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Contraction Principle on Bicomplex Valued Multiplicative Metric Spaces

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ABSTRACT: This paper aims to establish a foundation for the concepts of pairwise We introduce bicomplex valued multiplicative metric spaces in this study. We have updated the general background of bicomplex valued metric space and illustrated several well-known fixed point results in our study of complete bicomplex valued multiplicative metric space. Consequently, we have acquired some new findings concerning the complete multiplicative metric spaces of bicomplex values. We looked into fixed points in the bicomplex valued multiplicative metric space using the well-known Cauchy criteria Banach and Kannan contraction. Moreover, we give sufficient requirements for the common fixed point of a pair of contractive mappings in bicomplex valued multiplicative metric spaces. We also present several non-trivial cases to support the accuracy of our established findings.

Key Words: Contraction mapping, fixed point, common fixed point, multiplicative metric space.

Contents

1. Introduction

Segre [19] endeavored to create special algebras in a novel way. He proposed bicomplex numbers, tricomplex numbers, and other commutative generalizations of complex numbers as components of an endless collection of algebras. As a result Price [17] pioneered the field of function theory and bicomplex algebra. This topic holds significant relevance in various mathematical science domains, as well as other domains within the realm of science and technology, which have recently garnered a resurgence of interest. One notable study on the fundamental functions involving bicomplex numbers have been conducted by Luna-Elizarraras et al. [16].

Studies on the principle of Banach contraction [3] are ongoing and are a well-liked and useful tool for resolving problems in many mathematical analysis areas. Grossman and Katz [15] introduced multiplicative calculus, also referred to as non-Newtonian calculus, in the year 1972 by replacing the addition and subtraction functions with the multiplication function. In the year 2008, Bashirov et al. [7] defined the concept of real valued multiplicative metric by applying ideas from Grossman and Katz [15]. Ozavsar and Cevikel [12] discovered many fixed point theorems for contraction mappings of multiplicative metric spaces in 2012 as part of further study on the topological features of multiplicative metric spaces. In this regard, the following have made a significant contribution: For a characterization of multiplicative metric completeness in the year 2016, etc., see Rome and Sarwar [13].

Furthermore, a few common fixed point theorems were demonstrated by Choi et al. [8] concerning two mappings in bicomplex valued metric spaces that are poorly compatible. Jebril et al. [9] established a few common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued multiplicative metric spaces. This article continues these works. We show fixed point theorems for two contractive type mappings that fulfill a rational inequality and then analyze the bicomplex valued multiplicative metric spaces.

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Subsequently, we delineate fundamental concepts and symbols for subsequent use. We define $\mathbb{C}_0, \mathbb{C}_1$, and $\mathbb{C}_2$, respectively, to represent the set of real, complex, and bicomplex numbers.

2. Preliminaries

Bicomplex Number: The bicomplex number, as stated by Segre [19], is:

$$\bar{\xi} = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote the set of bicomplex numbers $\mathbb{C}_2$ as follows:

$$\mathbb{C}_2 = \left\{ \bar{\xi} : \bar{\xi} = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0 \right\},$$

i.e.,

$$\mathbb{C}_2 = \left\{ \bar{\xi} : \bar{\xi} = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1 \right\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.
In $\mathbb{C}_2$, there are four idempotent elements: $0, 1; e_1 = \frac{1+i_1 i_2}{2}$; and $e_2 = \frac{1-i_1 i_2}{2}$ out of which the nontrivial components e_1 and e_2, such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. One unique way to express every bicomplex number $z_1 + i_2 z_2$ is as the sum of e_1 and e_2, specifically

$$\bar{\xi} = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2)\, e_2.$$

Considering that any bicomplex number is represented by $\bar{\xi} = z_1 + i_2 z_2$. Consequently,

$$e^{\bar{\xi}} = e^{z_1}(cos(z_2) + i_2 sin(z_2))$$

defines the exponential function of a bicomplex variable by Luna-Elizarraras et al. [16].
Next, the idempotent depiction of bicomplex numbers and the complex coefficients $\bar{\xi}_1 = (z_1 - i_1 z_2)$ are the names given to this representation of $\bar{\xi}$. The bicomplex numbers $\bar{\xi}$ have idempotent components, denoted by and $\bar{\xi}_2 = (z_1 + i_1 z_2)$.
The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\bar{\xi}\| = \|z_1 + i_2 z_2\| &= \left\{ |z_1|^2 + |z_2|^2 \right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2} \right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where $\bar{\xi} = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.
The linear space $\mathbb{C}_2$ with respect to the defined norm, called as Euclidean norm, is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space.
The definition of $\precsim_{i_2}$, the partial order relation on $\mathbb{C}_2$, is:
If $\mathbb{C}_2$ is the set of bicomplex numbers, and $\bar{\xi} = z_1 + i_2 z_2, \bar{\eta} = w_1 + i_2 w_2 \in \mathbb{C}_2$, then $\bar{\xi} \precsim_{i_2} \bar{\eta}$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$,

