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Quotient space Henstock-Kurzweil integration on time scales

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ABSTRACT: We introduce Henstock-Kurzweil integrability for functions whose values lies in the quotient space on time scales. We define the Henstock-Kurzweil integral with respect to the Δ-derivative and ∇-derivative namely the quotient Henstock-Kurzweil Δ-integral and quotient Henstock-Kurzweil ∇-integral respectively. Result establishing the criterion of integrability is observed, and few properties of the integrals are formulated. Relations between quotient Henstock-Kurzweil integral and quotient Riemann integral, and quotient Henstock-Kurzweil integral and Banach Henstock-Kurzweil integral are also presented.

In addition, as a linear combination of the Δ- and ∇-integrals we introduce the quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integral and conclude with a theorem depicting the relation between the three integrals.

Key Words: Banach space, quotient space, Henstock-Kurzweil integral, Δ-integral, ∇-integral, time scales.

Contents

1. Introduction and preliminaries

The objective of this paper is to introduce the Henstock-Kurzweil integration for functions whose values lie in the quotient space in the context of time scale calculus.

In this section, we present a survey of the existing literature as preliminary, forming the foundation on which we define the integral. We briefly explore the theory of time scale calculus followed by the theory of classical quotient space.

S. Hilger, in 1988, as part of his Würzburg doctoral degree [9] (also view [10, 1990] and [11, 1997]) introduced the theory of time scale calculus (initially known as measure chain calculus), this version of calculus unifies and extends discrete and continuous calculus; the theory proves immensely useful when dealing with hybrid models [1, 2025]. S. Hilger, as theoretical framework, presented a system of three axioms (view [10] and [11]) and stated that any set, say T, that satisfies these three axioms were to be called time scales. He further concluded that any closed subset T of $\mathbb{R}$ forms a time scale [10]. Significant core concept of the theory are the formulation of the jump operators [10]- the forward jump operator (σ) and the backward jump operator (ρ). σ is defined as $\sigma(t) = \inf\{\tilde{t} \in T : \tilde{t} > t\}$ given σ is a mapping, $\sigma : T \to T$. ρ is defined as $\rho(t) = \sup\{\tilde{t} \in T : \tilde{t} < t\}$ given ρ is a mapping, $\rho : T \to T$. A non-maximal element $t \in T$ is said to be right-scattered, if $\sigma(t) > t$, and right-dense, if $\sigma(t) = t$. We call a non-minimal element $t \in T$ left-scattered, if $\rho(t) > t$, and left-dense, if $\rho(t) = t$. $t \in T$ is said to be dense if $\sigma(t) = t = \rho(t)$, isolated if $\rho(t) < t < \sigma(t)$.

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Using the notion of the forward jump operator, S. Hilger in [10] formulated the delta derivative (Δ-derivative). A decade later, C. D. Ahlbrandt et al. [2, 2000] introduced the alpha derivative (α-derivative) consisting both the Δ-derivative and nabla derivative (∇-derivative, initially denoted by ρ-derivative [2]) as special cases. ∇-derivative was officially named the nabla derivative by F. M. Atici et al. [3, 2002]. Reader is referred to [10] and [1] for more details.

Various notions of integration of Banach valued functions (which we will refer to as Banach functions) and real valued functions (which we will refer to as real functions) in their constructive sense are discussed in literature for both continuous (ordinary) and time scale calculus including the Henstock-Kurzweil integration.

The Henstock-Kurzweil integral, sometimes called the gauge integral and the generalized Riemann integral, as it's latter name suggest is a Riemann-type integral introduced independently by J. Kurzweil [13, 1957] and R. Henstock [8, 1961]. Preserving the intuitive approach of the Riemann integral, the formulation of the Henstock-Kurzweil integral is only slightly different from the formulation of the Riemann integral- instead of taking size of partition to be lesser than a positive constant δ we consider δ to be a positive valued function. This alteration has enormous advantage in application. Reader is referred to [12] for better insight.

For real functions in time scale calculus (domain is T)- the Henstock-Kurzweil integral preserving the intuitive approach of the Riemann integral was introduced by A. Peterson et al. [18, 2006]; for unbounded time scale intervals this integral was generalized by S. Avsec et al. [4, 2006] in the same year. A study exploring the relation between Henstock integral (as given in [12, Definition 4.12]) and Henstock delta integral was concluded by J. M. Park et al. [16, 2013] (also view [17, 2013]). Using covering theory approach, the Henstock-Kurzweil integral was defined by B. S. Thompson [26, 2008].

For Banach functions in continuous calculus (domain is $\mathbb{R}$)- the Henstock-Kurzweil integral was first introduced by S. S. Cao [5, 1992] (reader can also refer to [7, 1994], [24, 1999], [28, 2007] and [25, 2022] for more insight).

For Banach functions in time scale calculus (domain in T)- the Henstock-Kurzweil integral was defined by M. Cichoń [6, 2011].

As mentioned at the beginning, we motivate the development of the Henstock-Kurzweil integral for quotient valued functions (which we will call quotient functions) on time scales, preserving the intuitive approach of the Riemann integration. This formulation of the integral can be applied to continuous, discrete and hybrid models. In a recent paper [21, 2025] the theory of Riemann integration for quotient functions on time scales has been introduced.

We briefly explore classical quotient space theory (reader may refer [15, 1998]), before introducing the integral.

Let A be a vector space over the scalar field $\mathbb{R}$, and let B be a subspace of A. The quotient space, denoted by A/B, is a vector space whose underlying set is the collection $\{a + \mathrm{B} : \ a \in \mathrm{A}\}$. Here $a + \mathrm{B} = \{a + b : \ b \in \mathrm{B}\}$, $a \in \mathrm{A}$ are called the cosets of B. Since two cosets of B are either identical or disjoint, the quotient space A/B is the set of all distinct cosets of B.
Operations of cosets are defined as-
Addition: $\text{“}+\text{”} : \mathrm{A/B} \times \mathrm{A/B} \longrightarrow \mathrm{A/B}$,

$$(a_1 + \mathrm{B}) + (a_2 + \mathrm{B}) = (a_1 + a_2) + \mathrm{B},$$

and scalar multiplication: $\text{“}\cdot\text{”} : \mathbb{R} \times \mathrm{A/B} \longrightarrow \mathrm{A/B}$,

$$r \cdot (a + \mathrm{B}) = (r \cdot a) + \mathrm{B},$$

here $a_1, a_2, a \in \mathcal{V}$ and $r \in \mathbb{R}$.

