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Analytic solution of the generalized Bratu-type fractional differential equation utilizing the q-HATM within the framework of the Yang-Abdel-Cattani operator

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ABSTRACT: In this work, we use the Yang-Abdel-Cattani (YAC) fractional derivative operator to examine the generalized Bratu-type fractional differential equation. We use q-homotopy analysis transform method (q-HATM) which helps us to deal with fractional calculus problems much more effectively. We obtain an analytical solution of the defined problem by using the above told method. Furthermore, we investigate and analyze some special cases of the generalized fractional Bratu equation, demonstrating the versatility of the solution in many contexts with the help of a few graphical illustrations. These cases show how easily and reliably the solution technique works under different conditions. All in all, our study brings new knowledge to the study of Bratu-type equations using fractional calculus.

Key Words: q-homotopy analysis transform method, Yang-Abdel-Cattani differential operator, generalized Bratu-type fractional differential equation.

Contents

1. Introduction

A great tool for uncovering the hidden characteristics of different physical and material processes involving derivatives and integrals of complex orders is fractional calculus. The theory of fractional differential equations does a great job of translating natural reality in an organized and practical way. Fractional differential equation is quite useful for modeling the phenomena defined by linear or nonlinear differential equations ([1]- [5]). In order to develop mathematical modeling of several physical events, varieties of them are crucial tools and roles in physics, dynamical systems, control systems, engineering, and mathematics. Physical science includes mathematical physics governed by non-linear partial differential dynamical equations. Numerous phenomena in fluid mechanics, hydrodynamics, optics, and plasma physics depend on analytical solutions to these dynamical equations ([6]- [9]). Numerous methods have been used in recent years to study fractional-order partial differential equations, including the Laplace variational iteration method ([10]- [12]), the reduced differential transform method [13], the Laplace decomposition method [14], the homotopy analysis method [15], the variational iteration method ([16]- [18]), the q-homotopy analysis transform method [19,20], the homotopy perturbation transform method [21,22], and others.

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Fractional calculus allows mathematics to work with integration or differentiation for any possible order beyond 1 and 0, respectively. Where classical calculus has one important interpretation, fractional calculus can be understood in several different ways ([23]- [27]). Riemann, Kilbas, Caputo, and Samko are well-known mathematicians who worked on fractional operators with singular kernels. Other researchers, for example, Miller-Ross, Atangana-Baleanu, Yang, and Wiman, have looked into fractional operators that use non-singular kernels [28,29]. In the process of modeling with differential equations, it is necessary to separate them into two different categories. A fractional differential equation may be classified as linear or non-linear. In general, linear fractional differential equations are much simpler to solve than nonlinear ones. The literature has limited methods for resolving models with nonlinear fractional differential equations.

Numerous scientific and engineering domains employ nonlinear differential equations, including the Bratu-type equation. Many areas of science and engineering use the Bratu equation. It participates greatly in thermal combustion, discussing the ignition of fuels, heat generation in chemical reactions, and the expansion of stars described by the Chandrasekhar model. It has also been applied in chemical reaction kinetics, radiative heat transfer, and a new area like nanotechnology ([30]– [33]). The formula finds important use in environmental science, nuclear physics, materials science, studies of diffusion processes, reactor neutron transport, electrochemical systems, and pollutant dispersal. In addition to its use in certain situations, the Bratu equation gives vital information about the properties and actions of nonlinear differential equations ([34]– [36]). Under the category of differential equations in fractional calculus, the nonlinear fractional differential equation is a complex and fascinating mathematical expression. Fractional calculus makes it possible to depict complex physical processes with memory and anomalous diffusion by extending existing concepts of differentiation and integration to non-integer orders.

Many applications in science and engineering depend on the importance of the Bratu equation. The modeling capabilities of the Bratu equation is improved by generalizing it to fractional-order forms, particularly when fractional operators with generalized kernels are used [37,38]. The Yang–Abdel–Cattani (YAC) fractional derivative is one such operator that provides more versatility for characterizing memory-dependent processes [39]. However, because of their intrinsic nonlinearity and the operator's complexity, solving such generalized fractional models presents substantial analytical difficulties. The creation of precise and effective analytical techniques becomes essential in this situation. A viable approach for tackling these issues is offered by the q-Homotopy Analysis Transform Method (q-HATM), which combines the advantages of the Homotopy Analysis Method with the Laplace transform. Mittag-Leffler function helps us to expand the effects of the classical fractional Bratu model. By adding the Mittag-Leffler function, modelling becomes more flexible, and equations that correspond to certain behaviors seen in the system being studied may be customized. An effective way to improve a framework and handle the complexities of fractional calculus is to include the Mittag-Leffler function in the generalization of an equation.

