Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.77569

Deffered Rough $\mathcal{I}$-Statistical Convergence in $\mathcal{L}$-Fuzzy Normed Spaces

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ABSTRACT: The present article devoted to explore the concept of deffered rough statistical convergence of sequences through ideals in $\mathcal{L}$-fuzzy normed spaces. We introduce the notions of deffered rough ideal statistical limit points, deffered rough ideal cluster points, and deffered rough ideal boundedness within these spaces. Additionally, we examine the theory of convexity and closedness in relation to the set of approximate statistical limit points.

Key Words: Deffered Convergence, ideals, rough statistical convergence, $\mathcal{L}$-fuzzy normed space.

Contents

1. Introduction

Statistical convergence is a generalized summability approach using natural density to usual convergence, which was established by Fast [12] in 1951. The researchers have explored the numerous new results and have influenced the development of new methodologies in statistical convergence approach. The ideal convergence ($\mathcal{I}$-convergence) is one of the generalization of statistical convergence which was introduced by Kostryko *et al.* [17]. Ideal convergence has inspired numerous researchers to explore this direction, including extending ideal convergence from sequences of real numbers to sequences of functions, broadening its scope for sequences, and various other related advancements [6,7,16,19]. In addition, concept of rough convergence is considered as an important advanvement in convergence which handles the approximate solution got using the numerical analysis. Phu [25] introduced concept of rough convergence for sequences on finite-dimensional normed linear spaces, which was later extended on infinite-dimensional normed linear spaces [26]. Aytar [5] further advanced the field by proposing rough statistical convergence as a generalized variant of convergence. The concept of rough convergence has since inspired extensive research on various sequence spaces and among others [2,3,4,8,20,24]. Küçükaslan and Yilmaztürk [18] presented the concept of deferred statistical convergence as generalization of statistical convergence using deferred Cesàro mean, which was given by Agnew [1]. Numerous research contributions in this direction can be seen in [10,15,21,22].

The statistical convergence of sequences on $\mathcal{L}$-fuzzy normed spaces was explored by Yapali [29]. $\mathcal{L}$-fuzzy normed spaces [9] broaden the concept of classical normed spaces by using a complete lattice $\mathcal{L}$ instead of the unit interval $[0, 1]$.This approach offers a flexible framework for addressing problems involving uncertainty and ordered structures. The development of $\mathcal{L}$-fuzzy sets [11], which utilize a complete lattice $\mathcal{L}$, introduces the necessary flexibility and generality. $\mathcal{L}$-fuzzy normed spaces emerge from the convergence of classical functional analysis, lattice theory, and fuzzy mathematics, offering several advantages over traditional fuzzy normed spaces. Yapali *et al.* [30] expanded this concept by studying lacunary statistical convergence sequences, which is a refined version of statistical convergence, within the framework of $\mathcal{L}$-fuzzy normed spaces. Aykut *et al.* [23] explored rough statistical convergence in $\mathcal{L}$-fuzzy normed spaces. Additionally, Khan *et al.* [14] investigated ideal convergence within the context of $\mathcal{L}$-fuzzy normed spaces. Rahaman and Mursaleen [27] presented rough deferred statistical convergence for difference sequences on $\mathcal{L}$-fuzzy normed space. Jan and Jalal [13] worked on lacunary

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Submitted June 30, 2025. Published August 12, 2025
2010 *Mathematics Subject Classification:* 40A35, 03E72, 40E15, 06D72.

Δ-statistical convergence on thses spaces. Recent research highlights the effectiveness of $\mathcal{L}$-fuzzy normed spaces, in addressing challenges where classical fuzzy approaches fall short. In present article aims to define and investigate the fundamental properties of deffered rough ideal statistical convergence within the framework of $\mathcal{L}$-fuzzy normed spaces, offering a novel perspective on statistical convergence.

2. Preliminaries

The essential terminology and concepts required for deffered Rough $\mathcal{I}$-statistical convergence in $\mathcal{L}$-fuzzy normed spaces are reviewed in the present section.

Definition 2.1 *A real sequence $y = \{y_m\}$ is statistical convergence to $y_0 \in \mathbb{R}$ if to each $\varepsilon > 0$, set $\{m \in \mathbb{N} : |y_m - y_0| \geq \varepsilon\}$ has natural density zero. The natural density of set K_n, characterized as $d(K_n) = \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ indicates elements present in $K_n = \{m \in \mathbb{N} : m \leq n\}$.*

Definition 2.2 *[17] Let $\mathbb{Y} \neq \emptyset$. A family $\mathcal{I} \subset P(\mathbb{Y})$ of subsets from $\mathbb{Y}$ is said to be an ideal in $\mathbb{Y}$ provided, (i) $\emptyset \in \mathcal{I}$ (ii) If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$ and (iii) For each $A \in \mathcal{I}, B \subset A$ we have $B \in \mathcal{I}$. Here, $P(\mathbb{Y})$ represents the power set of $\mathbb{Y}$, which is the collection of all subsets of $\mathbb{Y}$.*

An admissible ideal in $\mathbb{Y}$ is defined as non-trivial ideal $(\mathcal{I} \neq P(\mathbb{Y}))$ that is a proper subset of $P(\mathbb{Y})$ and encompasses all singleton sets. This ideal must be distinct from $P(\mathbb{Y})$ itself.
Consider $\mathcal{I}$ as non-trivial admissible ideal in set of natural numbers throughout the article.

Definition 2.3 *[17] Let $\mathbb{Y} \neq \emptyset$. A non-empty family $\mathcal{F} \subset P(\mathbb{Y})$ of subsets from $\mathbb{Y}$ is said to be a filter in $\mathbb{Y}$ provided, (i) $\emptyset \notin \mathcal{F}$ (ii) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ and (iii) For each $B \in \mathcal{F}, A \subset B$ we have $B \in \mathcal{F}$.*

Every ideal $\mathcal{I}$ is connected with a filter $\mathcal{F}(\mathcal{I})$ defined by $\mathcal{F}(\mathcal{I}) = \{M \subseteq \mathbb{Y} : M^c \in \mathcal{I}\}$ where $M^c = \mathbb{Y} - M$.