that is, $\bar{\xi} \precsim_{i_2} \bar{\eta}$, if any of the subsequent circumstances holds true:

(1) $z_1 = w_1, z_2 = w_2$,
(2) $z_1 \prec w_1, z_2 = w_2$,
(3) $z_1 = w_1, z_2 \prec w_2$, and
(4) $z_1 \prec w_1, z_2 \prec w_2$.

Specifically, we have employed certain norm characteristics and inequality that are often utilised in numerous works [10] [8].

Bicomplex valued metric space: The bicomplex valued metric spaces were defined by Choi et al. [8] as:

Definition 2.1 [8] Let $P(\neq \emptyset)$ be any set. Suppose the mapping $\eth : P \times P \to \mathbb{C}_2$ satisfies the following conditions:

1. $0 \precsim_{i_2} \eth(p,q)$ for all $p, q \in P$ (positivity),
2. $\eth(p,q) = 0$ if and only if $p = q$,
3. $\eth(p,q) = \eth(q,p)$ for all $p, q \in P$ (symmetry) and
4. $\eth(p,q) \precsim_{i_2} \eth(p,z) + \eth(z,q)$ for all $p, q, z \in P$ (triangle inequility).

Then the pair $(P, \eth)$ is called the bicomplex valued metric spaces.

Example 2.1 [10] Consider $P = \left\{0, \frac{1}{2}, 2\right\}$, define a bicomplex valued multiplicative metric $\eth : P \times P \to \mathbb{C}_2$ by $\eth(p,q) = (1 + i_2)\,|p - q|, \forall p, q \in P$.

According to the definition above of $\eth_m$ it is simple to confirm that $(P, \eth_m)$ is a bicomplex valued metric space.

3. Main Results

The concept of bicomplex valued multiplicative metric space, which is a generalisation of the concept of bicomplex valued metric space, was introduced in this portion of the study. The following is the definition of a bicomplex valued multiplicative metric space:

Definition 3.1 Let X be a nonempty set. Suppose the mapping $\eth_m : X \times X \to \mathbb{C}_2$ satisfies the following conditions:

1. $1 \precsim_{i_2} \eth_m(p,q)$ for all $p, q \in P$ (positivity),
2. $\eth_m(p,q) = 1$ if and only if $p = q$,
3. $\eth_m(p,q) = \eth_m(q,p)$ for all $p, q \in P$ (symmetry) and
4. $\eth_m(p,q) \precsim_{i_2} \eth_m(p,x) \cdot \eth_m(x,q)$ for all $p, q, x \in P$ (triangle inequility).

Then the pair $(P, \eth_m)$ is called bicomplex valued multiplicative metric space.

Example 3.1 Consider $P = \left\{0, \frac{1}{2}, 2\right\}$, define a bicomplex valued multiplicative metric $\eth_m : P \times P \to \mathbb{C}_2$ by $\eth_m(p,q) = e^{(1+i_2)|p-q|}, \forall p, q \in P$.

From the above definition of $\eth_m$ one can easily verify that $(P, \eth_m)$ is a bicomplex valued multiplicative metric spaces.

Definition 3.2 For a bicomplex valued multiplicative metric spaces $(P, \eth_m)$

1. A sequence $\{\vartheta_n\}$ in P is referred to as a convergent sequence and converges to a point x if for any $1 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $\eth_m(\vartheta_n, x) \prec_{i_2} r$ for all $n > n_0$ and we write $\lim_{n\to\infty} \vartheta_n = \vartheta$ or $\vartheta_n \to \vartheta$, as $n \to \infty$.
2. A sequence $\{\vartheta_n\}$ in P is referred to as a Cauchy sequence in $(P, \eth_m)$ if for any $1 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $\eth_m(\vartheta_n, \vartheta_{n+m}) \prec_{i_2} r$ for all $m, n \in \mathbb{N}$ and $n > n_0$.
3. A bicomplex valued multiplicative metric spaces $(P, \eth_m)$ is consider to be complete if every Cauchy sequence is convergent in P.