The goal of this research is to use q-HATM to solve the generalized Bratu-type fractional differential equation using the YAC operator analytically. In addition to making it easier to deal with nonlinear and nonlocal terms, this method provides a way to investigate the qualitative behavior of solutions under different fractional orders and parameters, advancing fractional modeling both theoretically and practically. This work holds significance because it advances the analytical treatment of generalized operator-based nonlinear fractional differential equations. Without the drawbacks of linearization or discretization, the q-Homotopy Analysis Transform Method (q-HATM) offers a strong and effective way to derive approximate analytical solutions to such complex equations. In addition to improving the theoretical comprehension of fractional-order models, this offers a useful computational foundation for upcoming research in applied mathematics, physics, and engineering. Furthermore, this work's significance and usefulness may be expanded beyond fields by applying its methods and findings to additional nonlinear fractional models.

Many applications in science and engineering depend on the importance of the Bratu equation. After realizing how important it is, we look at a general form called the fractional Bratu-type equation. Mittag-Leffler function helps us to to expand the effects of the classical fractional Bratu model.

$$D_{\varsigma}^{2n\beta}\omega(\varsigma)+\varepsilon E_{\alpha}(\omega)=0, \qquad 0<\varsigma\leq 1,\ 0<\beta\leq 1,\ n\in \mathrm{N}. \tag{1.1}$$

Here ε represents the constant, $E_{\alpha}(\omega)$ denotes the Mittag-Leffler function together with a real parameter α, ω is the unknown function, and $D_{\varsigma}^{2n\beta}=D^{\beta}D^{\beta}\ldots D^{\beta}(2n-times)$ where $D^{\beta}\omega=\frac{d^{\beta}\omega}{d\varsigma^{\beta}}$.

In this investigation, the powerful method called q-HATM is applied to obtain solutions to the extended Bratu-type equation. By joining the Laplace transform technique with the Homotopy analysis (HAM), the q-HATM increases the rate at which the convergence rate improves. In particular, when $q=1$ reduces q-HAM to the usual HAM. To handle fractional derivatives, one uses the Laplace transform. The q-HAM method, which was suggested by El-Tawil and Huseen [20], makes HAM more flexible. If n is greater than 1, the homotopy parameter must be used. The first parameter, q, is chosen from the range $(0,1/r)$.

This paper is divided into six sections: We provide an illustrated outline in Section 1. Some fundamental definitions are discussed in Section 2. The basic q-Homotopy analysis transform technique is defined in Section 3. We outline the key findings of our study in Section 4. Section 5 discusses certain special examples of generalized Bratu-type equations with graphical representation. This study comes to its conclusion in the last section.

2. Preliminaries

Definition 2.1: [40] The Rabotnov exponential function of order β is defined as

$$\xi\left(\lambda\mu^{\beta}\right)=\sum_{n=0}^{\infty}\frac{\lambda^{n}\mu^{(n+1)(\beta+1)-1}}{\Gamma\left((n+1)\left(\beta+1\right)\right)}, \quad \mu\in\mathbb{C}, \quad \beta,\lambda\in\mathbb{R}^{+}. \tag{2.1}$$

Definition 2.2: [40] The Yang-Abdel-Cattani fractional derivative of order β for μ on $D^{1}[a,b]$, $\varsigma>0$, $\lambda\in\mathbb{R}^{+}$, $0<\beta\leqslant 1$ is defined as

$${}_{0}^{YAC}D_{\varsigma}^{\beta}\left(\mu(\varsigma)\right)=\int_{0}^{\varsigma}\xi_{\beta}\left(-\lambda(\varsigma-t)^{\beta}\right)\mu'\left(t\right)dt, \tag{2.2}$$

where, the Rabotnov exponential function of order β is denoted by the symbol ξ_{β}.

Definition 2.3: [40] The Yang-Abdel-Cattani fractional integral of order β for μ on $D^{1}[a,b]$, $\varsigma>0$, $\lambda\in\mathbb{R}^{+}$, $0<\beta\leqslant 1$ is defined as

$$I_{YAC}^{\beta}\left(\mu(\varsigma)\right)=\int_{0}^{\varsigma}\xi_{\beta}\left(-\lambda(\varsigma-t)^{\beta}\right)\mu\left(t\right)dt, \tag{2.3}$$

where, the Rabotnov exponential function of order β is denoted by the symbol ξ_{β}.

Definition 2.4: [40] The Laplace transform of the Yang-Abdel-Cattani fractional derivative is defined as

$$L\left({}_{0}^{YAC}D_{\varsigma}^{\beta}\left(\mu(\varsigma)\right)\right)=\frac{1}{s^{\beta+1}}\frac{sL\left[\mu(\varsigma)\right]-\mu(0)}{1+\lambda s^{-(\beta+1)}}. \tag{2.4}$$

Definition 2.5: [40] The Laplace transform of the Yang-Abdel-Cattani fractional integral is defined as

$$L\left(I_{YAC}^{\beta}\left(\mu(\varsigma)\right)\right)=\frac{1}{s^{\beta+1}}\frac{L\left[\mu(\varsigma)\right]}{1+\lambda s^{-(\beta+1)}}. \tag{2.5}$$