Definition 2.4 *[17] A sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be ideal convergent ($\mathcal{I}$ -convergent) to ζ^* if for every $\varepsilon > 0$, we have $\{n \in \mathbb{N} : |y_m - \zeta^*| \geq \varepsilon\} \in \mathcal{I}$. Here, ζ^* is termed as $\mathcal{I}$-limit of sequence $y = \{y_m\}$.*

Further, we would like to highlight the concept of rough convergence.

Definition 2.5 *[25] Let $(\mathbb{Y}, \|\,.\,\|)$ be a normed linear space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be rough convergent (r-convergent) to $\zeta^* \in \mathbb{Y}$ for some non-negative real number r if there exists $m_0 \in \mathbb{N}$ for every $\varepsilon > 0$ such that $\|y_m - \xi\| < r + \varepsilon$ for all $m \geq m_0$.*
It is denoted symbolically by $y_m \xrightarrow{r} \zeta^*$ or $r - \lim\limits_{m\to\infty} y_m = \zeta^*$, where r is called roughness degree of rough convergence of the sequence $y = \{y_m\}$.

Definition 2.6 *[24] A sequence $y = \{y_m\}$ is said to be rough ideal convergent ($r - \mathcal{I}$-convergent) to ζ^*, where r is non-negative real number, if for every $\varepsilon > 0$,*

$$\{m \in \mathbb{N} : |y_m - \zeta^*\,| \geq r + \varepsilon\} \in \mathcal{I}.$$

It is denoted symbolically by $y_m \xrightarrow{r-\mathcal{I}} \zeta^*$ or $r - \mathcal{I} - \lim\limits_{m\to\infty} y_m = \zeta^*$.

The concept of deferred statistical convergence is considered using deferred Cesàro mean [1] rather than Cesàro mean which is used with statistical convergence.

Definition 2.7 *Deferred Cesaro mean for a sequence $y = \{y_m\}$ is given by*

$$(D_{p,q}y)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} y_k;$$

$n = 1, 2, 3, \ldots$ where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are sequences of non-negative integers. Also, $p(n) < q(n)$ and $\lim\limits_{n\to\infty} q(n) = \infty$.

For $M \subseteq \mathbb{N}$, deferred density for the set M is given by $D_{p,q}(K) = \lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{i \in \mathbb{N} : p(n) < i \leq q(n), i \in M\}|$.

Definition 2.8 *[18] A sequence $y = \{y_m\}$ is said to be deferred statistically convergent to y_0 if satisfies*

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{m \in \mathbb{N} : p(n) < m \leq q(n), |y_m - y_0| \geq \varepsilon\}| = 0.$$

For $p(n) = 0$ and $q(n) = n$, deferred statistical convergence coincides with the concept of statistical convergence for sequence $y = \{y_m\}$.

We will discuss $\mathcal{L}$-fuzzy normed spaces and convergence of sequences in these spaces through various setups.

Definition 2.9 *[28] Consider a complete lattice $\mathcal{L} = (L, \preccurlyeq)$ and set A referred as universe. A L-fuzzy set on M is characterized by the mapping $X : M \to L$. The collection of all L-sets for a given set M is represented by L^M.*
The intersection and union of two L-sets S_1 and S_2 on M can be depicted as

$$(S_1 \cap S_2)(y) = S_1(y) \wedge S_2(y),$$

$$(S_1 \cup S_2)(y) = S_1(y) \vee S_2(y),$$

for each $y \in M$.
In a complete lattice $\mathcal{L}$, 0_L and 1_L represent the minimum and maximum elements, respectively. For the lattice $(L, \preccurlyeq)$,the symbols $\succeq$, $\succ$ and $\prec$ used with their conventional meanings.

Definition 2.10 *[28] Consider $\mathcal{L} = (L, \preccurlyeq)$ as a complete lattice. A function $\Omega : L \times L \to L$ is termed as t-norm, if satisfies for $p, q, r, s \in L$: (i) $\Omega(p, q) = \Omega(q, p)$, (ii) $\Omega(\Omega(p, q), r) = \Omega(p, \Omega(q, r))$, (iii) $\Omega(p, 1_L) = \Omega(1_L, p) = p$, (iv) $p \preccurlyeq q$ and $r \preccurlyeq s$, implies $\Omega(p, r) \preccurlyeq \Omega(q, s)$.*

Definition 2.11 *[28] A function $\mathcal{N} : L \longrightarrow L$ is said to be negator on $\mathcal{L} = (L, \preccurlyeq)$ provided, (i) $\mathcal{N}(0_L) = 1_L$, (ii) $\mathcal{N}(1_L) = 0_L$, (iii) $p \preccurlyeq q$ implies $\mathcal{N}(q) \preccurlyeq \mathcal{N}(p)$ for $p, q \in L$.*
Additionally, if $\mathcal{N}(\mathcal{N}(p)) = p$ for every $p \in L$. Then, $\mathcal{N}$ is said to be involutive negator.

Definition 2.12 *[28] Let $\mathcal{L} = (L, \preccurlyeq)$ be a complete lattice and $\mathbb{Y}$ be a real vector space. Ω be a continuous t-norm on L and μ be a L-set on $\mathbb{Y} \times (0, \infty)$ satisfying*

(i) $\mu(p, s) \succ 0_L$ for all $p \in \mathbb{Y}$, $s > 0$,

(ii) $\mu(p, s) = 1_L$ for every $s > 0$ iff $p = \theta$,

(iii) $\mu(\alpha p, s) = \mu\left(p, \frac{s}{|\alpha|}\right)$ for all $p \in \mathbb{Y}$, $s > 0$ and $\alpha \in \mathbb{R} - \{\theta\}$,

(iv) $\Omega(\mu(p, s), \mu(q, t)) \preccurlyeq \mu(p + q, s + t)$, for all $p, q \in \mathbb{Y}$ and $s, t > 0$,

(v) $\lim_{s\to\infty} \mu(p, s) = 1_L$ and $\lim_{s\to 0} \mu(p, s) = 0_L$ for all $p \in \mathbb{Y} - \{\theta\}$,

(vi) Functions $f_p : (0, \infty) \to L$ such that $f_p(s) = \mu(p, s)$ are continuous.