Let A be a normed vector space, depending on whether B is a closed subspace of A or not, we obtain norm and semi-norm respectively. If A is a normed vector space and B is a subspace of A, then

$$\|a + \mathrm{B}\|_{\mathrm{A/B}} = \mathrm{dist}(a, \mathrm{B}) = \mathrm{dist}(a + \mathrm{B}, 0 + \mathrm{B}) = \inf_{b \in \mathrm{B}} \|a - b\|_{\mathrm{A}} = \inf_{b \in \mathrm{B}} \|a + b\|_{\mathrm{A}},$$

here $\|\cdot\|_{\mathrm{A/B}}$ is a semi-norm and $\|\cdot\|_{\mathrm{A}}$ is the norm of normed vector space A. The semi-norm ($\|\cdot\|_{\mathrm{A/B}}$) is nothing more than a pseudonorm. However, if we consider that B is a closed subspace of A then $\|\cdot\|_{\mathrm{A/B}}$

forms a norm, which we will call the quotient norm. The distance between two cosets $(a_1 + \mathrm{B})$ and $(a_2 + \mathrm{B})$ is defined as

$$\begin{aligned}\operatorname{dist}(a_1 + \mathrm{B}, a_2 + \mathrm{B}) &= \|(a_1 + \mathrm{B}) - (a_2 + \mathrm{B})\|_{\mathrm{A/B}} = \|(a_1 - a_2) + \mathrm{B}\|_{\mathrm{A/B}} \\ &= \inf_{b \in \mathrm{B}} \|(a_1 - a_2) - b\|_{\mathrm{A}} = \inf_{b \in \mathrm{B}} \|(a_1 - a_2) + b\|_{\mathrm{A}}.\end{aligned}$$

Note that $\operatorname{dist}(a_1, a_2 + \mathrm{B}) = \operatorname{dist}(a_1 + \mathrm{B}, a_2 + \mathrm{B})$.

The quotient space on which we define the integral is constructed as- $\mathfrak{X}/\mathrm{B}$, where $\mathfrak{X}$ is a Banach space and B is a closed subspace of $\mathfrak{X}$, hence our constructed quotient space $\mathfrak{X}/\mathrm{B}$ is itself a Banach space.

Below we present the definition of Henstock-Kurzweil integral for Banach functions $(\mathfrak{X})$ on time scales as defined by M. Cichoń [6, Definition 2.5], followed by our definition of the Henstock-Kurzweil integral for quotient functions $(\mathfrak{X}/\mathrm{B})$ on time scales.

Given T is a time scale, intervals on T will be defined as-

$$[d,e]_T = \{t \in T \,:\, d \le t \le e\}; \quad [d,e)_T = \{t \in T \,:\, d \le t < e\};$$
$$(d,e]_T = \{t \in T \,:\, d < t \le e\}; \quad (d,e)_T = \{t \in T \,:\, d < t < e\}.$$

Let $P([d,e]_T)$ denote the collection of all possible partitions of $[d,e]_T$. Let $J = \{d = t_0 < t_1 < \ldots < t_j = e\} \in P([d,e]_T)$. Δ-subintervals are taken to be of the form $[t_{k-1}, t_k)_T$, $1 \le k \le j$. Δ-tag is chosen arbitrarily as $\vartheta_k \in [t_{k-1}, t_k)_T$. We will call the point-interval collection $J = \left\{\left(\vartheta_k, [t_{k-1}, t_k)_T\right)\right\}_{k=1}^{j}$ as Δ-tagged partition.

The construction of the Δ-gauge and β^{Δ}-fine partition are from [18, pp. 164]. Consider a positive function $\beta^{\Delta} : [d,e]_T \to \mathbb{R}$ defined on $[d,e]_T$. We write $\beta^{\Delta}(t) = \big(\beta_L(t), \beta_R(t)\big)$, here β^{Δ} will be called the Δ-gauge for $[d,e]_T$ provided $\beta_L(t) > 0$ on $(d,e]_T$, $\beta_R(t) > 0$ on $[d,e)_T$, $\beta_L(d) \ge 0$, $\beta_R(e) \ge 0$ and $\beta_R(t) \ge \sigma(t) - t$ for all $t \in [d,e)_T$. For any Δ-gauge, say β^{Δ}, a Δ-tagged partition $J = \left\{\left(\vartheta_k, [t_{k-1}, t_k)_T\right)\right\}_{k=1}^{j}$ is a β^{Δ}-fine Δ-tagged partition provided

$$\vartheta_k - \beta_L(\vartheta_k) \le t_{k-1} < t_k \le \vartheta_k + \beta_R(\vartheta_k)$$

for all $k = 1, 2, \ldots, j$. When $T = \mathbb{R}$, then $\beta_L = \beta_R$.

Definition 1.1 *[6] Let $q : [d,e]_{\mathrm{T}} \to \mathfrak{X}$, then q is said to be Banach Henstock-Kurzweil Δ-integrable on $[d,e]_{\mathrm{T}}$ if there exists an $\overline{I} \in \mathfrak{X}$ such that for any $\varepsilon > 0$ there exists Δ-gauge, β^{Δ}, on $[d,e]_{\mathrm{T}}$; hence for every β^{Δ}-fine Δ-tagged partition* J*, we have*

$$\left\| \sum_{k=1}^{j} (t_k - t_{k-1}) \cdot q(\vartheta_k) - \overline{I} \right\|_{\mathfrak{X}} < \varepsilon.$$

Here $\overline{I} = \overline{\mathrm{HK}} \int_d^e q(t)\Delta t$, where $\overline{\mathrm{HK}} \int_d^e q(t)\Delta t$ is called the Banach Henstock-Kurzweil Δ-integral.