Definition 2.6: [38] The Mittag-Leffler function in power series form is defined as

$$E_{\alpha}(\mu)=\sum_{n=0}^{\infty}\frac{\mu^{n}}{\Gamma(\alpha n+1)}, 0<\alpha<1. \tag{2.6}$$

Definition 2.7: [41] The Prabhakar function is defined as

$$E_{\alpha,\beta}^{\gamma}(\mu)=\sum_{m=0}^{\infty}\frac{(\gamma)_m}{\Gamma(\alpha m+\beta)}\frac{\mu^m}{m!},\quad \alpha,\beta>0,\gamma\neq 0, \tag{2.7}$$

where,

$$(\gamma)_m=\begin{cases}1 & if\,,m=0,\\ \gamma(\gamma+1)...(\gamma+m-1) & if\,,m=1,2,3...\end{cases}$$

Lemma 2.8: [42] Let $\mu\in W^{1,n}(0,\infty)\cap C^{n-1}([0,\infty))\,,\,n\in N$. Then the Laplace transform of the n^{th} order Yang-Abdel-Cattani fractional derivative is defined as

$$L\left\{{}_0^{YAC}D_{\varsigma}^{n\beta}(\mu(\varsigma))\right\}=\frac{1}{(s^{\beta+1}+\lambda)}\left\{s^nL[\mu(\varsigma)]-\sum_{m=1}^{n}s^{n-m}\mu^{(m-1)}(0)\right\},\quad s>0. \tag{2.8}$$

Lemma 2.9: [42] Let $\alpha,\beta>0,\ \delta\in\mathbb{R}$ and k denotes positive integer then

$$L\left\{\varsigma^{\beta-1}E_{\alpha,\beta}^{-k}(\delta\varsigma^{\alpha})\right\}(s)=s^{-\beta}\left(1-\delta s^{-\alpha}\right)^k,\ \ s>0. \tag{2.9}$$

3. Basic idea of q-Homotopy Analysis Transform approach

To show how q-HATM works, let us use a generalized fractional nonlinear partial differential equation shown as:

$$D_{\varsigma}^{\beta}\Psi(x,\varsigma)+R\Psi(x,\varsigma)+N\Psi(x,\varsigma)=\phi(x,\varsigma)\,,\qquad n-1<\beta\leq n. \tag{3.1}$$

Taking the Laplace transform of equation (3.1), we obtain

$$\frac{1}{(s^{\beta+1}+\lambda)}\{sL[\Psi(x,\varsigma)]-\Psi(x,0)\}+L\{R\Psi(x,\varsigma)\}+L\{N\Psi(x,\varsigma)\}=L\{\phi(x,\varsigma)\}\,,$$

or

$$L[\Psi(x,\varsigma)]-\frac{\Psi(x,0)}{s}+\left(\frac{s^{\beta+1}+\lambda}{s}\right)(L\{R\Psi(x,\varsigma)\}+L\{N\Psi(x,\varsigma)\}-L\{\phi(x,\varsigma)\})=0$$

Let N be defined as

$$\begin{aligned}\mathrm{N}[\gamma(x,\varsigma;q)]=L[\gamma(x,\varsigma;q)]-\frac{\gamma(x,0)}{s}+\left(\frac{s^{\beta+1}+\lambda}{s}\right)(L\{R\gamma(x,\varsigma)\}\\+L\{N\gamma(x,\varsigma)\}-L\{\phi(x,\varsigma)\})\,,\end{aligned} \tag{3.2}$$

where, $\gamma(x,\varsigma;q)$ is a function of x,ς,q and $q\in[0,1/r]$ is an embedding parameter.
Now, define a homotopy as follows:

$$(1-rq)\,L[\gamma(x,\varsigma;q)-\Psi_0(x,\varsigma)]=hqH(x,\varsigma)\,\mathrm{N}[\gamma(x,\varsigma;q)]\,, \tag{3.3}$$

where, $r\geq 1$, q is the embedding parameter, $H(x,\varsigma)$ denotes a non-zero auxiliary function, and is $h\neq 0$ an auxiliary parameter.

Here, we can see that, when $q=0$, $\gamma(x,\varsigma;0)=\Psi_0(x,\varsigma)$ and $q=1/r$, $\gamma(x,\varsigma,1/r)=\Psi(x,\varsigma)$.