Lemma 3.1 *Let $(P, \eth_m)$ be a bicomplex valued multiplicative metric spaces and $\{\vartheta_n\}$ be a sequence in P. Then $\{\vartheta_n\}$ is convergent sequence and converges to a point ϑ if and only if* $\lim_{n\to\infty} \|\eth_m(\vartheta_n, \vartheta)\| = 1$.

Proof: *Let (ϑ_n) be a convergent sequence and converges to a point ϑ and let $\varepsilon > 1$ be any real number. Suppose*

$$\bar{\xi} = \frac{\varepsilon}{2} + i_1\frac{\varepsilon}{2} + i_2\frac{\varepsilon}{2} + i_1 i_2\frac{\varepsilon}{2}.$$

Then clearly $1 \prec_{i_2} \bar{\xi} \in \mathbb{C}_2$ *and for this* $\bar{\xi}$ *there exists* $n_0 \in \mathbb{N}$ *such that* $Ɗ_m(\vartheta_n, \vartheta) \prec_{i_2} \bar{\xi}$ *for all* $n \geq n_0$. *Therefore, we have*

$$\|Ɗ_m(\vartheta_n, \vartheta)\| < \|\bar{\xi}\| = \varepsilon, \text{ whenever } n \geq n_0$$

. *And this implies,*

$$\lim_{n\to\infty} \|Ɗ(\vartheta_n, \vartheta)\| = 1$$

.

Conversly let $\lim_{n\to\infty} \|Ɗ(\vartheta_n, \vartheta)\| = 1$. *Then for* $1 \prec_{i_2} \bar{\xi} \in \mathbb{C}_2$, *there exists a real number* $\varepsilon > 1$, *such that for all* $r \in \mathbb{C}_2$

$$\|\bar{\xi}\| < \varepsilon \implies \bar{\xi} \prec_{i_2} r.$$

Then for this $\varepsilon > 1$, *there exists a natural number* $n_0 \in \mathbb{N}$ *such that*

$$\|Ɗ_m(\vartheta_n, \vartheta)\| < \varepsilon \quad \forall n \in n_0.$$

Hence $\{\vartheta_n\}$ *converges to a point* ϑ. □

Lemma 3.2 *Let* $(P, Ɗ_m)$ *be a bicomplex valued multiplicative metric spaces and* $\{\vartheta_n\}$ *be a sequence in* P. *Then* $\{\vartheta_n\}$ *is a Cauchy sequence in* P *if and only if* $\lim_{n,m\to\infty} \|Ɗ_m(\vartheta_n, \vartheta_{n+m})\| = 1$.

Proof: *Let* (ϑ_n) *be a multiplicative Cauchy sequence and let* $\varepsilon > 1$ *be any real number. Suppose*

$$\bar{\xi} = \frac{\varepsilon}{2} + i_1\frac{\varepsilon}{2} + i_2\frac{\varepsilon}{2} + i_1 i_2\frac{\varepsilon}{2}.$$

Then clearly $1 \prec_{i_2} \bar{\xi} \in \mathbb{C}_2$ *and for this* $\bar{\xi}$ *there exists* $n_0 \in \mathbb{N}$ *such that* $Ɗ_m(\vartheta_n, \vartheta_m) \prec_{i_2} \bar{\xi}$ *for all* $m, n \geq n_0$. *Therefore, we have*

$$\|Ɗ_m(\vartheta_n, \vartheta_m)\| < \|\bar{\xi}\| = \varepsilon, \text{ whenever } n, m \geq n_0$$

. *And this implies,*

$$\lim_{m,n\to\infty} \|Ɗ(\vartheta_n, \vartheta_m)\| = 1$$

.