The set of all Banach Henstock-Kurzweil Δ-integrable functions on $[d,e]_T$ will be denoted by $\mathrm{HK}^{\Delta}\left\{[d,e]_T; \mathfrak{X}\right\}$.

Let $K = \{d = t_0 < t_1 < \ldots < t_j = e\} \in P([d,e]_T)$. ∇-subintervals are taken to be of the form $(t_{k-1}, t_k]_T$, $1 \le k \le j$. ∇-tag is chosen arbitrarily as $\xi_k \in (t_{k-1}, t_k]_T$. We will call the point-interval collection $K = \left\{\left(\xi_k, (t_{k-1}, t_k]_T\right)\right\}_{k=1}^{j}$ as ∇-tagged partition.

The construction of the ∇-gauge and γ^{∇}-fine partition are from [18, pp. 164]. Consider a positive function $\gamma^{\nabla} : [d,e]_T \to \mathbb{R}$ defined on $[d,e]_T$. We write $\gamma^{\nabla}(t) = \big(\gamma_L(t), \gamma_R(t)\big)$, here γ^{∇} will be called the ∇-gauge for $[d,e]_T$ provided $\gamma_L(t) > 0$ on $(d,e]_T$, $\gamma_R(t) > 0$ on $[d,e)_T$, $\gamma_L(d) \ge 0$, $\gamma_R(e) \ge 0$ and $\gamma_R(t) \ge t - \rho(t)$ for all $t \in (d,e]_T$. For any ∇-gauge, say γ^{∇}, a ∇-tagged partition $K = \left\{\left(\xi_k, (t_{k-1}, t_k]_T\right)\right\}_{k=1}^{j}$ is a γ^{∇}-fine ∇-tagged partition provided

$$\xi_k - \gamma_L(\xi_k) \le t_{k-1} < t_k \le \xi_k + \gamma_R(\xi_k)$$

for all $k = 1, 2, \ldots, j$. When $T = \mathbb{R}$, then $\gamma_L = \gamma_R$.

Definition 1.2 *Let $q : [d,e]_{\mathrm{T}} \to \mathfrak{X}$, then q is said to be Banach Henstock-Kurzweil ∇-integrable on $[d,e]_{\mathrm{T}}$ if there exists an $\underline{I} \in \mathfrak{X}$ such that for any $\varepsilon > 0$ there exists ∇-gauge, γ^{∇}, on $[d,e]_{\mathrm{T}}$; hence for every γ^{∇}-fine ∇-tagged partition K, we have*

$$\left\| \sum_{k=1}^{j} (t_k - t_{k-1}) \cdot q(\xi_k) - \underline{I} \right\|_{\mathfrak{X}} < \varepsilon.$$

Here $\underline{I} = \underline{\mathrm{HK}} \int_d^e q(t)\nabla t$, where $\underline{\mathrm{HK}} \int_d^e q(t)\nabla t$ is called the Banach Henstock-Kurzweil ∇-integral.

We proceed to define the Henstock-Kurzweil Δ-integral and Henstock-Kurzweil ∇-integral for quotient functions on time scales, which we will call the quotient Henstock-Kurzweil Δ-integral and quotient Henstock-Kurzweil ∇-integral.

2. Henstock-Kurzweil integration for quotient functions

Let $P([d,e]_T)$ denote the collection of all possible partitions of $[d,e]_T$.

2.1. Quotient Henstock-Kurzweil Δ-integral

Considering J to be a β^{Δ}-fine Δ-tagged partition, the formulation of the quotient Henstock-Kurzweil Δ-sum, $\overline{\mathrm{HK}}_Q(q + \mathrm{B}; J)$, given $q + \mathrm{B}$ be a quotient function is as follows-

$$\overline{\mathrm{HK}}_Q(q + \mathrm{B}; J) := \sum_{k=1}^{j} (t_k - t_{k-1}) \cdot (q(\vartheta_k) + \mathrm{B}).$$

Definition 2.1 *Let $q + \mathrm{B} : [d,e]_{\mathrm{T}} \to \mathfrak{X}/\mathrm{B}$, then $q + \mathrm{B}$ is said to be quotient Henstock-Kurzweil Δ-integrable on $[d,e]_{\mathrm{T}}$ if there exists an $\overline{I} + \mathrm{B} \in \mathfrak{X}/\mathrm{B}$ such that for any $\varepsilon > 0$ there exists Δ-gauge, β^{Δ}, on $[d,e]_{\mathrm{T}}$; hence for every β^{Δ}-fine Δ-tagged partition J, we have*

$$\left\| \overline{\mathrm{HK}}_Q(q + \mathrm{B}; J) - (\overline{I} + \mathrm{B}) \right\|_{\mathfrak{X}/\mathrm{B}} < \varepsilon,$$

$$\begin{aligned}
\left\| \sum_{k=1}^{j} (t_k - t_{k-1}) \cdot (q(\vartheta_k) + \mathrm{B}) - (\overline{I} + \mathrm{B}) \right\|_{\mathfrak{X}/\mathrm{B}} &= \left\| \left(\sum_{k=1}^{j} (t_k - t_{k-1}) \cdot q(\vartheta_k) - \overline{I} \right) + \mathrm{B} \right\|_{\mathfrak{X}/\mathrm{B}} \\
&= \inf_{b \in \mathrm{B}} \left\| \left(\sum_{k=1}^{j} (t_k - t_{k-1}) \cdot q(\vartheta_k) - \overline{I} \right) + b \right\|_{\mathfrak{X}}, \\
&= \inf_{b \in \mathrm{B}} \left\| \left(\sum_{k=1}^{j} (t_k - t_{k-1}) \cdot q(\vartheta_k) - \overline{I} \right) - b \right\|_{\mathfrak{X}}.
\end{aligned}$$

Here $\overline{I} + \mathrm{B} = \overline{\mathrm{HK}}_Q \int_d^e q(t)\Delta t$, which we will call the quotient Henstock-Kurzweil Δ-integral.