Then, triplet $(\mathbb{Y}, \mu, \Omega)$ is known as $\mathcal{L}$-fuzzy normed space or $\mathcal{L}$-normed space.

Example 2.1 *[23] Let $(\mathbb{R}, |\,.\,|)$ be a normed space, Ω be a continuous t-norm given by $\Omega(a_1, a_2) = a_1 a_2$ for $a_1, a_2 \in L$, and μ be a L-fuzzy set on $\mathbb{R} \times (0, \infty)$ defined by $\mu(a, s) = \frac{s}{s+|a|}$ for all $s > 0$ and $a \in \mathbb{R}$. Then, triplet $(\mathbb{R}, \mu, \Omega)$ is a $\mathcal{L}$-fuzzy normed space.*

Definition 2.13 *[28] Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be convergent to $\zeta^* \in \mathbb{Y}$ corresponding to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, there exists m_0 such that $\mu(y_m - \zeta^*; s) \succ \mathcal{N}(\varepsilon)$ for $m \geq m_0$.*

Definition 2.14 *[29] Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be statistically convergent to $\zeta^* \in \mathbb{Y}$ corresponding to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, we have $d(\{m \in \mathbb{N} : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}) = 0$.*
It is denoted symbolically by $y_m \xrightarrow{St_{\mathcal{L}}} \zeta^*$ or $St_{\mathcal{L}} - \lim_{m\to\infty} y_m = \zeta^*$.

Definition 2.15 *[23] Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be rough convergent to $\zeta^* \in \mathbb{Y}$ for some $r \geq 0$ corresponding to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, there exists m_0 such that $\mu(y_m - \zeta^*; r + s) \succ \mathcal{N}(\varepsilon)$ for all $m \geq m_0$.*

Definition 2.16 *[23] Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be rough statistically convergent to $\zeta^* \in \mathbb{Y}$ for some $r \geq 0$ corresponding to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, we have $d(\{m \in \mathbb{N} : \mu(y_m - \zeta^*; r + s) \nsucc \mathcal{N}(\varepsilon)\}) = 0$.*
It is denoted symbolically by $y_m \xrightarrow{r-St_{\mathcal{L}}} \zeta^*$ or $r - St_{\mathcal{L}} - \lim_{m\to\infty} y_m = \zeta^*$.

Definition 2.17 *[14] Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be ideal convergent to $\zeta^* \in \mathbb{Y}$ corresponding to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, we have $\{m \leq n : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\} \in \mathcal{I}$.*
It is denoted symbolically by $y_m \xrightarrow{\mathcal{I}_{\mathcal{L}}} \zeta^*$ or $\mathcal{I}_{\mathcal{L}} - \lim_{m\to\infty} y_m = \zeta^*$.

Definition 2.18 *Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be ideal statistically convergent to $\zeta^* \in \mathbb{Y}$ corresponding to fuzzy norm μ if for every or every $\varepsilon \in L - \{0_L\}$, $\delta > 0$ and $s \in \mathbb{R}^+$, we have*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{m \leq n : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\} \right| > \delta \right\} \in \mathcal{I}.$$

It is denoted symbolically by $y_m \xrightarrow{St^{\mathcal{I}}_{\mathcal{L}}} \zeta^*$ or $St^{\mathcal{I}}_{\mathcal{L}} - \lim_{m\to\infty} y_m = \zeta^*$.

Definition 2.19 *Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be deffered statistically convergent to $\zeta^* \in \mathbb{Y}$ corresponding to fuzzy norm μ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, satisfying*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left| \{m \in \mathbb{N} : p(n) < m \leq (n), \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\} \right| = 0.$$

It is denoted symbolically by $y_m \xrightarrow{D_{p,q}-St_{\mathcal{L}}} \zeta^*$ or $D_{p,q} - St_{\mathcal{L}} - \lim_{m\to\infty} y_m = \zeta^*$.

Definition 2.20 *Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y = \{y_m\} \in \mathbb{Y}$ is said to be deffered rough statistically convergent to $\zeta^* \in \mathbb{Y}$ corresponding to fuzzy norm μ for some $r \geq 0$ if for every $\varepsilon \in L - \{0_L\}$ and $s \in \mathbb{R}^+$, we have*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left| \{m \in \mathbb{N} : p(n) < m \leq (n), \mu(y_m - \zeta^*; r + s) \nsucc \mathcal{N}(\varepsilon)\} \right| = 0.$$

It is denoted symbolically by $y_m \xrightarrow{D_{p,q}-r-St_{\mathcal{L}}} \zeta^*$ or $D_{p,q} - r - St_{\mathcal{L}} - \lim_{m\to\infty} y_m = \zeta^*$.

3. Main Results

In this section, we will address deffered rough ideal statistical convergence in a $\mathcal{L}$-Fuzzy normed space and demonstrate its important properties.