Conversly let $\lim_{m,n\to\infty} \|Ɗ(\vartheta_n, \vartheta_m)\| = 1$. *Then for* $1 \prec_{i_2} \bar{\xi} \in \mathbb{C}_2$, *there exists a real number* $\varepsilon > 1$, *such that for all* $r \in \mathbb{C}_2$

$$\|\bar{\xi}\| < \varepsilon \implies \bar{\xi} \prec_{i_2} r.$$

Then for this $\varepsilon > 1$, *there exists a natural number* $n_0 \in \mathbb{N}$ *such that*

$$\|Ɗ_m(\vartheta_n, \vartheta_m)\| < \varepsilon \quad \forall n, m \in n_0.$$

Hence $\{\vartheta_n\}$ *is a Cauchy sequence.* □

Theorem 3.1 *Let* $(P, Ɗ_m)$ *be a complete bicomplex valued multiplicative metric space and* $T : P \to P$ *be a mapping satisfying:*

$$Ɗ_m(T\vartheta, Ty) \precsim_{i_2} (Ɗ_m(\vartheta, y))^{\alpha} \tag{3.1}$$

for all $\vartheta, y \in P$, *where* $\alpha \in [0, 1)$. *Then* T *has a unique fixed point.*

Proof: *Let* T *satisfy Equation (3.1),* $\vartheta_0 \in P$ *be an arbitrary point and define the sequence* $\{\vartheta_n\}$ *by* $\vartheta_n = T^n\vartheta_0$. *From (3.1), we get*

$$\begin{aligned} Ɗ_m(\vartheta_n, \vartheta_{n+1}) &= Ɗ_m(T\vartheta_{n-1}, T\vartheta_n) \\ &\precsim_{i_2} (Ɗ_m(\vartheta_{n-1}, \vartheta_n))^{\alpha}. \end{aligned} \tag{3.2}$$

Using again Equation (3.1), we have

$$\daleth_m(\vartheta_{n-1},\vartheta_n) \precsim_{i_2} \left(\daleth_m(\vartheta_{n-2},\vartheta_{n-1})\right)^{\alpha},$$

and by Equation (3.2), we get

$$\daleth_m(\vartheta_n,\vartheta_{n+1}) \precsim_{i_2} \left(\daleth_m(\vartheta_{n-2},\vartheta_{n-1})\right)^{\alpha^2}.$$

If we continue this process, we obtain

$$\daleth_m(\vartheta_n,\vartheta_{n+1}) \precsim_{i_2} \left(\daleth_m(\vartheta_0,\vartheta_1)\right)^{\alpha^n}. \tag{3.3}$$

Using triangle inequility and Equation (3.3) for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
\daleth_m(\vartheta_n,\vartheta_m) &\precsim_{i_2} \daleth_m(\vartheta_n,\vartheta_{n+1})\cdot\daleth_m(\vartheta_{n+1},\vartheta_m)\\
&\precsim_{i_2} \daleth_m(\vartheta_n,\vartheta_{n+1}\cdot\daleth_m(\vartheta_{n+1},\vartheta_{n+2})\cdot\daleth_m(\vartheta_{n+2},\vartheta_m)\\
&\precsim_{i_2} \daleth_m(\vartheta_n,\vartheta_{n+1})\cdot\daleth_m(\vartheta_{n+1},\vartheta_{n+2})\cdot\daleth_m(\vartheta_{n+2},\vartheta_{n+3})\cdot\ \cdots\ \cdot\daleth_m(\vartheta_{m-1},\vartheta_m)\\
&\precsim_{i_2} (\daleth_m(\vartheta_0,\vartheta_1))^{\alpha^n}\cdot(\daleth_m(\vartheta_0,\vartheta_1))^{\alpha^{n+1}}\cdot(\daleth_m(\vartheta_0,\vartheta_1))^{\alpha^{n+2}}\cdot\cdots\cdot(\daleth_m(\vartheta_0,\vartheta_1))^{\alpha^{m-1}}\\
&\precsim_{i_2} (\daleth_m(\vartheta_0,\vartheta_1))^{(\alpha^n+\alpha^{n+1}+\alpha^{n+2}+\cdots+\alpha^{m-1})}\\
&\precsim_{i_2} (\daleth_m(\vartheta_0,\vartheta_1))^{\alpha^n[1+\alpha+(\alpha)^2+(\alpha)^3+\cdots+(\alpha)^{m-n-1}]}\\
&\precsim_{i_2} (\daleth_m(\vartheta_0,\vartheta_1))^{\frac{\alpha^n}{1-\alpha}}.
\end{aligned}$$

Thus, we have

$$\|\daleth_m(\vartheta_n,\vartheta_m)\| \le \|\daleth_m(\vartheta_0,\vartheta_1)\|^{\frac{\alpha^n}{1-\alpha}}. \tag{3.4}$$

