4.8461×10^{-05}	2.2924×10^{-06}	4.7650×10^{-05}	4.8461×10^{-05}	2.2924×10^{-06}	4.7650×10^{-05}
0	0	0	0	0	0	0
5	4.8461×10^{-05}	2.2924×10^{-06}	4.7650×10^{-05}	4.8461×10^{-05}	2.2924×10^{-06}	4.7650×10^{-05}
10	6.1343×10^{-05}	2.9018×10^{-06}	6.0316×10^{-05}	6.1343×10^{-05}	2.9018×10^{-06}	6.0316×10^{-05}
15	6.3317×10^{-05}	2.9952×10^{-06}	6.2257×10^{-05}	6.3317×10^{-05}	2.9952×10^{-06}	6.2257×10^{-05}
20	6.3589×10^{-05}	3.0081×10^{-06}	6.2525×10^{-05}	6.3589×10^{-05}	3.0081×10^{-06}	6.2525×10^{-05}

Table 3: Comparison analysis with $q-HATM$ [36] at
$q=0.1$, $t=0.5$, $\eta=0.5$, $\mu_1=0.1$, $\mu_3=-1$, $\mu_4=1$, $\mu_5=0.5$ and $\zeta=1$ for example 3.

	$LTDM$		$q-HATM$ [36]	
κ	$\mathrm{Re}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Im}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Re}\left[\Phi_{(3)}\right]^{Abs}$	$\mathrm{Im}\left[\Phi_{(3)}\right]^{Abs}$
-20	2.8697×10^{-07}	1.1297×10^{-05}	2.8697×10^{-07}	1.1297×10^{-05}
-15	3.1274×10^{-07}	1.2192×10^{-05}	3.1274×10^{-07}	1.2192×10^{-05}
-10	2.6194×10^{-07}	1.0101×10^{-05}	2.6194×10^{-07}	1.0101×10^{-05}
-5	1.4701×10^{-07}	5.5378×10^{-06}	1.4701×10^{-07}	5.5378×10^{-06}
0	3.9101×10^{-09}	3.8166×10^{-07}	3.9101×10^{-09}	3.8166×10^{-07}
5	1.5388×10^{-07}	6.2077×10^{-06}	1.5388×10^{-07}	6.2077×10^{-06}
10	2.6617×10^{-07}	1.0514×10^{-05}	2.6617×10^{-07}	1.0514×10^{-05}
15	3.1329×10^{-07}	1.2246×10^{-05}	3.1329×10^{-07}	1.2246×10^{-05}
20	2.8371×10^{-07}	1.0980×10^{-05}	2.8371×10^{-07}	1.0980×10^{-05}

Finally, the exact solution (exponential soliton) of equation (4.1) is given as

$$\Phi\left(\kappa,t\right)=\eta e^{i(q\kappa+\eta t)}. \tag{4.16}$$

5. Results and Discussions

In the present study, we analyze Tables I–III and observe that the numerical results obtained using the proposed technique closely align with those derived from the q-homotopy analysis transform method (q-HATM) [36]. The essence of the achieved outcomes for the weakly nonlinear Schrödinger equation (NLSE) with parabolic law nonlinearity and external potential is illustrated through 2D plots. The dynamic behavior of different fractional orders is demonstrated using the 2D graphs. To validate the reliability of the proposed technique, numerical simulations were conducted using MATLAB. These simulations confirm the effectiveness of the method by comparing the obtained and exact solutions at $\zeta=1$ with the q-HATM results at $t=0.5$. The solutions are shown to be both accurate and efficient, as evident from the comparison of $\Phi_{(3)}$ solutions obtained via the Laplace transform decomposition method (LTDM) with their corresponding exact solutions. The response of the results achieved through LTDM is presented in terms of real and imaginary components, as well as the absolute behavior, depicted through 2D plots in Figures 1–3.