The set of all quotient Henstock-Kurzweil Δ-integrable functions on $[d,e]_T$ will be denoted by $\mathrm{HK}_Q^{\Delta}\{[d,e]_T; \mathfrak{X}/\mathrm{B}\}$.

Example 2.1 *1. When $\mathrm{T} = \mathbb{R}$, the quotient Henstock-Kurzweil Δ-integral coincides with the usual quotient Henstock-Kurzweil integral (domain in $\mathbb{R}$).*

2. When $\mathrm{T} = r\mathbb{Z}$, here $r \in \mathbb{R}$ and $d, e \in r\mathbb{Z}$. If $q + \mathrm{B} \in \mathrm{HK}_Q^{\Delta}\{[d,e]_{\mathrm{T}}; \mathfrak{X}/\mathrm{B}\}$, then

$$\overline{\mathrm{HK}}_Q \int_d^e q(t)\Delta t = r \cdot \sum_{z=\frac{d}{r}}^{\frac{e}{r}-1} \big(q(rz) + \mathrm{B}\big).$$

If $r = 1$, $\mathrm{T} = \mathbb{Z}$ and

$$\overline{\mathrm{HK}}_Q \int_d^e q(t)\Delta t = \sum_{z=d}^{e-1} \big(q(z) + \mathrm{B}\big).$$

2.2. Quotient Henstock-Kurzweil ∇-integral

Considering K to be a γ^{∇}-fine ∇-tagged partition, the formulation of the quotient Henstock-Kurzweil ∇-sum, $\underline{\mathrm{HK}}_Q(q+\mathrm{B};K)$, given $q+\mathrm{B}$ be a quotient function is as follows-

$$\underline{\mathrm{HK}}_Q(q+\mathrm{B};K) := \sum_{k=1}^{j}(t_k - t_{k-1}) \cdot (q(\xi_k)+\mathrm{B}).$$

Definition 2.2 *Let $q+\mathrm{B} : [d,e]_{\mathrm{T}} \to \mathfrak{X}/\mathrm{B}$, then $q+\mathrm{B}$ is said to be quotient Henstock-Kurzweil ∇-integrable on $[d,e]_{\mathrm{T}}$ if there exists an $\underline{I}+\mathrm{B} \in \mathfrak{X}/\mathrm{B}$ such that for any $\varepsilon > 0$ there exists ∇-gauge, γ^{∇}, on $[d,e]_{\mathrm{T}}$; hence for every γ^{∇}-fine ∇-tagged partition K, we have*

$$\left\|\underline{\mathrm{HK}}_Q(q+\mathrm{B};K) - (\underline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} < \varepsilon,$$

$$\begin{aligned}
\left\|\sum_{k=1}^{j}(t_k - t_{k-1}) \cdot (q(\xi_k)+\mathrm{B}) - (\underline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} &= \left\|\left(\sum_{k=1}^{j}(t_k - t_{k-1}) \cdot q(\xi_k) - \underline{I}\right) + \mathrm{B}\right\|_{\mathfrak{X}/\mathrm{B}} \\
&= \inf_{b\in \mathrm{B}} \left\|\left(\sum_{k=1}^{j}(t_k - t_{k-1}) \cdot q(\xi_k) - \underline{I}\right) + b\right\|_{\mathfrak{X}}, \\
&= \inf_{b\in \mathrm{B}} \left\|\left(\sum_{k=1}^{j}(t_k - t_{k-1}) \cdot q(\xi_k) - \underline{I}\right) - b\right\|_{\mathfrak{X}}.
\end{aligned}$$

Here $\underline{I}+\mathrm{B} = \underline{\mathrm{HK}}_Q \int_d^e q(t)\nabla t$, which we will call the quotient Henstock-Kurzweil ∇-integral.

The set of all quotient Henstock-Kurzweil ∇-integrable functions on $[d,e]_T$ will be denoted by $\mathrm{HK}_Q^{\nabla}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$.

Example 2.2 *1. When $\mathrm{T} = \mathbb{R}$, the quotient Henstock-Kurzweil ∇-integral coincides with the usual quotient Henstock-Kurzweil integral (domain in $\mathbb{R}$).*

2. When $\mathrm{T} = r\mathbb{Z}$, here $r \in \mathbb{R}$ and $d,e \in r\mathbb{Z}$. If $q+\mathrm{B} \in \mathrm{HK}_Q^{\nabla}\{[d,e]_{\mathrm{T}};\mathfrak{X}/\mathrm{B}\}$, then

$$\underline{\mathrm{HK}}_Q \int_d^e q(t)\nabla t = \alpha \cdot \sum_{z=\frac{d}{r}+1}^{\frac{e}{r}} \big(q(rz)+\mathrm{B}\big).$$

If $r = 1$, $\mathrm{T} = \mathbb{Z}$ and

$$\underline{\mathrm{HK}}_Q \int_d^e q(t)\nabla t = \sum_{z=d+1}^{e} \big(q(z)+\mathrm{B}\big).$$

3. Properties of quotient Henstock-Kurzweil integral

In this section, we discuss and formulate some properties of the quotient Henstock-Kurzweil integral (we limit our results to the Δ-integral (Definition 2.1) since the ∇-integral results can be obtained very similarly using Definition 2.2).