Here, we can see that, when q expands from 0 to $1/r$, the output $\gamma(x,\varsigma;q)$ evolves from the initial approximation $\Psi_0(\varsigma)$ to exact result $\Psi(x,\varsigma)$.
Expanding $\gamma(x,\varsigma;q)$, we get

$$\gamma(x,\varsigma;q)=\Psi_0(x,\varsigma)+\sum_{i=1}^{\infty}\Psi_i(x,\varsigma)q^i,$$

where,

$$\Psi_i(x,\varsigma) = \frac{1}{i!}\left.\frac{\partial^i \gamma(x,\varsigma;q)}{\partial q^i}\right|_{q=0} \tag{3.4}$$

We can choose $\Psi_0(x,\varsigma), h, H(x,\varsigma)$ and r appropriately such that the series defined in (3.4) converges at $q = 1/r$, hence

$$\Psi(x,\varsigma) = \Psi_0(x,\varsigma) + \sum_{i=1}^{\infty} \Psi_i(x,\varsigma)\left(\frac{1}{r}\right)^i. \tag{3.5}$$

Introducing the vector $\bar{\Psi}_i = \{\Psi_0, \Psi_1, \Psi_2, ..., \Psi_i\}$ and differentiated with equation (3.3) with respect to q and adding $q = 0$ and dividing the result by $i!$, we get

$$L[\Psi_i(x,\varsigma) - \kappa_i \Psi_{i-1}(x,\varsigma)] = hH(x,\varsigma)\Re_i(\Psi_{i-1}(x,\varsigma)), \tag{3.6}$$

where,

$$\Re_i(\Psi_{i-1}(x,\varsigma)) = \frac{1}{(i-1)!}\left.\frac{\partial^{i-1} N(x,\varsigma;q)}{\partial q^{i-1}}\right|_{q=0} \tag{3.7}$$

and

$$\kappa_i = \begin{cases} 0; & i \le 1 \\ r; & otherwise \end{cases} \tag{3.8}$$

Lastly, by taking the inverse Laplace transform in (3.6), we get

$$\Psi_i(x,\varsigma) = \kappa_i \Psi_{i-1}(x,\varsigma) + hH(x,\varsigma) L^{-1}\Re_i(\Psi_{i-1}(x,\varsigma)), \tag{3.9}$$

by choosing appropriate values for $h, H(x,\varsigma)$ and r, we can easily find the solution as:

$$\Psi(x,\varsigma) = \lim_{N\to\infty} \sum_{i=0}^{\infty} \Psi_i(x,\varsigma)\left(\frac{1}{r}\right)^i. \tag{3.10}$$

4. Methodology for Solving the Generalized Fractional Bratu-Type Equation

Theorem 4.1 *Examine the generalized Bratu equation that follows:*

$${}_{0}^{YAC}D_{\varsigma}^{2n\beta}\omega(\varsigma) + \varepsilon E_\alpha(\omega) = 0,\ n-1 < \beta \le n,\ 0 < \varsigma \le 1,\ n \in \mathrm{N}.$$

Here ε is constant, $E_\alpha(\omega)$ denotes the Mittag-Leffler function with a real parameter α, ω is the unknown function, and $D^\beta\omega = \frac{d^\beta\omega}{d\varsigma^\beta}$ denotes the YAC fractional derivative.

Theorem 4.1 *Given that a generalized Bratu fractional differential equation is*

$${}_{0}^{YAC}D_{\varsigma}^{2n\beta}\omega(\varsigma) + \varepsilon E_\alpha(\omega) = 0. \tag{4.1}$$

Applying the Laplace transform on both sides of Equation (4.1),

$$\frac{1}{(s^{\beta+1}+\lambda)}\left\{s^{2n}L[\omega(\varsigma)] - \sum_{m=1}^{2n} s^{2n-m}\omega^{(m-1)}(0)\right\} = -\varepsilon L\{E_\alpha(\omega)\},$$

$$L[\omega(\varsigma)] = \frac{1}{s^{2n}}\sum_{m=1}^{2n} s^{2n-m}\omega^{(m-1)}(0) - \varepsilon\left(\frac{s^{\beta+1}+\lambda}{s^{2n}}\right)L\{E_\alpha(\omega)\}.$$

Consider N *in such a manner*

$$\mathrm{N}[\gamma(\varsigma;q)] = L[\gamma(\varsigma;q)] - \frac{1}{s^{2n}}\sum_{m=1}^{2n} s^{2n-m}\omega^{(m-1)}(0) + \varepsilon\left(\frac{s^{\beta+1}+\lambda}{s^{2n}}\right)L\{E_\alpha(\omega)\}, \tag{4.2}$$

where $\gamma(\varsigma;q)$ is a function of ς, q and $q \in [0, 1/r]$ is an embedding parameter.
Now, define a homotopy,

$$(1-rq)\,L\left[\gamma(\varsigma;q)-\omega_0(\varsigma)\right] = hqH(\varsigma)\,\mathrm{N}\left[\gamma(\varsigma;q)\right], \tag{4.3}$$

where, $r \geq 1$, q represents the embedding parameter, $H(x,\varsigma)$ is non vanishing auxiliary function, $h \neq 0$ denotes the auxiliary constant, and $\omega_0(\varsigma)$ refers to the initial approximation of the function $\omega(\varsigma)$.
Here, we can see that, when q expands from 0 to $1/r$, the output $\gamma(\varsigma;q)$ evolves from the initial approximation $\omega_0(\varsigma)$ to exact result $\omega(\varsigma)$.