Definition 3.1 *Let $(\mathbb{Y},\mu,\Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then, a sequence $y=\{y_m\}\in\mathbb{Y}$ is said to be deffered ideal statistically convergent to $\zeta^*\in\mathbb{Y}$ corresponding to fuzzy norm μ if for every $\varepsilon\in\mathcal{L}-\{0_{\mathcal{L}}\}$, $\delta>0$ and $s\in\mathbb{R}^+$, satisfies*

$$\left\{n\in\mathbb{N}:\frac{1}{q(n)-p(n)}\left|\{m\in\mathbb{N}:\mu(y_m-\zeta^*;s)\not\succ\mathcal{N}(\varepsilon)\}\right|\geq\delta\right\}\in\mathcal{I}.$$

It is denoted symbolically by $y_m\xrightarrow{D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}}\zeta^*$ or $D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-\lim_{m\to\infty}y_m=\zeta^*$.

Definition 3.2 *Let $(\mathbb{Y},\mu,\Omega)$ be a $\mathcal{L}$-fuzzy normed space and $r\geq 0$. Then, a sequence $y=\{y_m\}\in\mathbb{Y}$ is said to be deffered rough ideal statistically convergent to $\zeta^*\in\mathbb{Y}$ corresponding to fuzzy norm μ if for every $\varepsilon\in\mathcal{L}-\{0_{\mathcal{L}}\}$, $\delta>0$ and $s\in\mathbb{R}^+$, we have*

$$\left\{n\in\mathbb{N}:\frac{1}{q(n)-p(n)}\left|\{m\leq n:\mu(y_m-\zeta^*;r+s)\not\succ\mathcal{N}(\varepsilon)\}\right|\geq\delta\right\}\in\mathcal{I}.$$

It is denoted symbolically by $y_m\xrightarrow{D_{p,q}-r-St_{\mathcal{L}}^{\mathcal{I}}}\zeta^*$ or $D_{p,q}-r-St_{\mathcal{L}}^{\mathcal{I}}-\lim_{m\to\infty}y_m=\zeta^*$.

In general, $D_{p,q}-r-St_{\mathcal{L}}^{\mathcal{I}}$ limit of a sequence may not be unique. Therefore, we consider the set $D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-LIM_{\mu}^{r}(y)$ which represents the set of all deffered rough ideal statistical limit points of the sequence $y=\{y_m\}$ corresponding to fuzzy norm μ in $\mathcal{L}$-fuzzy normed space $(\mathbb{Y},\mu,\Omega)$ as

$$D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-LIM_{\mu}^{r}(y)=\left\{\zeta^*:y_m\xrightarrow{D_{p,q}-r-St_{\mathcal{L}}^{\mathcal{I}}}\zeta^*\right\}.$$

Moreover, sequence $y=\{y_m\}$ is ζ^*-convergent provided $D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-LIM_{\mu}^{r}(y)\neq\emptyset$. For sequence $y=\{y_m\}$ of real numbers, a set of rough limit points is observed as

$$D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-LIM_{\mu}^{r}(y)=\left\{D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-\limsup_{m\to\infty}y_m-r,\ D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-\liminf_{m\to\infty}y_m+r\right\}.$$

Example 3.1 *Let $(\mathbb{R},|\,.\,|)$ be a normed space, Ω be a continuous t-norm defined by $\Omega(a_1,a_2)=a_1a_2$ for $a_1,a_2\in L$, and μ be a L-fuzzy set on $\mathbb{R}\times(0,\infty)$ defined by $\mu(a,s)=\frac{s}{s+|a|}$ for all $s>0$ and $a\in\mathbb{R}$. As μ satisfies above conditions then triplet $(\mathbb{R},\mu,\Omega)$ becomes $\mathcal{L}$-fuzzy normed space. Consider ideal in $\mathbb{N}$ which contains the sets with natural density zero. Take a sequence $y=\{y_m\}$ such that*

$$y_m=\begin{cases}(-1)^m & m\neq n^2\\ m & \textit{otherwise.}\end{cases},\ n\in\mathbb{N}$$

Then,

$$D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-LIM_{\mu}^{r}(y)=\begin{cases}\emptyset & r<1\\ [1-r,r-1] & \textit{otherwise.}\end{cases}$$

Definition 3.3 *A sequence $y=\{y_m\}$ in $\mathcal{L}$-fuzzy normed space $(\mathbb{Y},\mu,\Omega)$ is said to be $D_{p,q}-\mathcal{I}-St_{\mathcal{L}}$-bounded if for every $\varepsilon\in\mathcal{L}-\{0_{\mathcal{L}}\}$ and $\delta>0$, there exists $H>0$ satisfying*

$$\left\{n\in\mathbb{N}:\frac{1}{q(n)-p(n)}\left|\{m\in\mathbb{N}:\mu(y_m;H)\not\succ\mathcal{N}(\varepsilon)\}\right|\geq\delta\right\}\in\mathcal{I}.$$

Next, we demonstrate important properties of deffered rough ideal statistical convergence in a $\mathcal{L}$-fuzzy normed spaces.

Theorem 3.1 *Let $(\mathbb{Y},\mu,\Omega)$ be $\mathcal{L}$-fuzzy normed space. Then, sequence $y=\{y_m\}$ in $\mathbb{Y}$ is said to be deffered ideal statistically bounded or $D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}$-bounded if and only if $D_{p,q}-St_{\mathcal{L}}^{\mathcal{I}}-LIM_{\mu}^{r}(y)\neq\emptyset$ for some $r>0$.*

Proof: *Necessary Part:*
Let sequence $y = \{y_m\}$ be deffered ideal statistically bounded or $\mathcal{I} - St_{\mathcal{L}}$-bounded $\mathcal{L}$-fuzzy normed space. Then, for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in \mathbb{R}^+$ satisfying

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Being admissible ideal $\mathcal{I}$, we have $M = \mathbb{N} \setminus G \neq \emptyset$, where

$$G = \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\}.$$

Choose $m \in M$, then

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; s) \nsucc \mathcal{N}(\varepsilon)\}\right|\right\} < \delta$$

$$\Longrightarrow \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; s) \succ \mathcal{N}(\varepsilon)\}\right|\right\} \geq 1 - \delta. \tag{3.1}$$