Since $\alpha \in [0,1)$, taking limits as $n \to \infty$, then $\frac{\alpha^n}{1-\alpha} \to 0$. So from (3.4) it follows that

$$\|\daleth_m(\vartheta_0,\vartheta_1)\|^{\frac{\alpha^n}{1-\alpha}} \to 1.$$

This means that

$$\|\daleth_m(\vartheta_n,\vartheta_m)\| \to 1,\ \text{as } n \to \infty.$$

So $\{\vartheta_n\}$ is bicomplex valued Cauchy sequence by lemma (3.2). Completeness of $(P, \daleth_m)$ gives us that there is an element $u \in P$ such that $\{\vartheta_n\}$ is bicomplex valued convergent to u, i.e. $\lim_{n\to\infty}\{\vartheta_n\} = u$. Now we will show that u is a fixed point of T, i.e. $Tu = u$. For any $n \in \mathbb{N}$, we get

$$\begin{aligned}
\daleth_m(u,Tu) &\precsim_{i_2} \daleth_m(u,\vartheta_{n+1})\cdot\daleth_m(\vartheta_{n+1},Tu)\\
&= \daleth_m(u,T\vartheta_n)\cdot\daleth_m(T\vartheta_n,Tu)\\
&\precsim_{i_2} \daleth_m(u,T\vartheta_n)\cdot\daleth_m(\vartheta_n,u)^{\alpha}\\
&\precsim_{i_2} \daleth_m(u,\vartheta_{n+1})\cdot\daleth_m(\vartheta_n,u)^{\alpha}.
\end{aligned}$$

Since ϑ_n converges to u as $n \to \infty$, it follows from the latter inequality that

$$\begin{aligned}
\daleth_m(u,Tu) &\precsim_{i_2} \daleth_m(u,u)\cdot\daleth_m(u,u)^{\alpha}\\
&= 1 \text{ as } \daleth_m(u,u) = 1.
\end{aligned}$$

It follows that

$$\|\mathfrak{D}_m(u, Tu)\| \leq 1,$$

which is a contradiction as $\|\mathfrak{D}_m(u, Tu)\| \geq 1$. *So we conclude that* $\mathfrak{D}_m(u, Tu) = 1$, *i.e.* $Tu = u$. *Therefore* u *is a fixed point of* T.
Finally, we prove the uniqueness. Let $w \neq u$ *be another fixed point of* T, *i.e* $Tw = w$. *Using* (3.1), *we have*

$$\mathfrak{D}_m(u, w) = \mathfrak{D}_m(Tu, Tw) \precsim_{i_2} (\mathfrak{D}_m(u, w))^{\alpha}.$$

Hence

$$\|\mathfrak{D}_m(u, w)\| \leq (\|\mathfrak{D}_m(u, w)\|)^{\alpha}$$

which is a contradiction as $\alpha \in [0, 1)$. *Thus, we conclude that* $u = w$ *and so* u *is a unique fixed point of* T. □

Example 3.2 In the example (3.1), let we consider a mapping $T : P \to P$ such as follows:

$$T(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0, & \text{if } x = \frac{1}{2} \\ \frac{1}{2}, & \text{if } x = 2 \end{cases}$$

Let us choose any positive real number k such that $0.34 \leq \alpha < 1$. Then T satisfy $\mathfrak{D}_m(Tx, Ty) \precsim_{i_2} (\mathfrak{D}_m(x, y))^{\alpha}$ holds for all $x, y \in P$, where $0.34 \leq \alpha < 1$. Hence $x = 0$ is the unique fixed point of T.

Theorem 3.2 *Let* $(P, \mathfrak{D}_m)$ *be a complete bicomplex valued multiplicative metric space and let* $T : P \to P$ *and* $G : P \to P$ *be two mapping that satisfies:*

$$\mathfrak{D}_m(T\vartheta, Gy) \precsim_{i_2} (\mathfrak{D}_m(T\vartheta, \vartheta) \cdot \mathfrak{D}_m(Gy, y))^{\alpha} \tag{3.5}$$

for all $\vartheta, y \in P$, *where* α *be any real number with* $0 \leq 2\alpha < 1$. *Then* T *and* G *have a common unique fixed point on* P.