Expanding $\gamma(\varsigma;q)$, we get

$$\gamma(\varsigma;q) = \omega_0(\varsigma) + \sum_{i=1}^{\infty} \omega_i(\varsigma) q^i, \tag{4.4}$$

where,

$$\omega_i(\varsigma) = \frac{1}{i!}\left.\frac{\partial^i \gamma(\varsigma;q)}{\partial q^i}\right|_{q=0}$$

We can choose $\omega_0(\varsigma), h,\ H(\varsigma)$ and r appropriately such that the series defined in (4.4) converges at $q = 1/r$, hence

$$\omega(\varsigma) = \omega_0(\varsigma) + \sum_{i=1}^{\infty} \omega_i(\varsigma)\left(\frac{1}{r}\right)^i. \tag{4.5}$$

Introducing the vector $\bar{\omega}_i = \{\omega_0, \omega_1, \omega_2, ..., \omega_i\}$ and differentiated with equation (4.3) with respect to q and adding $q = 0$ and dividing the result by $i!$, we get

$$L\left[\omega_i(\varsigma) - \kappa_i \omega_{i-1}(\varsigma)\right] = hH(\varsigma)\,\Re_i\left(\omega_{i-1}(\varsigma)\right), \tag{4.6}$$

where,

$$\Re_i\left(\omega_{i-1}(\varsigma)\right) = \frac{1}{(i-1)!}\left.\frac{\partial^{i-1} N(\varsigma;q)}{\partial q^{i-1}}\right|_{q=0} \tag{4.7}$$

and

$$\kappa_i = \begin{cases} 0; & i \leq 1 \\ r; & otherwise \end{cases} \tag{4.8}$$

Using (4.2) in (4.7), we get

$$\begin{aligned}\Re_i\left(\omega_{i-1}(\varsigma)\right) = L\left(\omega_{i-1}\right) - &\left[\tfrac{1}{s^{2n}}\sum_{m=1}^{2n} s^{2n-m}\omega^{(m-1)}(0)\right]\left(1-\tfrac{\kappa_i}{r}\right) \\ &+\varepsilon\left(\tfrac{s^{\beta+1}+\lambda}{s^{2n}}\right) L\left\{E_\alpha(\omega_{i-1})\right\},\end{aligned} \tag{4.9}$$

lastly, by taking inverse Laplace transform on (4.6), we get

$$\omega_i(\varsigma) = \kappa_i \omega_{i-1}(\varsigma) + hH(\varsigma)\,L^{-1}\Re_i\left(\omega_{i-1}(\varsigma)\right), \tag{4.10}$$

by choosing appropriate values for $h, H(x,\varsigma)$ and r, we can easily find the solution as:

$$\omega(\varsigma) = \lim_{N\to\infty}\sum_{i=0}^{\infty} \omega_i(\varsigma)\left(\frac{1}{r}\right)^i. \tag{4.11}$$

5. Particular Cases

We address a few specific instances of generalized Bratu-type fractional differential equations by entering specified values for n and α.

5.1. Case-I

By putting $n = 1$ in Theorem 1, we get

$$ {}_{0}^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) + \varepsilon E_{\alpha}(\omega) = 0,\ 0 < \beta \leq 1,\ 0 < \varsigma \leq 1, \tag{5.1} $$

we know that for the Mittag-Leffler function

$$ E_{\alpha}(\omega) = 1 + \frac{\omega}{\Gamma(\alpha+1)} + \frac{\omega^2}{\Gamma(2\alpha+1)} + ..., $$

neglecting the higher order terms, equation (5.1) becomes

$$ {}_{0}^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) + \varepsilon\left(1 + \frac{\omega}{\Gamma(\alpha+1)} + \frac{\omega^2}{\Gamma(2\alpha+1)}\right) = 0,\ 0 < \beta \leq 1,\ 0 < \varsigma \leq 1, \tag{5.2} $$

with $\omega(0) = \omega'(0) = 0$.
Using the process described above and taking $h = 1$, we get the following results:

$$ \omega_i(\varsigma) = \kappa_i \omega_{i-1}(\varsigma) + H(\varsigma) L^{-1}\Re_i(\omega_{i-1}(\varsigma)), \tag{5.3} $$

where,

$$ \kappa_i = \begin{cases} 0; & i \leq 1 \\ r; & otherwise \end{cases} $$

and

$$ \begin{aligned} \Re_i(\omega_{i-1}(\varsigma)) = L(\omega_{i-1}) - \left[\tfrac{1}{s^2}\sum_{m=1}^{2} s^{2-m}\omega^{(m-1)}(0)\right]\left(1 - \tfrac{\kappa_i}{r}\right) \\ + \varepsilon\left(\tfrac{s^{\beta+1}+\lambda}{s^2}\right)\left(1 + \tfrac{\omega}{\Gamma(\alpha+1)} + \tfrac{\omega^2}{\Gamma(2\alpha+1)}\right). \end{aligned} $$