Let $K = \{m \in \mathbb{N} : \mu(y_m; s) \succ \mathcal{N}(\varepsilon)\}$.
Also,

$$\begin{aligned} \mu(y_m; r + s) &\succeq \Omega\left((\mu(0, r), \mu(y_m, s)\right) \\ &= \Omega\left(1_{\mathcal{L}}, \mu(y_m; s)\right) \\ &\succ \mathcal{N}(\varepsilon). \end{aligned}$$

Thus $K \subset \{m \in \mathbb{N} : \mu(y_m; r + s) \succ \mathcal{N}(\varepsilon)\}$.
Using (3.1), it implies $1 - \delta \leq \frac{|K|}{q(n) - p(n)} \leq \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; r + s) \succ \mathcal{N}(\varepsilon)\}\right|$.
Therefore,

$$\frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta.$$

That gives

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \subset G \in \mathcal{I}.$$

Hence, $0 \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$. Therefore, $D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y) \neq \emptyset$.

Sufficient Part:
Let $D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y) \neq \emptyset$ for some $r > 0$. Then, $\xi \in \mathbb{Y}$ exists such that $\xi \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$. Hence, for all $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in \mathbb{R}^+$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m - \xi; s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Consequently, almost all y_m's are enclosed within a ball having centre ξ in $\mathcal{L}$-fuzzy norm space, which imply that $y = \{y_m\}$ is $D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}}$-statistically bounded in $\mathcal{L}$-fuzzy normed space. □

The algebraic characterization of $r - \mathcal{I} - St_{\mathcal{L}}$-convergence provides a useful tool for analyzing and understanding the behavior of sequences in $\mathcal{L}$-fuzzy normed space.
Next, we discuss a $r - \mathcal{I} - St_{\mathcal{L}}$-convergent sequence $y = \{y_m\} \subset \mathbb{Y}$ can be characterized algebraically by the following results in $\mathcal{L}$-fuzzy normed space.

Theorem 3.2 *Let $\{y_m\}$ and $\{z_m\}$ be sequences in $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$ and $\mathcal{I}$ be an admissible ideal, then next results holds:*
(i) If $y_m \xrightarrow{D_{p,q} - r - St_{\mathcal{L}}^{\mathcal{I}}} y_0$ *and* $\alpha \in \mathbb{N}$ *then* $\alpha y_m \xrightarrow{D_{p,q} - r - St_{\mathcal{L}}^{\mathcal{I}}} \alpha y_0$.
(ii) If $y_m \xrightarrow{D_{p,q} - r - \mathcal{I} - St_{\mathcal{L}}} y_0$ *and* $z_m \xrightarrow{D_{p,q} - r - St_{\mathcal{L}}^{\mathcal{I}}} z_0$*, then* $(y_m + z_m) \xrightarrow{D_{p,q} - r - St_{\mathcal{L}}^{\mathcal{I}}} (y_0 + z_0)$.

Proof: Proof is obvious, so we won't include it. □

In the next theorem, we will show the set $D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ is closed.

Theorem 3.3 *Let $y = \{y_m\}$ be a sequence and r is some non-negative real number. Then, set of deffered rough ideal limit points i.e $D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ of a sequence $y = \{y_m\}$ in a $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$ is a closed set.*

Proof: If $r = 0$ then the result is obvious as $D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ is either $\emptyset$ or singleton set.
Let $D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y) \neq \emptyset$ for some $r > 0$.
Consider convergent $x = \{x_m\}$ sequence in $(\mathbb{Y}, \mu, \Omega)$, which converges to $x_0 \in \mathbb{Y}$ corresponding to fuzzy norm μ. For $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in \mathbb{R}^+$ we get $m_0 \in \mathbb{N}$ such that

$$\mu\left(x_m - x_0; \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon) \text{ for } \ m \geq m_0.$$

Let us take $x_{m_1} \in D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ and $\delta > 0$ then

$$S = \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\left\{m \in \mathbb{N} : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Being $\mathcal{I}$ admissible we get $G = \mathbb{N} \setminus S \neq \emptyset$. Choose $m \in G$, then

$$\frac{1}{q(n) - p(n)} \left|\left\{m \in \mathbb{N} : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \not\succ \mathcal{N}(\varepsilon)\right\}\right| < \delta$$

$$\Rightarrow \frac{1}{q(n) - p(n)} \left|\left\{m \in \mathbb{N} : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)\right\}\right| \geq 1 - \delta.$$

Put $B = \left\{m \in \mathbb{N} : \mu\left(y_m - x_{m_1}; r + \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)\right\}$. For $j \in B$ with $j \geq m_0$, we have

$$\begin{aligned}\mu(y_j - x_0; r + s) &\succeq \Omega\left(\mu\left(y_j - x_{m_1}; r + \frac{s}{2}\right), \mu\left(x_{m_1} - x_0; \frac{s}{2}\right)\right) \\ &\succ \mathcal{N}(\varepsilon).\end{aligned}$$

Therefore,

$$j \in \{m \in \mathbb{N} : \mu\left(y_m - x_0; r + s\right) \succ \mathcal{N}(\varepsilon)\}.$$

Hence, $B \subset \{m \in \mathbb{N} : \mu\left(y_m - x_0; r + s\right) \succ \mathcal{N}(\varepsilon)\}$, that provides

$$1 - \delta \leq \frac{|B_n|}{q(n) - p(n)} \leq \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu\left(y_m - x_0; r + s\right) \succ \mathcal{N}(\varepsilon)\}\right|.$$

Therefore, $\frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu\left(y_m - x_0; r + s\right) \not\succ \mathcal{N}(\varepsilon)\}\right| < \delta$.
Then,

$$\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu\left(y_m - x_0; r + s\right) \not\succ \mathcal{N}(\varepsilon)\}\right| \geq \delta\} \subset A \in \mathcal{I}.$$

Hence, $x_0 \in D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ in $(\mathbb{Y}, \mu, \Omega)$. □

The convexity of the set $D_{p,q} - St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ is explained in the next result.