Proof: *Suppose that* $\bar{\vartheta}_0 \in X$, *then we set*

$$\begin{array}{rcl} \vartheta_1 = T(\vartheta_0) & , & \vartheta_2 = G(\bar{\vartheta}_1) \\ \vartheta_3 = T(\vartheta_2) & , & \vartheta_4 = G(\vartheta_3) \\ \vdots & & \vdots \\ \vartheta_{2n+1} = T(\vartheta_{2n}) & , & \vartheta_{2n+2} = G(\vartheta_{2n+1}). \end{array}$$

Then by using (3.5) *we get*

$$\begin{aligned} \mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n+2}) &= \mathfrak{D}_m(T\vartheta_{2n}, G\vartheta_{2n+1}) \\ &\precsim_{i_2} (\mathfrak{D}_m(T\vartheta_{2n}, \vartheta_{2n}) \cdot \mathfrak{D}_m(G\vartheta_{2n+1}, \vartheta_{2n+1}))^{\alpha} \\ &= (\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n}) \cdot \mathfrak{D}_m(\vartheta_{2n+2}, \vartheta_{2n+1}))^{\alpha} \\ &= (\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n}) \cdot \mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n+2}))^{\alpha} \\ \implies (\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n+2}))^{1-\alpha} &\precsim_{i_2} (\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n}))^{\alpha} \\ \implies \mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n+2}) &\precsim_{i_2} (\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n}))^{\frac{\alpha}{1-\alpha}}. \end{aligned} \tag{3.6}$$

Let $\frac{\alpha}{1-\alpha} = \beta$. *So clearly* $\beta < 1$, *as* $2\alpha < 1$. *Now from* (3.6) *we get*

$$\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n+2}) \precsim_{i_2} (\mathfrak{D}_m(\vartheta_{2n+1}, \vartheta_{2n}))^{\beta}. \tag{3.7}$$

Letting $2n = p$, so from equation (3.7) it follows that

$$\begin{aligned}
\daleth_m(\vartheta_{p+1}, \vartheta_{p+2}) &\precsim_{i_2} (\daleth_m(\vartheta_{p+1}, \vartheta_p))^{\beta} \\
&\precsim_{i_2} (\daleth_m(\vartheta_p, \vartheta_{p-1}))^{\beta^2} \\
&\vdots \\
&\precsim_{i_2} (\daleth_m(\vartheta_1, \vartheta_0))^{\beta^{p+1}}.
\end{aligned} \tag{3.8}$$

By using triangle inequility and equation (3.8) for all $p, q \in \mathbb{N}$ with $p < q$, we have

$$\begin{aligned}
\daleth_m(\vartheta_p, \vartheta_q) &\precsim_{i_2} \daleth_m(\vartheta_p, \vartheta_{p+1}) \cdot \daleth_m(\vartheta_{p+1}, \vartheta_q) \\
&\precsim_{i_2} \daleth_m(\vartheta_p, \vartheta_{p+1}) \cdot \daleth_m(\vartheta_{p+1}, \vartheta_{p+2}) \cdot \daleth_m(\vartheta_{p+2}, \vartheta_q) \\
&\precsim_{i_2} \daleth_m(\vartheta_p, \vartheta_{p+1}) \cdot \daleth_m(\vartheta_{p+1}, \vartheta_{p+2}) \cdot \daleth_m(\vartheta_{p+2}, \vartheta_{p+3}) \cdot \ \cdots \ \cdot \daleth_m(\vartheta_{q-1}, \vartheta_q) \\
&\precsim_{i_2} (\daleth_m(\vartheta_0, \vartheta_1))^{\beta^p} \cdot (\daleth_m(\vartheta_0, \vartheta_1))^{\beta^{p+1}} \cdot (\daleth_m(\vartheta_0, \vartheta_1))^{\beta^{p+2}} \cdot \cdots \cdot (\daleth_m(\vartheta_0, \vartheta_1))^{\beta^{q-1}} \\
&= (\daleth_m(\vartheta_0, \vartheta_1))^{(\beta^p + \beta^{p+1} + \beta^{p+2} + \cdots + \beta^{q-1})} \\
&= (\daleth_m(\vartheta_0, \vartheta_1))^{\beta^p[1 + \beta + (\beta)^2 + (\beta)^3 + \cdots + (\beta)^{q-p-1}]} \\
&\precsim_{i_2} (\daleth_m(\vartheta_0, \vartheta_1))^{\frac{\beta^p}{1-\beta}}.
\end{aligned}$$