By equation (5.3), we get

$$ \omega_0(\varsigma) = 0, $$

$$ \omega_1(\varsigma) = H\varepsilon\varsigma^{1-\beta}E_{\beta+1,2-\beta}^{-1}(-\lambda\varsigma^{\beta+1}), $$

$$ \begin{aligned} \omega_2(\varsigma) = (r + H + \varepsilon H)\,\omega_1 + \tfrac{\varepsilon^2 H^2}{\Gamma(\alpha+1)}\varsigma^{2-2\beta}E_{\beta+1,3-2\beta}^{-2}(-\lambda\varsigma^{\beta+1}) \\ + \tfrac{\varepsilon^2 H^2}{\Gamma(2\alpha+1)}\varsigma^{4-3\beta}E_{\beta+1,5-3\beta}^{-3}(-\lambda\varsigma^{\beta+1}), \end{aligned} $$

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By choosing appropriate values for $H(x, \varsigma)$ and r, we can easily find the solution as:

$$ \omega(\varsigma) = \lim_{N\to\infty}\sum_{i=0}^{\infty}\omega_i(\varsigma)\left(\frac{1}{r}\right)^i. $$

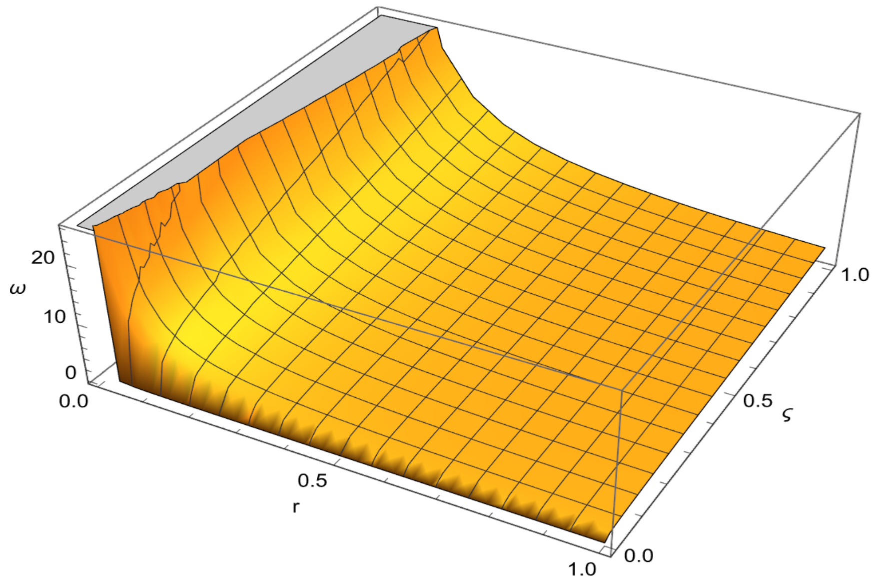

Figure 1: The graph of solution for Case-I at $\varepsilon = 1$, $\alpha = -1$, $\lambda = 1$, $H = -1$ and $\beta = 0.7$.

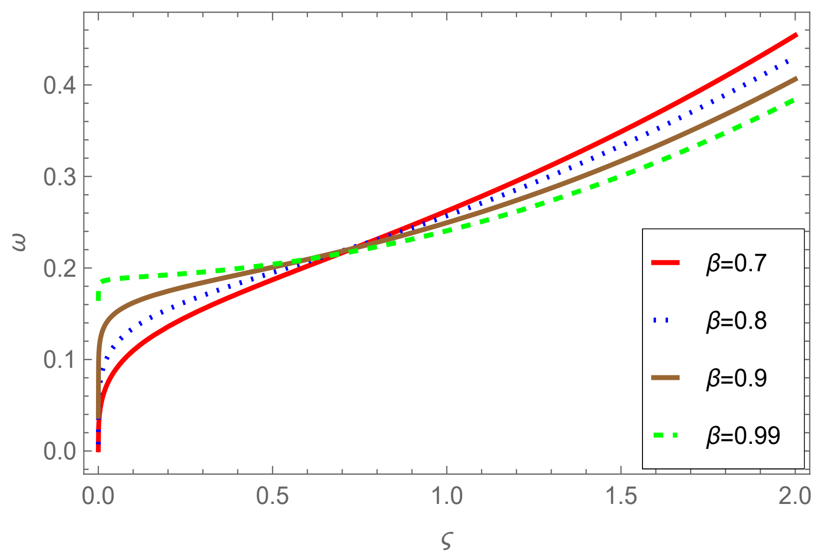

Figure 2: The graph of solution for Case-I at $\varepsilon = 0.5$, $\alpha = -1$, $\lambda = 0.5$, $H = -1$ and $r = 5$ for different values of β