Theorem 3.4 *Let $y = \{y_m\}$ be any sequence in a $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$ then the set $St^{\mathcal{I}}_{\mathcal{L}} - LIM^r_\mu(y)$ is a convex set for some positive number r.*

Proof: Consider $\zeta_1, \zeta_2 \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$. For convexity we will show $(1-\omega)\zeta_1 + \omega\zeta_2 \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$ for any $\omega \in (0,1)$.
As $\zeta_1, \zeta_2 \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$, then $m \in \mathbb{N}$ exists for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in \mathbb{R}^+$ such that

$$S_0 = \left\{ m \in \mathbb{N} : \mu\left(y_m - \zeta_1; \frac{r+s}{2(1-\omega)} \right) \nsucc \mathcal{N}(\varepsilon) \right\},$$

and

$$S_1 = \left\{ m \in \mathbb{N} : \mu\left(y_m - \zeta_2; \frac{r+s}{2\omega} \right) \nsucc \mathcal{N}(\varepsilon) \right\}.$$

For $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in S_0 \cup S_1 \} \right| \geq \delta \right\} \in \mathcal{I}.$$

Take $0 < \delta_1 < 1$ with $0 < 1 - \delta_1 < \delta$. Then

$$S = \left\{ n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in S_0 \cup S_1 \} \right| \geq \delta_1 \right\} \in \mathcal{I}.$$

For $n \notin S$

$$\frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in S_0 \cup S_1 \} \right| < 1 - \delta_1.$$

$$\frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \notin S_0 \cup S_1 \} \right| \geq 1 - (1-\delta_1) = \delta_1.$$

Thus, $\{ m \in \mathbb{N} : m \notin S_0 \cup S_1 \} \neq \emptyset$. Let $m \in (S_0 \cup S_1)^c = S_0^c \cap S_1^c$.
Then,

$$\begin{aligned}
\mu\left(y_m - [(1-\omega)\zeta_1 + \omega\zeta_2]; r+s \right) &= \mu[(1-\omega)(y_m - \zeta_1) + \omega(y_m - \zeta_2); r+s] \\
&\succeq \Omega\left(\mu\left((1-\omega)(y_m - \zeta_1); \frac{r+s}{2} \right), \mu\left(\omega(y_m - \zeta_2); \frac{r+s}{2} \right) \right) \\
&= \Omega\left(\mu\left(y_m - \zeta_1; \frac{r+\varepsilon}{2(1-\omega)} \right), \mu\left(y_m - \zeta_2; \frac{r+\varepsilon}{2\omega} \right) \right) \\
&\succ \mathcal{N}(\varepsilon).
\end{aligned}$$

This implies $S_0^c \cap S_1^c \subset {B_n}^c$ where $B_n = \{ m \in \mathbb{N} : \mu\left(y_{m_0} - [(1-\omega)\zeta_1 + \omega\zeta_2]; r+s \right) \nsucc \mathcal{N}(\varepsilon) \}$.
For $n \notin S$,

$$\delta_1 \leq \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \notin S_0 \cup S_1 \} \right| \leq \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \notin B_n \} \right|$$

or

$$\frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in B_n \} \right| < 1 - \delta_1 < \delta.$$

Hence, $S^c \subset \left\{ n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in B_n \} \right| < \delta \right\}$.
Since $S^c \in \mathcal{F}(\mathcal{I})$, then $\left\{ n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in B_n \} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I})$. Therefore, $\left\{ n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left| \{ m \in \mathbb{N} : m \in B_n \} \right| \geq \delta \right\} \in \mathcal{I}$. This implies that $D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$ is a convex set. □

Theorem 3.5 *A sequence $y = \{y_m\}$ in a $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$ is deffered rough ideal statistically convergent to $\zeta^* \in \mathbb{Y}$ corresponding to the norm μ for some $r > 0$ if there exists a sequence $z = \{z_m\}$ in $\mathbb{Y}$ such that $D_{p,q} - \mathcal{I} - St_{\mathcal{L}} - \lim_{m\to\infty} z_m = \zeta^*$ in $\mathbb{Y}$ and for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in \mathbb{R}^+$ have $\mu(y_m - z_m; r+s) \succ \mathcal{N}(\varepsilon)$ for all $m \in \mathbb{N}$.*

Proof: Consider $z = \{z_m\}$ sequence from $\mathbb{Y}$, which is deffered $\mathcal{I}$-statistically convergent to ζ^* and $\mu(y_m - z_m; r+s) \succ \mathcal{N}(\varepsilon)$.
For $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in \mathbb{R}^+$ the set

$$M = \left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu(z_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Define

$$S_1 = \{m \in \mathbb{N} : \mu(z_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}$$
$$S_2 = \{m \in \mathbb{N} : \mu(y_m - z_m; r) \nsucc \mathcal{N}(\varepsilon)\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Take $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Then

$$S = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| \geq \delta_1\right\} \in \mathcal{I},$$

For $n \notin S$

$$\frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| < 1 - \delta_1$$
$$\frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \notin S_1 \cup S_2\}\right| \geq \delta_1.$$

Thus $\{m \in \mathbb{N} : m \notin S_1 \cup S_2\} \neq \emptyset$.
Let $m \in (S_1 \cup S_2)^c = S_1^c \cap S_1^c$.
Then,

$$\begin{aligned} \mu(y_m - \zeta^*; r+s) &\succeq \Omega\left(\mu(y_m - z_m; r), \mu(z_m - \zeta^*; s)\right) \\ &\succ \mathcal{N}(\varepsilon). \end{aligned}$$

Which gives $S_1^c \cap S_2^c \subset B^c$, where

$$B = \{m \in \mathbb{N} : \mu(y_m - \zeta^*; r+\varepsilon) \nsucc \mathcal{N}(\varepsilon)\}.$$