Thus, we have

$$\|\daleth_m(\vartheta_p, \vartheta_q)\| \leq \|\daleth_m(\vartheta_0, \vartheta_1)\|^{\frac{\beta^n}{1-\beta}}. \tag{3.9}$$

Since $\beta \in [0, 1)$, taking limits as $p \to \infty$, then $\frac{\beta^n}{1-\beta} \to 0$. So from (3.9) it follows that

$$\|\daleth_m(\vartheta_0, \vartheta_1)\|^{\frac{\beta^n}{1-\beta}} \to 1.$$

This means that

$$\|\daleth_m(\vartheta_p, \vartheta_q)\| \to 1, \text{ as } p, q \to \infty.$$

So $\{\vartheta_p\}$ is bicomplex valued Cauchy sequence by lemma (3.2). Completeness of $(P, \daleth_m)$ gives us that there exist an element $v \in P$ such that $\{\vartheta_n\}$ is bicomplex valued convergent to v, i.e. $\lim_{p\to\infty} \{\vartheta_p\} = v \implies \lim_{n\to\infty} \{\vartheta_{2n}\} = v$.
Now we will show that u is a fixed point of T, i.e. $Tu = u$. For any $n \in \mathbb{N}$, we get

$$\begin{aligned}
\daleth_m(Tu, u) &\precsim_{i_2} \daleth_m(Tu, \vartheta_{2n}) \cdot \daleth_m(\vartheta_{2n}, u) \\
&= \daleth_m(Tu, G\vartheta_{2n-1}) \cdot \daleth_m(\vartheta_{2n}, u) \\
&\precsim_{i_2} (\daleth_m(Tu, u) \cdot \daleth_m(G\vartheta_{2n-1}, \vartheta_{2n-1}))^{\alpha} \cdot \daleth_m(\vartheta_{2n}, u) \\
&\precsim_{i_2} (\daleth_m(Tu, u))^{\alpha} \cdot (\daleth_m(\vartheta_{2n}, \vartheta_{2n-1}))^{\alpha} \cdot \daleth_m(\vartheta_{2n}, u) \\
&\precsim_{i_2} (\daleth_m(Tu, u))^{\alpha} \cdot (\daleth_m(\vartheta_1, \vartheta_0))^{\alpha^{2n-1}} \cdot \daleth_m(\vartheta_{2n}, u)
\end{aligned}$$

Since $\alpha^{2n-1} \to 0$ and ϑ_n converges to u when $n \to \infty$, then it follows from the latter inequality that

$$\begin{aligned}
\daleth_m(Tu, u) &\precsim_{i_2} (\daleth_m(Tu, u))^{\alpha} \cdot \daleth_m(u, u), \text{ as } \lim_{n\to\infty} (\daleth_m(\vartheta_1, \vartheta_0))^{\alpha^{2n-1}} = 1 \\
&\precsim_{i_2} (\daleth_m(Tu, u))^{\alpha}, \text{ as } \daleth_m(u, u) = 1 \\
\implies (\daleth_m(Tu, u))^{1-\alpha} &\precsim_{i_2} 1 \\
\implies (\daleth_m(Tu, u)) &\precsim_{i_2} (1)^{\frac{1}{1-\alpha}} = 1.
\end{aligned}$$

It follows that

$$\|\mathrm{D}_m(Tu,u)\| \leq 1,$$

which contradicts the fact that $\|\mathrm{D}_m(Tu,u)\| \geq 1$*. So we conclude that* $\mathrm{D}_m(Tu,u)=1$*, i.e.* $Tu=u$*. Therefore* u *is a fixed point of* T*. Similarly we can show that* u *is a fixed point of* G*, i.e.* $Gu=u$*. Hence* u *is a common fixed point of* T *and* G*, i.e.* $Tu=Gu=u$*.*
Finally, we prove the uniqueness. Let $w \neq u$ *be a another common fixed point of* T *and* G*, i.e* $Tw=w=Gw$*. Using (3.1), we have*

$$\begin{aligned} \mathrm{D}_m(u,w) &= \mathrm{D}_m(Tu,Gw) \\ &\precsim_{i_2} (\mathrm{D}_m(Tu,u)\cdot \mathrm{D}_m(Gw,w))^{\alpha} \\ &= (1)^{\alpha}=1 \end{aligned}$$

and

$$\|\mathrm{D}_m(u,w)\| \leq 1.$$

which is a contradiction. Thus, we conclude that $\|\mathrm{D}_m(u,w)\|=1 \implies \mathrm{D}_m(u,w)=1 \implies u=w$ *and so* u *is a unique common fixed point of* T *and* G*. Hence* T *and* G *have a unique common fixed point on* P*.* □