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5.2. Case-II

Putting $n = 1$ and $\alpha = 1$ in Theorem 1, we get

$$ {}_{0}^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) + \varepsilon e^{\omega} = 0, 0 < \beta \leq 1, 0 < \varsigma \leq 1, \tag{5.4} $$

we know that the series expansion of the exponential function is

$$ e^{\omega} = 1 + \omega + \frac{\omega^2}{2!} + \frac{\omega^3}{3!}..., $$

neglecting the higher-order terms, equation (5.4) becomes

$$ {}_{0}^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma)+\varepsilon\left(1+\omega+\frac{\omega^{2}}{2!}\right)=0,\ 0<\beta\leq 1,\ 0<\varsigma\leq 1, \tag{5.5}$$

with, $\omega(0)=\omega'(0)=0$.
Using the process described above and taking $h=1$, we get the following results:

$$\omega_i(\varsigma)=\kappa_i\omega_{i-1}(\varsigma)+H(\varsigma)L^{-1}\Re_i(\omega_{i-1}(\varsigma)), \tag{5.6}$$

where,

$$\kappa_i=\begin{cases}0; & i\leq 1\\ r; & otherwise\end{cases}$$

and

$$\begin{aligned}\Re_i(\omega_{i-1}(\varsigma))=L(\omega_{i-1})-&\left[\tfrac{1}{s^2}\sum_{m=1}^{2}s^{2-m}\omega^{(m-1)}(0)\right]\left(1-\tfrac{\kappa_i}{r}\right)\\ &+\varepsilon\left(\tfrac{s^{\beta+1}+\lambda}{s^2}\right)\left(1+\omega+\tfrac{\omega^2}{2!}\right).\end{aligned}$$

By equation (5.3), we get

$$\omega_0(\varsigma)=0,$$

$$\omega_1(\varsigma)=H\varepsilon\varsigma^{1-\beta}E_{\beta+1,2-\beta}^{-1}(-\lambda\varsigma^{\beta+1}),$$

$$\begin{aligned}\omega_2(\varsigma)=(r+H+\varepsilon H)\omega_1+\varepsilon^2H^2\varsigma^{2-2\beta}E_{\beta+1,3-2\beta}^{-2}(-\lambda\varsigma^{\beta+1})\\ +\tfrac{\varepsilon^2H^2}{2!}\varsigma^{4-3\beta}E_{\beta+1,5-3\beta}^{-3}(-\lambda\varsigma^{\beta+1}),\end{aligned}$$

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By choosing appropriate values for $H(x,\varsigma)$ and r, we can easily find the solution as:

$$\omega(\varsigma)=\lim_{N\to\infty}\sum_{i=0}^{\infty}\omega_i(\varsigma)\left(\frac{1}{r}\right)^i.$$

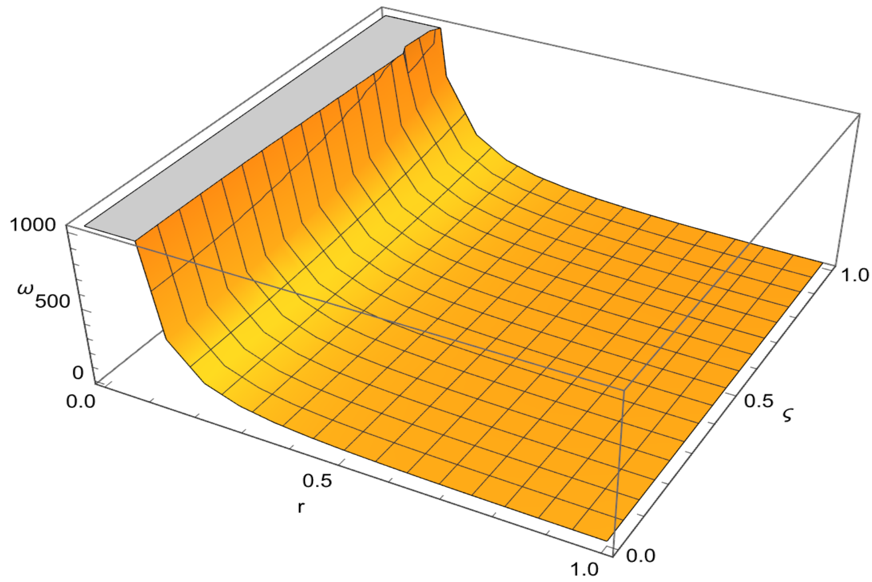

Figure 3: The graph of solution for Case-II at $\varepsilon = -2$, $\lambda = 1$, $H = -1$ and $\beta = 0.7$