So for $n \notin S$,

$$\delta_1 \leq \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \notin S_1 \cup S_2\}\right| \leq \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \notin B\}\right|$$

or

$$\frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \in B\}\right| < 1 - \delta_1 < \delta.$$

Therefore, $S^c \subset \left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \in B\}\right| < \delta\right\}$. Since $S^c \in \mathcal{F}(\mathcal{I})$ then we get

$$\left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \in B\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}),$$

i.e $\left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : m \in B\}\right| \geq \delta\right\} \in \mathcal{I}$. Hence, $y_m \xrightarrow{r-\mathcal{I}-St_\theta^{\mathcal{L}}} \zeta^*$. □

Theorem 3.6 *Let $y = \{y_m\}$ be a sequence in a $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \eta)$ then there does not exist two elements $\beta_1, \beta_2 \in D_{p,q}-\mathcal{I}-St_{\mathcal{L}}-LIM_\mu^r(y)$ for $r > 0$ and $\varepsilon \in \mathcal{L}-\{0_{\mathcal{L}}\}$ such that $\mu(\beta_1 - \beta_2; cr) \succ \mathcal{N}(\varepsilon)$ for $c > 2$.*

Proof: We shall use contradiction to support our conclusion. Assume there exists two elements $\beta_1, \beta_2 \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$ such that

$$\mu(\beta_1 - \beta_2; cr) \not\succ \mathcal{N}(\varepsilon) \text{ for } c > 2. \tag{3.2}$$

As $\beta_1, \beta_2 \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^{r}(y)$ then for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in \mathbb{R}^+$. Define,

$$S_1 = \left\{m \in \mathbb{N} : \mu\left(y_m - \beta_1; r + \frac{s}{2}\right) \not\succ \mathcal{N}(\varepsilon)\right\}$$

$$S_2 = \left\{m \in \mathbb{N} : \mu\left(y_m - \beta_2; r + \frac{s}{2}\right) \not\succ \mathcal{N}(\varepsilon)\right\}.$$

Then,

$$\frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| \leq \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1\}\right| + \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_2\}\right|.$$

By the property of $\mathcal{I}$-convergence, we have

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| \leq \mathcal{I} - \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1\}\right| + \mathcal{I} - \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_2\}\right| = 0.$$

Thus,

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| \geq \delta\right\} \in \mathcal{I}, \text{ for all } \delta > 0.$$

Take $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$.
Let

$$K = \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| \geq \delta_1\right\} \in \mathcal{I}.$$

Now for $n \notin K$

$$\frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \in S_1 \cup S_2\}\right| < 1 - \delta_1,$$

$$\frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : m \notin S_1 \cup S_2\}\right| \geq 1 - (1 - \delta_1) = \delta_1.$$

This implies $\{m \in \mathbb{N} : m \notin S_1 \cup S_2\} \neq \emptyset$. Then for $m \in S_1^c \cap S_2^c$ we have

$$\mu(\beta_1 - \beta_2; 2r + s) \succeq \Omega\left(\mu\left(y_m - \beta_2 : r + \frac{s}{2}\right), \mu\left(y_m - \beta_1; r + \frac{s}{2}\right)\right) \succ \mathcal{N}(\varepsilon).$$

Hence,

$$\mu(\beta_1 - \beta_2; 2r + s) \succ \mathcal{N}(\varepsilon). \tag{3.3}$$

Therefore, from (3.3) we have $\mu(\beta_1 - \beta_2; cr) \succ \mathcal{N}(\varepsilon)$ for $c > 2$. which results contradiction to (3.2). Consequently, there fails to exist the elements with condition $\mu(\beta_1 - \beta_2; cr) \not\succ \mathcal{N}(\varepsilon)$ for $c > 2$. □

Definition 3.4 *Consider $(\mathbb{Y}, \mu, \Omega)$ as a $\mathcal{L}$-fuzzy normed space. Then $\gamma \in \mathbb{Y}$ is said to be deffered rough $\mathcal{I}$-statistical cluster point of the sequence $y = \{y_m\} \in \mathbb{Y}$ corresponding to fuzzy norm μ for some $r > 0$ if for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in \mathbb{R}^+$*

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m - \gamma; r + s) \not\succ \mathcal{N}(\varepsilon)\}\right| < \delta\right\} \notin \mathcal{I}.$$

In this case, γ is known as deffered rough ideal statistically cluster point of a sequence $y = \{y_m\}$.

Let $\varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(y)$ represents the collection of deffered rough ideal statistically cluster points of sequence $y = \{y_m\}$ corresponding to fuzzy norm μ of sequence $y = \{y_m\}$ in a $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$. If $r = 0$, then we get deffered ideal statistically cluster point corresponding to fuzzy norm μ in $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$ *i.e* $\varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(y) = \varGamma^{D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(y)$.

Theorem 3.7 *Let $(\mathbb{Y}, \mu, \Omega)$ be a $\mathcal{L}$-fuzzy normed space. Then $\varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)(\mu)}(x)$, set of all r-$\mathcal{I}$-statistical-cluster points corresponding to fuzzy norm μ of a sequence $x = \{x_m\}$ is closed for some $r > 0$.*

Proof: If $\varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(\mu)(x) = \emptyset$, then we have nothing to prove.
Assume, $\varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(\mu)(x) \neq \emptyset$. Take a sequence $y = \{y_m\} \subseteq \varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(\mu)(x)$ such that $y_m \xrightarrow{\mu} y_0$. It is sufficient to show that $y_0 \in \varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(x)$.
As $y_m \xrightarrow{\mu} y_0$, then for every $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ and $s \in \mathbb{R}^+$, there exists $m_\varepsilon \in \mathbb{N}$ such that $\mu\left(y_m - y_0; \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)$ for $m \geq m_\varepsilon$.
Let us choose some $m_0 \in \mathbb{N}$ such that $m_0 \geq m_\varepsilon$. Then we have $\mu\left(y_{m_0} - y_0; \frac{s}{2}\right) \succ \mathcal{N}(\varepsilon)$.
Again as $y = \{y_m\} \subseteq \varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(x)$, we have $y_{m_0} \in \varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(x)$.