Theorem 3.1 *If $q+\mathrm{B} \in \mathrm{HK}_Q^{\Delta}\{[d,e]_{\mathrm{T}};\mathfrak{X}/\mathrm{B}\}$, then the value of the integral $\overline{I}+\mathrm{B}$ is unique and well-defined.*

Proof: Suppose $q+\mathrm{B} \in \mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$ has two integral values, say $\overline{I}_1+\mathrm{B}$ and $\overline{I}_2+\mathrm{B}$; and let $\varepsilon > 0$. There exists Δ-gauge, β_1^{Δ}, on $[d,e]_T$ such that for any β_1^{Δ}-fine Δ-tagged partition J_1, we have

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_1) - (\overline{I}_1+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} < \frac{\varepsilon}{2},$$

and there exists Δ-gauge, β_2^{Δ}, on $[d,e]_T$ such that for any β_2^{Δ}-fine Δ-tagged partition J_2, we have

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_2)-(\bar{I}_2+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{\varepsilon}{2}.$$

Now let Δ-gauge $\beta^{\Delta}(t)=\beta_1^{\Delta}(t)\cap\beta_2^{\Delta}(t)$ on $[d,e]_T$ such that for any β^{Δ}-fine Δ-tagged partition J, we obtain

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)-(\bar{I}_1+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{\varepsilon}{2} \text{ and } \left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)-(\bar{I}_2+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{\varepsilon}{2}.$$

It follows from triangle inequality that

$$\begin{aligned}\left\|(\bar{I}_1+\mathrm{B})-(\bar{I}_2+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}\leq\left\|(\bar{I}_1+\mathrm{B})-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)\right\|_{\mathfrak{X}/\mathrm{B}}+\big\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)\\ -(\bar{I}_2+\mathrm{B})\big\|_{\mathfrak{X}/\mathrm{B}}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, hence we conclude that the integral is unique.
The value of the integral $\bar{I}+\mathrm{B}$ of quotient function $q+\mathrm{B}$ is well-defined by the fact that addition and scalar multiplication of elements of quotient space are well defined (view [15, pp. 50]). □

Without knowing the actual value of the integral, we can prove the integrability of a function via the criterion of integrability stated as-

Theorem 3.2 *A quotient function $q+\mathrm{B}:[d,e]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable on $[d,e]_{\mathrm{T}}$ if and only if for any $\varepsilon>0$ there exists a Δ-gauge, β^{Δ}, such that for all β^{Δ}-fine Δ-tagged partitions J_1 and J_2 of $[d,e]_{\mathrm{T}}$, we have*

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_1)-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_2)\right\|_{\mathfrak{X}/\mathrm{B}}<\varepsilon.$$

Proof: ($\Rightarrow$) If $q+\mathrm{B}\in\mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$, for all $\varepsilon>0$ there exists Δ-gauge, β^{Δ}, hence for any β^{Δ}-fine Δ-tagged partitions J_1 and J_2, we obtain

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_1)-\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{\varepsilon}{2} \text{ and}$$

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_2)-\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{\varepsilon}{2}$$

Thus, we have

$$\begin{aligned}\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_1)-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_2)\right\|_{\mathfrak{X}/\mathrm{B}}&\leq\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_1)-\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right\|_{\mathfrak{X}/\mathrm{B}}\\&+\left\|\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_2)\right\|_{\mathfrak{X}/\mathrm{B}}\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

($\Leftarrow$) For each $n\in\mathbb{N}$, choose a Δ-gauge, β_n^{Δ}, so that for any two β_n^{Δ}-fine Δ-tagged partitions J_1 and J_2 of $[d,e]_T$, we have

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_1)-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_2)\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{1}{n}.$$

Replacing β_n^{Δ} by $\bigcap_{m=1}^{n}\beta_m^{\Delta}$, we may assume that $\beta_{n+1}^{\Delta}\subset\beta_n^{\Delta}$. For each n, fix a β_n^{Δ}-fine Δ-tagged partition J_n. Note that for $m>n$, since $\beta_m^{\Delta}\subset\beta_n^{\Delta}$, J_m is a β_n^{Δ}-fine Δ-tagged partition of $[d,e]_T$. Thus,

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_n)-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_m)\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{1}{n},$$

which implies that the sequence $\left\{\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_n)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{X}/\mathrm{B}$ (Banach space), and hence converges. Let $\overline{I}$ be the limit of this sequence, hence

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_n)-\overline{I}\right\|_{\mathfrak{X}/\mathrm{B}}<\frac{1}{n}.$$

To show that $\overline{I}=\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t$.

Fix $\varepsilon>0$ and choose $N>\frac{2}{\varepsilon}$. Let J be a β_N^{Δ}-fine Δ-tagged partition of $[d,e]_T$. Then,

$$\begin{aligned}\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)-\overline{I}\right\|_{\mathfrak{X}/\mathrm{B}} &\le \left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)-\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_N)\right\|_{\mathfrak{X}/\mathrm{B}}\\ &\quad+\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J_N)-\overline{I}\right\|_{\mathfrak{X}/\mathrm{B}}\\ &<\frac{1}{N}+\frac{1}{N}<\varepsilon.\end{aligned}$$

Therefore, $q+\mathrm{B}\in\mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$. □

Below we formulate a theorem establishing the relation between quotient Henstock-Kurzweil Δ-integral and quotient Riemann Δ-integral (defined in [21, Definition 1.1.1]).

Considering $J=\left\{\left(\vartheta_k,[t_{k-1},t_k)_T\right)\right\}_{k=1}^{j}$ to be a Δ-tagged partition. Mesh of partition J is defined as: mesh(J)=$\max_{1\le k\le j}[t_k-t_{k-1}]>0$. For some $\delta>0$, J_δ will represent a partition of $[d,e]_T$ with mesh δ satisfying the property: for each $k=1,\dots,j$ we have either $t_k-t_{k-1}\le\delta$ or $t_k-t_{k-1}>\delta\wedge\rho(t_k)=t_{k-1}$ (here $\wedge$ stand for 'and'). Hereafter, J_δ will mean a Δ-tagged partition with mesh δ.

Definition 3.1 *[21] Let $q+\mathrm{B}:[d,e]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}$, then $q+\mathrm{B}$ is said to be quotient Riemann Δ-integrable on $[d,e]_{\mathrm{T}}$ if there exists an $\overline{I}+\mathrm{B}\in\mathfrak{X}/\mathrm{B}$ such that for any $\varepsilon>0$ there exists a $\delta>0$ hence for every Δ-tagged partition J_δ, we have*

$$\left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B})-(\overline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}<\varepsilon.$$

Here $\overline{I}+\mathrm{B}=\overline{\mathrm{R}}_Q\int_d^e q(t)\Delta t$ which is called the quotient Riemann Δ-integral.