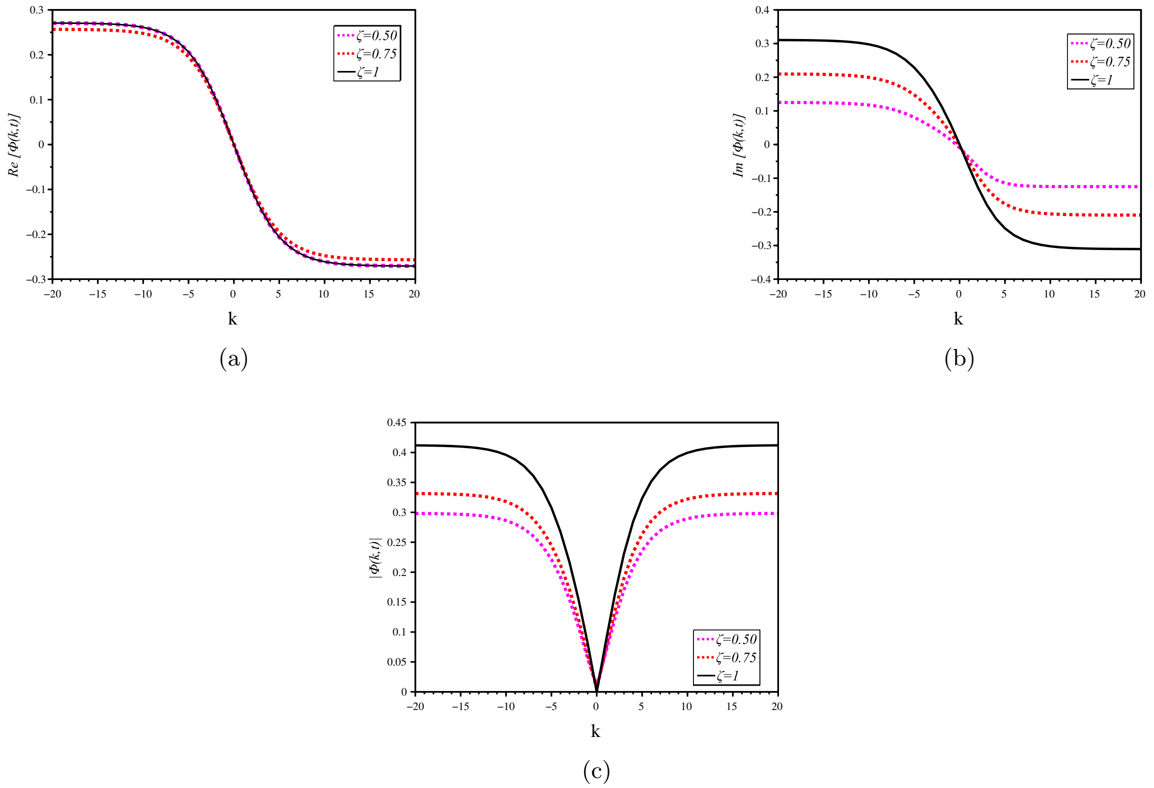

Figure 2: 2D plots for different values of ζ with parameters $q = 0.2$, $t = 2$, $\mu_1 = 0.1$, $\mu_2 = -1$, $\mu_4 = 1$, and $\mu_5 = 0.5$ for Example 2.

6. Conclusion

In this work, the fractional weakly nonlinear Schrödinger equation is analyzed using the Laplace transform decomposition method (LTDM), which facilitates the construction of both analytical and numerical solutions. To visualize the effects of fractional dynamics, two-dimensional plots are presented for varying values of the fractional order ζ. Furthermore, the numerical results demonstrate excellent agreement with the exact solutions, providing valuable insights and significant findings for the proposed problem. These results enhance our understanding of dynamic and nonlinear models and offer a deeper comprehension of the implications of nonlinear Schrödinger equations. The findings confirm that the proposed method is highly effective and efficient in deriving approximate solutions to nonlinear problems, with broad applicability in various scientific and engineering fields.

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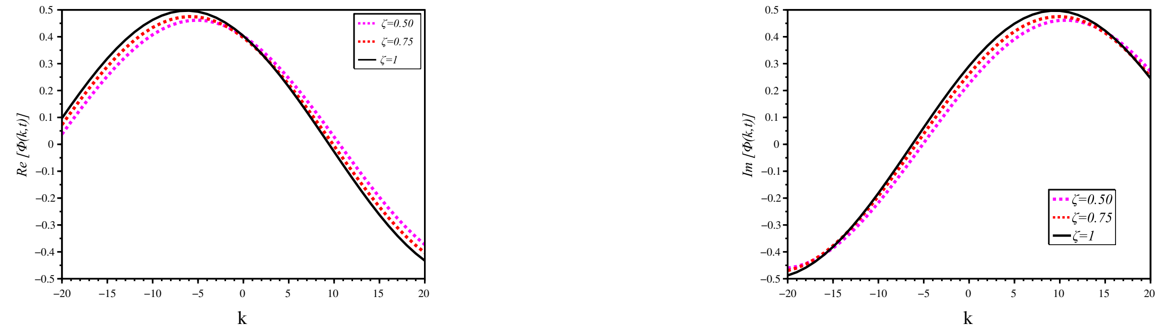

(a) (b)

Figure 3: 2D plots for different values of ζ with parameters $q = 0.1$, $t = 2$, $\eta = 0.5$, $\mu_1 = 0.1$, $\mu_3 = -1$, $\mu_4 = 1$, and $\mu_5 = 0.5$ for example 3.

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