Corollary 3.1 *Let* (P,D_m) *be a complete bicomplex valued multiplicative metric space and let* $T:P\to P$ *be a self mapping that satisfying:*

$$\mathrm{D}_m(T\vartheta,Ty) \precsim_{i_2} (\mathrm{D}_m(T\vartheta,\vartheta)\cdot \mathrm{D}_m(Ty,y))^{\alpha} \tag{3.10}$$

for all $\vartheta, y \in P$*, where* α *be any real number with* $0 \leq 2\alpha < 1$*. Then* T *has a unique fixed point on* P*.*
Proof: *We can easily prove this result by applying the theorem (3.2) and taking* $T=S$*.* □

Theorem 3.3 *Let* (P,D_m) *be a complete bicomplex valued multiplicative metric space and let* $T:P\to P$ *be a self mapping that satisfying:*

$$\mathrm{D}_m(T^n\vartheta,T^ny) \precsim_{i_2} (\mathrm{D}_m(T^n\vartheta,\vartheta)\cdot \mathrm{D}_m(T^ny,y))^{\alpha} \tag{3.11}$$

for all $\vartheta, y \in P$*, where* α *be any real number with* $0 \leq 2\alpha < 1$*. Then* T *has a unique fixed point on* P*.*

Proof: *By corollary (3.1) there is a unique fixed point* $\vartheta \in P$ *such that*

$$T\vartheta = \vartheta \tag{3.12}$$

$$\text{i.e. } T^2\vartheta = T(T\vartheta)=T\vartheta=\vartheta$$
$$\vdots$$
$$\text{i.e. } T^n\vartheta = \vartheta. \tag{3.13}$$

Therefore we have

$$\begin{aligned} &\mathrm{D}_m(T\vartheta,\vartheta)=\mathrm{D}_m(TT^n\vartheta,T^n\vartheta)=\mathrm{D}_m(T^nT\vartheta,T^n\vartheta)\precsim_{i_2}(\mathrm{D}_m(T^nT\vartheta,T\vartheta)\cdot \mathrm{D}_m(T^n\vartheta,\vartheta))^{\alpha} \\ &\text{i.e. } \mathrm{D}_m(T\vartheta,\vartheta)\precsim_{i_2}(\mathrm{D}_m(TT^n\vartheta,T\vartheta)\cdot \mathrm{D}_m(T^n\vartheta,\vartheta))^{\alpha} \\ &\text{i.e. } \mathrm{D}_m(T\vartheta,\vartheta)\precsim_{i_2}(\mathrm{D}_m(T\vartheta,T\vartheta)\cdot \mathrm{D}_m(\vartheta,\vartheta))^{\alpha}\ [\text{by using (3.12)}] \\ &\text{i.e. } \mathrm{D}_m(T\vartheta,\vartheta)\precsim_{i_2}(\mathrm{D}_m(\vartheta,\vartheta)\cdot \mathrm{D}_m(\vartheta,\vartheta))^{\alpha}\quad [\text{by using (3.13)}] \\ &\text{i.e. } \|\mathrm{D}_m(T\vartheta,\vartheta)\|\leq 1\quad [\text{ as } \mathrm{D}_m(\vartheta,\vartheta)=1] \\ &\text{i.e. } \|\mathrm{D}_m(T\vartheta,\vartheta)\|=1\quad [\text{ as } \|\mathrm{D}_m(T\vartheta,\vartheta)\|\not< 1] \\ &\text{i.e. } T\vartheta=\vartheta. \end{aligned}$$

This complete the proof of this corollary. □

4. Conclusion and Future Prospective

This paper presents a novel concept of a complete bicomplex valued multiplicative metric space, where we have updated the general background of bicomplex valued metric space and demonstrated some well-known fixed point results. It is anticipated that this paper, which is the first on the topic, will draw researchers for additional studies and applications. The findings of our study demonstrate the singularity of a fixed point under various contraction conditions. We believe that these findings will significantly advance this area of study in the future. If we apply the concepts described in this paper to future studies on alternative metric spaces, such as bicomplex valued controlled metric space and bicomplex valued cone metric space, we may find intriguing results.

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