As mentioned in Section 1, for the formulation of the quotient Riemann integral instead of taking a positive valued function β^{Δ} (the Δ-gauge) as defined in Definition 2.1 we instead consider a positive constant δ for the integral.

Theorem 3.3 *If $q+\mathrm{B}:[d,e]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}$ is quotient Riemann Δ-integrable on $[d,e]_{\mathrm{T}}$ with $\overline{\mathrm{R}}_Q\int_d^e q(t)\Delta t=\overline{I}+\mathrm{B}$, then $q+\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable on $[d,e]_{\mathrm{T}}$ with $\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t=\overline{I}+\mathrm{B}$.*

Proof: Let $\varepsilon>0$. Since $q+\mathrm{B}$ is quotient Riemann Δ-integrable, hence there exists $\delta>0$ such that given J_δ is any Δ-tagged partition of $[d,e]_T$, we have

$$\left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B})-(\overline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}<\varepsilon$$

Now, we choose our Δ-gauge to be $\beta^{\Delta}(t)=\delta$. Hence J_δ is a β^{Δ}-fine Δ-tagged partition and

$$\left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B})-(\overline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $\overline{I}+\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable and $\overline{I}+\mathrm{B}=\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t$. □

Theorem 3.4 *1. Let $q+\mathrm{B}:[\tilde{t},\sigma(\tilde{t})]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[\tilde{t},\sigma(\tilde{t})]_{\mathrm{T}};\mathfrak{X}/\mathrm{B}\}$, then*

$$\overline{\mathrm{HK}}_Q\int_{\tilde{t}}^{\sigma(\tilde{t})} q(t)\Delta t=(\sigma(\tilde{t})-\tilde{t})\cdot\big(q(\tilde{t})+\mathrm{B}\big).$$

2. Let $q+\mathrm{B}:[\tilde{t},\sigma(\tilde{t})]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}\in \mathrm{HK}_Q^{\nabla}\{[\tilde{t},\sigma(\tilde{t})]_{\mathrm{T}};\mathfrak{X}/\mathrm{B}\}$, then

$$\underline{\mathrm{HK}}_Q\int_{\tilde{t}}^{\sigma(\tilde{t})} q(t)\nabla t=(\sigma(\tilde{t})-\tilde{t})\cdot\big(q(\sigma(\tilde{t}))+\mathrm{B}\big).$$

3. Let $q+\mathrm{B}:[\rho(\tilde{t}),\tilde{t}]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[\rho(\tilde{t}),\tilde{t}]_{\mathrm{T}};\mathfrak{X}/\mathrm{B}\}$, then

$$\overline{\mathrm{HK}}_Q\int_{\rho(\tilde{t})}^{\tilde{t}} q(t)\Delta t=\big(\tilde{t}-\rho(\tilde{t})\big)\cdot\big(q(\rho(\tilde{t}))+\mathrm{B}\big).$$

4. Let $q+\mathrm{B}:[\rho(\tilde{t}),\tilde{t}]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[\rho(\tilde{t}),\tilde{t}]_{\mathrm{T}};\mathfrak{X}/\mathrm{B}\}$, then

$$\underline{\mathrm{HK}}_Q\int_{\rho(\tilde{t})}^{\tilde{t}} q(t)\Delta t=\big(\tilde{t}-\rho(\tilde{t})\big)\cdot\big(q(\tilde{t})+\mathrm{B}\big).$$

Basics properties of quotient Henstock-Kurzweil Δ-integrable functions are stated below-

1. Linearity: Let $q+\mathrm{B},p+\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$ and $r_1,r_2\in\mathbb{R}$, then $\big[r_1\cdot q\ \oplus\ r_2\cdot p\big]\in \mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$ and

$$\overline{\mathrm{HK}}_Q\int_d^e\big[r_1\cdot q\ \oplus\ r_2\cdot p\big](t)\Delta t=r_1\cdot\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\ \oplus\ r_2\cdot\overline{\mathrm{HK}}_Q\int_d^e p(t)\Delta t.$$

2. Subinterval: Let $q+\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$, then $q+\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable on every subinterval of $[d,e]_T$.

3. Δ-Additivity: Let $q+\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[d,c]_T;\mathfrak{X}/\mathrm{B}\}$ and $q+\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[c,e]_T;\mathfrak{X}/\mathrm{B}\}$, then $q+\mathrm{B}\in \mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$ and

$$\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t=\overline{\mathrm{HK}}_Q\int_d^c q(t)\Delta t\oplus\overline{\mathrm{HK}}_Q\int_c^e q(t)\Delta t.$$

   Following Property 3, if c is right-scattered, then $\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t$ does depend on the value $(q(c)+\mathrm{B})\cdot(\sigma(c)-c)$.

4. ∇-Additivity: Let $q+\mathrm{B}\in \mathrm{HK}_Q^{\nabla}\{[d,c]_T;\mathfrak{X}/\mathrm{B}\}$ and $q+\mathrm{B}\in \mathrm{HK}_Q^{\nabla}\{[c,e]_T;\mathfrak{X}/\mathrm{B}\}$, then $q+\mathrm{B}\in \mathrm{HK}_Q^{\nabla}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$ and

$$\underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t=\underline{\mathrm{HK}}_Q\int_d^c q(t)\nabla t\oplus\underline{\mathrm{HK}}_Q\int_c^e q(t)\nabla t.$$

   Following Property 4, if c is left-scattered, then $\underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t$ does depend on the value $(q(c)+\mathrm{B})\cdot(c-\rho(c))$.

We formulate a theorem which depicts the relation between Banach Hensock-Kurzweil Δ-integral and quotient Hensock-Kurzweil Δ-integral.