$$\Longrightarrow \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\left\{m \in \mathbb{N} : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}\right| < \delta\right\} \notin \mathcal{I}. \tag{3.4}$$

Consider $G = \left\{m \in \mathbb{N} : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}$. Choose $j \in G^c$, then we have $\mu(x_j - y_{m_0}; r + \frac{s}{2}) \succ \mathcal{N}(\varepsilon)$.
Now,

$$\begin{aligned} \mu\left(x_j - y_0; r + s\right) &\succeq \Omega\left(\mu\left(x_j - y_{m_0}; r + \frac{s}{2}\right), \mu\left(y_{m_0} - y_0; r + \frac{s}{2}\right)\right) \\ &\succ \mathcal{N}(\varepsilon). \end{aligned}$$

Hence
$\left\{m \in \mathbb{N} : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \succ \mathcal{N}(\varepsilon)\right\} \subseteq \{m \in \mathbb{N} : \mu(x_m - y_0; r + s) \succ \mathcal{N}(\varepsilon)\}$.
$\Longrightarrow \{m \in \mathbb{N} : \mu(x_m - y_0; r + s) \nsucc \mathcal{N}(\varepsilon)\} \subseteq \left\{m \leq n : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}$.
Thus,

$$\begin{aligned} &\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(x_m - y_0; r + s) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta\} \\ &\qquad \subseteq \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\left\{m \in \mathbb{N} : \mu(x_m - y_{m_0}; r + \frac{s}{2}) \nsucc \mathcal{N}(\varepsilon)\right\}\right| < \delta\right\}. \end{aligned}$$

From (3.4), we have

$$\left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(x_m - y_0; r + \varepsilon) \nsucc \mathcal{N}(\varepsilon)\}\right| < \delta\right\} \notin \mathcal{I}.$$

Therefore, $y_0 \in \varGamma^{r-D_{p,q}}_{St^{\mathcal{I}}_{\mathcal{L}}(\mu)}(x)$. Hence, the result proved. □

Theorem 3.8 *Let $y = \{y_m\}$ be a sequence in $\mathcal{L}$-fuzzy normed space $(\mathbb{Y}, \mu, \Omega)$, which is deffered ideal statistically convergent to ζ^* and $\overline{B\left(\zeta^*, \varepsilon, r\right)} = \{y \in \mathbb{Y} : \mu\left(y - \zeta^*; r\right) \nsucc \mathcal{N}(\varepsilon)\}$ is a closed ball for some $r > 0$ and $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$ then $\overline{B\left(\zeta^*, \varepsilon, r\right)} \subset \mathcal{I} - St^{\mathcal{L}}_{\theta} - LIM^r_\mu(y)$.*

Proof: Since y_m is an deffered ideal statistically convergent to ζ^* corresponding to fuzzy norm μ *i.e* $y_m \xrightarrow{D_{p,q}-St^{\mathcal{I}}_{\mathcal{L}}} \zeta^*$, then for $\varepsilon \in \mathcal{L} - \{0_{\mathcal{L}}\}$, $\delta > 0$ and $s \in \mathbb{R}^+$

$$S = \left\{n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : \mu(y_m - \zeta^*; s) \nsucc \mathcal{N}(\varepsilon)\}\right| > \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is admissible so $G = \mathbb{N} \setminus S \neq \emptyset$, then for $m \in G^c$,

$$\frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu(y_m - \zeta^*; s) \not\succ \mathcal{N}(\varepsilon)\}\right| < \delta.$$

$$\Rightarrow \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu(y_m - \zeta^*; s) \succ \mathcal{N}(\varepsilon)\}\right| \geq 1-\delta.$$

Put $C = \{m \in \mathbb{N} : \mu\left(y_m - \zeta^*; s\right) \succ \mathcal{N}(\varepsilon)\}$ for $j \geq m$. For $j \in C$, $\mu\left(y_j - \zeta^*; s\right) \succ \mathcal{N}(\varepsilon)$.
Let $y^* \in \overline{B\left(\zeta^*, \lambda, r\right)}$. We will prove $y \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_y^r$

$$\begin{aligned} \mu(y_j - y^*; r+s) &\succeq \Omega\left(\mu(y_j - \zeta^*, \varepsilon), \mu(y^* - \zeta^*, r)\right) \\ &\succ \mathcal{N}(\varepsilon). \end{aligned}$$

Hence, $C \subset \{m \in \mathbb{N} : \mu\left(y_m - y^*; r+s\right) \succ \mathcal{N}(\varepsilon)\}$, which implies that

$$1-\delta \leq \frac{|C|}{q(n)-p(n)} \leq \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu\left(y_m - y^*; r+s\right) \succ \mathcal{N}(\varepsilon)\}\right|.$$

Therefore, $\frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu\left(y_m - y^*; r+s\right) \not\succ \mathcal{N}(\varepsilon)\}\right| < \delta$.
Then,

$$\left\{n \in \mathbb{N} : \frac{1}{q(n)-p(n)} \left|\{m \in \mathbb{N} : \mu\left(y_m - y^*; r+s\right) \not\succ \mathcal{N}(\varepsilon)\}\right| \geq \delta\right\} \subset S \in \mathcal{I}.$$

which gives that $y^* \in D_{p,q} - St_{\mathcal{L}}^{\mathcal{I}} - LIM_{\mu}^r(y)$ in $(\mathbb{Y}, \mu, \Omega)$. □

Acknowledgments

We would like to express our sincere gratitude to the referees for their suggestions to enhanced the quality of this work.

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