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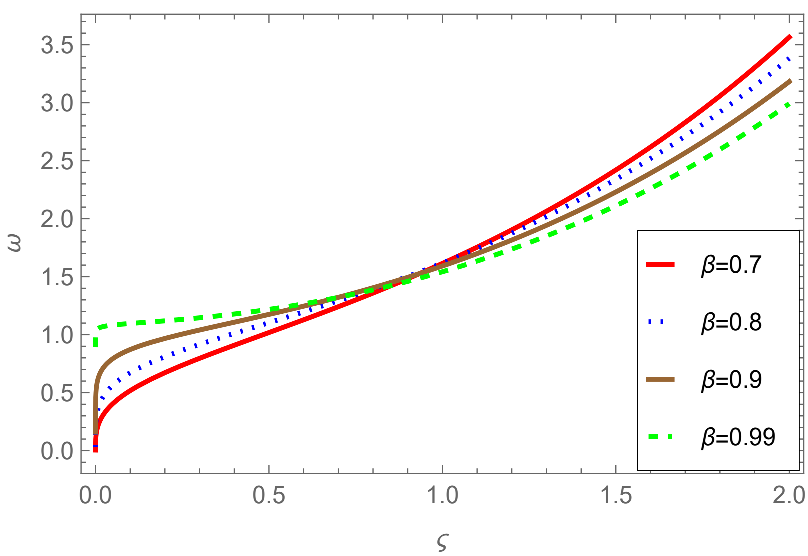

Figure 4: The graph of solution for Case-II at $\varepsilon = -2$, $\lambda = 0.5$, $H = -1$ and $r = 5$ for different values of β

.

Subcase-I If we put $\varepsilon = -2$ in equation (5.4), we get

$$ {}_{0}^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) - 2e^{\omega} = 0, 0 < \beta \leq 1, 0 < \varsigma \leq 1, \tag{5.7}$$

neglecting the higher order terms, equation (5.7) becomes

$$ {}_{0}^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) - 2\left(1 + \omega + \frac{\omega^2}{2!}\right) = 0,\ \ 0 < \beta \leq 1, 0 < \varsigma \leq 1, \tag{5.8}$$

with, $\omega(0) = \omega'(0) = 0$.
Now, following the same process as case-II, we get

$$\omega_0(\varsigma) = 0,$$

$$\omega_1(\varsigma) = -2H\varsigma^{1-\beta} E^{-1}_{\beta+1,2-\beta}(-\lambda\varsigma^{\beta+1}),$$

$$\omega_2(\varsigma) = (r-H)\,\omega_1 + 4H^2\varsigma^{2-2\beta} E^{-2}_{\beta+1,3-2\beta}(-\lambda\varsigma^{\beta+1})$$
$$+2H^2\varsigma^{4-3\beta} E^{-3}_{\beta+1,5-3\beta}(-\lambda\varsigma^{\beta+1}),$$

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By choosing appropriate values for $H(x,\varsigma)$ and r, we can easily find the solution as:

$$\omega(\varsigma) = \lim_{N\to\infty} \sum_{i=0}^{\infty} \omega_i(\varsigma)\left(\frac{1}{r}\right)^i.$$

Subcase-II If we put $\varepsilon = -1$ and replacing e^{ω} by $e^{2\omega}$ in equation (5.4) , we get

$$ {}_0^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) - e^{2\omega} = 0, \quad 0 < \beta \leqslant 1, \quad 0 < \varsigma \leqslant 1, \tag{5.9}$$

neglecting the higher order terms, equation (5.8) becomes

$$ {}_0^{YAC}D_{\varsigma}^{2\beta}\omega(\varsigma) - \left(1 + 2\omega + \frac{2\omega^2}{2!}\right) = 0,\ 0 < \beta \leq 1,\ 0 < \varsigma \leq 1, \tag{5.10}$$

with, $\omega(0) = \omega'(0) = 0$.
Now, following the same process as case-II, we get

$$\omega_0(\varsigma) = 0,$$

$$\omega_1(\varsigma) = -H\varsigma^{1-\beta} E^{-1}_{\beta+1,2-\beta}(-\lambda\varsigma^{\beta+1}),$$

$$\omega_2(\varsigma) = r\omega_1 + 2H^2\varsigma^{2-2\beta} E^{-2}_{\beta+1,3-2\beta}(-\lambda\varsigma^{\beta+1})$$
$$+2H^2\varsigma^{4-3\beta} E^{-3}_{\beta+1,5-3\beta}(-\lambda\varsigma^{\beta+1}),$$

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By choosing appropriate values for $H(x,\varsigma)$ and r, we can easily find the solution as:

$$\omega(\varsigma) = \lim_{N\to\infty} \sum_{i=0}^{\infty} \omega_i(\varsigma)\left(\frac{1}{r}\right)^i.$$

6. Conclusion

In this study, we established an efficient analytical framework for fractional calculus issues by effectively examining the extended Bratu-type fractional differential equation using the YAC fractional derivative operator. We produced analytical responses and examined several particular cases using q-HATM, demonstrating the adaptability and usefulness of the suggested approach. The results offer new insights into the use of fractional derivative operators, particularly the YAC operator, in solving Bratu-type equations and underscore its potential for broader applications in fractional differential equations. We also plot some graphs to express the behavior of the results obtained.

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