Theorem 3.5 *Let $q:[d,e]_{\mathrm{T}}\to\mathfrak{X}$ be Banach Henstock-Kurzweil Δ-integral. If $g:=q+\mathrm{B}:[d,e]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}$, then g is also quotient Henstock-Kurzweil Δ-integrable.*

Proof: Given $q \in \mathrm{HK}^{\Delta}\{[d,e]_T;\mathfrak{X}\}$, we will show that $g \in \mathrm{HK}_Q^{\Delta}\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$.

$$\begin{aligned}\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};J)-(\overline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} &= \left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B})-(\overline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} \\ &= \left\|\left(\sum_{k=1}^{j}(t_k-t_{k-1})\cdot q(\vartheta_k)\right)+\mathrm{B}-(\overline{I}+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} \\ &= \left\|\left(\sum_{k=1}^{j}(t_k-t_{k-1})\cdot q(\vartheta_k)-\overline{I}\right)+\mathrm{B}\right\|_{\mathfrak{X}/\mathrm{B}} \\ &\leq \left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot q(\vartheta_k)-\overline{I}\right\|_{\mathfrak{X}} \\ &< \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, hence we conclude. □

4. $\diamondsuit_\alpha$-integration for quotient functions

The formulation of the Δ-derivative and ∇-derivative led to the discussion and introduction of a combined dynamic derivative- the $\diamondsuit_\alpha$-derivative, defined as a linear combination of the Δ- and ∇-derivatives. This derivative offers a more improved and balanced approximation reducing computational spuriosity proving to be more reliable derivative (view [23, 2006] and [22, 2005]). The definition of the $\diamondsuit_\alpha$-derivative as presented in [23] is defined as a linear combination of the standard Δ- and ∇-derivatives, hence the question arise- "How well-defined was this new derivative?", this question beautifully phrased in [27, 2007] as- "*The question remains, however, as to whether the $\diamondsuit_\alpha$ derivative is a well-defined dynamic derivative upon which a calculus on time scale can be built.*". This question was answered in a paper by J. W. Roger Jr et al. [20, 2007], where they re-defined the $\diamondsuit_\alpha$-derivative independently of the standard Δ- and ∇- derivatives and also proved the equivalence of the two definitions.

Various notions of $\diamondsuit_\alpha$-integration in their constructive sense are defined in literature- the Riemann $\diamondsuit_\alpha$-integral was defined by A. B. Malinowska et al. [14, 2009], the Riemann-Stieltjes $\diamondsuit_\alpha$-integral was defined by D. Zhao [29, 2015], the Lebesgue-Stieltjes $\diamondsuit_\alpha$-integral was defined by G. Qin et al. [19, 2021], and the quotient Riemann $\diamondsuit_\alpha$-integral defined in [21].

We proceed to define the Henstock-Kurzweil $\diamondsuit_\alpha$-integral for quotient functions on time scales, which we will call the quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integral.

4.1. Quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integral

Considering $L = \{d = t_0 < t_1 < \ldots < t_j = e\} \in P([d,e]_T)$. Δ-subintervals and Δ-tags are denoted by $\vartheta_k \in [t_{k-1},t_k)_T$, $1 \leq k \leq j$. ∇-subintervals and ∇-tags are denoted by $\xi_k \in (t_{k-1},t_k]_T$, $1 \leq k \leq j$.

We introduce $\eta^{\diamondsuit_\alpha}$-gauge which will allocate β^{Δ}-gauge for the Δ-subintervals of $[d,e]_T$ and γ^{∇}-gauge for the ∇-subintervals of $[d,e]_T$. $\eta^{\diamondsuit_\alpha}$-fine $\diamondsuit_\alpha$-tagged partition will simply mean that L is β^{Δ}-fine Δ-tagged partition and γ^{∇}-fine ∇-tagged partition. The formulation of the quotient Henstock-Kurzweil $\diamondsuit_\alpha$-sum, $\mathrm{HK}_Q^{\diamondsuit}(q+\mathrm{B};L)$, given $q+\mathrm{B}$ be a quotient function is as follows-

$$\mathrm{HK}_Q^{\diamondsuit}(q+\mathrm{B};L) := \sum_{k=1}^{j}(t_k-t_{k-1})\cdot\left[\alpha\cdot(q(\vartheta_k)+\mathrm{B})+(1-\alpha)\cdot(q(\xi_k)+\mathrm{B})\right],$$

here $\alpha \in [0,1]$.

Definition 4.1 *Let $q+\mathrm{B} : [d,e]_{\mathrm{T}} \to \mathfrak{X}/\mathrm{B}$, then $q+\mathrm{B}$ is said to be quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integrable on $[d,e]_{\mathrm{T}}$ if there exists an $I_\diamondsuit+\mathrm{B} \in \mathfrak{X}/\mathrm{B}$ such that for any $\varepsilon > 0$ there exists $\eta^{\diamondsuit_\alpha}$-gauge on $[d,e]_{\mathrm{T}}$; hence for every $\eta^{\diamondsuit_\alpha}$-fine $\diamondsuit_\alpha$-tagged partition L, we have*

$$\left\|\mathrm{HK}_Q^{\diamondsuit}(q+\mathrm{B};L)-(I_\diamondsuit+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}} < \varepsilon,$$

$$\begin{aligned}
&\left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot[\alpha\cdot(q(\vartheta_k)+\mathrm{B})+(1-\alpha)\cdot(q(\xi_k)+\mathrm{B})]-(I_\diamondsuit+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}\\
&=\left\|\sum_{k=1}^{j}(t_k-t_{k-1})\cdot[\alpha\cdot q(\vartheta_k)+\mathrm{B}+(1-\alpha)\cdot q(\xi_k)+\mathrm{B}]-(I_\diamondsuit+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}\\
&=\left\|\left(\sum_{k=1}^{j}(t_k-t_{k-1})\cdot[\alpha\cdot q(\vartheta_k)+(1-\alpha)\cdot q(\xi_k)]+\mathrm{B}\right)-(I_\diamondsuit+\mathrm{B})\right\|_{\mathfrak{X}/\mathrm{B}}\\
&=\left\|\left(\sum_{k=1}^{j}(t_k-t_{k-1})\cdot[\alpha\cdot q(\vartheta_k)+(1-\alpha)\cdot q(\xi_k)]-I_\diamondsuit\right)+\mathrm{B}\right\|_{\mathfrak{X}/\mathrm{B}}\\
&=\inf_{b\in\mathrm{B}}\left\|\left(\sum_{k=1}^{j}(t_k-t_{k-1})\cdot[\alpha\cdot q(\vartheta_k)+(1-\alpha)\cdot q(\xi_k)]-I_\diamondsuit\right)+b\right\|_{\mathfrak{X}},\\
&=\inf_{b\in\mathrm{B}}\left\|\left(\sum_{k=1}^{j}(t_k-t_{k-1})\cdot[\alpha\cdot q(\vartheta_k)+(1-\alpha)\cdot q(\xi_k)]-I_\diamondsuit\right)-b\right\|_{\mathfrak{X}}.
\end{aligned}$$

Here $I_\diamondsuit+\mathrm{B}=\mathrm{HK}_Q^\diamondsuit\int_d^e q(t)\diamondsuit_\alpha t$, which we will call the quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integral.

The set of all quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integrable functions on $[d,e]_T$ will be denoted by $\mathrm{HK}_Q^\diamondsuit\{[d,e]_T;\mathfrak{X}/\mathrm{B}\}$.

Depending on the value of α we observe that the quotient Henstock-Kurzweil Δ-integral and quotient Henstock-Kurzweil ∇-integral are indeed special cases of the quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integral.

If $q+\mathrm{B}$ is given to be quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integrable, then taking $\alpha=1$ will mean that $q+\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable (Definition 4.1 equivalent to Definition 2.1) and taking $\alpha=0$ will mean that $q+\mathrm{B}$ is quotient Henstock-Kurzweil ∇-integrable (Definition 4.1 equivalent to Definition 2.2). $\alpha\in(0,1)$ will mean that $q+\mathrm{B}$ is both quotient Henstock-Kurzweil Δ-integrable and quotient Henstock-Kurzweil ∇-integrable given $q+\mathrm{B}$ is quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integrable, the inverse of this statement is proved below-

Theorem 4.1 *If $q+\mathrm{B}:[d,e]_{\mathrm{T}}\to\mathfrak{X}/\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable and quotient Henstock-Kurzweil ∇-integrable, then $q+\mathrm{B}$ is quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integrable and*

$$\mathrm{HK}_Q^\diamondsuit\int_d^e q(t)\diamondsuit_\alpha t=\alpha\cdot\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t+(1-\alpha)\cdot\underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t.$$

Proof: Suppose $q+\mathrm{B}$ is quotient Henstock-Kurzweil Δ-integrable and quotient Henstock-Kurzweil ∇-integrable on $[d,e]_T$, then given $\varepsilon>0$ there exists $\beta_\Delta>0$ and $\gamma_\nabla>0$ such that

$$\left\|\overline{\mathrm{HK}}_Q(q+\mathrm{B};L)-\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right\|_{\mathfrak{X}/\mathrm{B}}<\tfrac{\varepsilon}{2}\text{ and }\left\|\underline{\mathrm{HK}}_Q(q+\mathrm{B};L)-\underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t\right\|_{\mathfrak{X}/\mathrm{B}}<\tfrac{\varepsilon}{2}.$$

Letting $\eta^{\diamondsuit_\alpha}$-fine $=\min\{\beta^\Delta\text{-fine},\gamma^\nabla\text{-fine}\}>0$ hence taking L to be a $\eta^{\diamondsuit_\alpha}$-fine Δ-tagged partition we formulate

$$\begin{aligned}
&\left\|\mathrm{HK}_Q^\diamondsuit(q+\mathrm{B};L)-\left\{\alpha\cdot\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t+(1-\alpha)\cdot\underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t\right\}\right\|_{\mathfrak{X}/\mathrm{B}}\\
&=\left\|\alpha\left[\sum_{k=1}^{j}(t_k-t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B})-\overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right]+(1-\alpha)\left[\sum_{k=1}^{j}(t_k-t_{k-1})\cdot\right.\right.\\
&\left.\left.(q(\xi_k)+\mathrm{B})-\underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t\right]\right\|_{\mathfrak{X}/\mathrm{B}}
\end{aligned}$$

applying triangle inequality and homogeneity of the quotient norm we obtain,

$$\leq \left\|\alpha\left[\sum_{k=1}^{j}(t_k - t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B}) - \overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right]\right\| + \left\|(1-\alpha)\left[\sum_{k=1}^{j}\right.\right.$$

$$\left.\left.(t_k - t_{k-1})\cdot(q(\xi_k)+\mathrm{B}) - \underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t\right]\right\|_{\mathfrak{X}/\mathrm{B}}$$

$$= |\alpha|\left\|\sum_{k=1}^{j}(t_k - t_{k-1})\cdot(q(\vartheta_k)+\mathrm{B}) - \overline{\mathrm{HK}}_Q\int_d^e q(t)\Delta t\right\|_{\mathfrak{X}/\mathrm{B}} + |(1-\alpha)|\left\|\sum_{k=1}^{j}\right.$$

$$\left.(t_k - t_{k-1})\cdot(q(\xi_k)+\mathrm{B}) - \underline{\mathrm{HK}}_Q\int_d^e q(t)\nabla t\right\|_{\mathfrak{X}/\mathrm{B}} < \varepsilon.$$

Since $\varepsilon > 0$ arbitrary, hence we conclude. □

The $\diamondsuit_\alpha$-integral possesses similar properties as the Δ- and ∇-integrals, hence we limit the theory to the definition (definition of the quotient Henstock-Kurzweil $\diamondsuit_\alpha$-integral (Definition 4.1)) and theorem (Theorem 4.1) establishing the relation between the three integrals only.

5. Conclusion

This paper explores the theory of Henstock-Kurzweil integration for quotient valued functions on time scales and discuss a few fascinating results.

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