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On p-Symmetric Rings

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ABSTRACT: This article embodies the notion of p-symmetric rings using the concept of non-zero potent elements in a ring. It is proved that R is a p-symmetric ring if and only if $p^{n-1}Rp^{n-1}$ is a symmetric ring and p^{n-1} is left semicentral. Moreover, p-symmetric rings in terms of upper triangular matrix rings and left min-p-abel rings have been characterized. Furthermore, we introduce strongly p-symmetric rings and also provide a characterization of strongly p-symmetric rings in terms of strongly left min-p-abel rings. In particular, it is proved that R is a strongly left min-p-abel ring if and only if R is a strongly p-symmetric ring for each $p^{n-1} \in MP_l(R)$. Furthermore, it has been established that right p-reduced rings are p-symmetric rings.

Key Words: p-symmetric rings, strongly p-symmetric rings, left min-p-abel rings, p-reduced rings, potent elements.

Contents

1. Introduction

Throughout this paper all rings are associative with identity. Let R be a ring. We denote the centre of R, the set of all idempotents of R and the set of all nilpotent elements of R by $Z(R)$, $E(R)$ and $N(R)$ respectively. Moreover, the set of all $n \times n$ upper triangular matrix ring over R is denoted by $M_n(R)$. An element p of R is said to be a potent if $p^n = p$ for any $n \geq 2$. Let $PT(R)$ denote the set of all potent elements of R. It is obvious that all idempotents are potents but the converse is not true. For example, in the ring $R = M_2(\mathbb{R})$, $P = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ is a potent element in R as $P^3 = P$ for but not an idempotent. Also, an element $p \in PT(R)$ is called left minimal potent of R if Rp is a minimal left ideal of R. We denote the set of all left minimal potent elements of R by $MP_l(R)$. A ring is usually called reduced if it has no nilpotent elements other than zero. An element a is called left semicentral (resp., central) in R if $axa = xa$ (resp., $ax = xa$) for each $x \in R$. Following Lambek [5], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. Later on, Anderson and Comillo [1], used the term ZC_3 for symmetric ring. The investigation of symmetric ring is also covered by G. Marks [6]. Ouyang et al. [10], generalized the concept of symmetric rings by introducing weak symmetric rings. According to [10], a ring R is said to be weak symmetric if $abc \in N(R)$ implies $acb \in N(R)$ for all $a, b, c \in R$. Another generalization of symmetric rings has been introduced by Kafkas et al. [3] as central symmetric rings. They defined a ring R to be central symmetric if $abc = 0$ implies $bac \in Z(R)$ for any $a, b, c \in R$. Wei [11] introduced generalized weakly symmetric rings which further expands the idea of symmetric rings. According to Meng and Wei [7], a ring R is called (strongly) e-symmetric if $abc = 0$ implies ($aceb = 0$) $acbe = 0$, for any $a, b, c \in R$; e is an idempotent element of R. They have also studied some important properties of it (refer to [8]). Furthermore, Meng et al. [9] recently studied the notion of weak e-symmetric rings. Recently, Hoque and Saikia [2] studied the notion of t^2-symmetric ring using the concept of non-zero tripotent element t in a ring R. Also, they introduced a strong condition on this notion and called it

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strongly t^2-symmetric ring. They discussed some basic properties of (strongly) t^2-symmetric rings.

Wei [12], studied the concept of left minimal element and left minimal idempotent of a ring R. Extending this concept the left minimal potent element of R is defined using potent element. A ring R is called (strongly) left min-p-abel if either $MP_l(R) = \phi$ or every element $p \in MP_l(R)$ is (right) left semicentral. Following [7], a ring R is said to be Abel if all idempotents of R are central. Analogously we call a ring R to be p-abel if all potent elements of R are central. Following [4], a ring R is left quasi-duo if every maximal left ideal of R is an ideal. According to [12], a ring R is $MELT$ if every essential maximal left ideal of R is an ideal.

In this paper, we extend and generalize the structure of e-symmetric rings defined by F. Meng et al. [7] using the concept of non-zero potent elements of the ring by introducing the notions of p-symmetric rings and strongly p-symmetric rings. We characterize p-symmetric rings in terms of upper triangular matrix rings and left min-p-abel rings. We provide a characterization of strongly p-symmetric rings in terms of strongly left min-p-abel rings. We also introduce the notion of p-reduced ring as a subclass of reduced ring.

2. p-Symmetric Rings

In this section, we introduce the notion of p-symmetric rings. We discuss some basic properties of p-symmetric rings and study the characterizations of such rings with the help of upper triangular matrix rings. We also characterize p-symmetric rings in terms of left min-p-abel rings. We begin with the following definition.

Definition 2.1 *Let R be a ring and $p \in PT(R)$. Then, R is called a p-symmetric ring if and only if $abc = 0$ implies $acbp^{n-1} = 0$, for all $a, b, c \in R$ and $n \geq 2$.*

Example 2.1 *Let us consider the ring $R = M_2(\mathbf{Z_3})$. We know that $\mathbf{Z_3}$ is a reduced ring. Since every reduced ring is also a symmetric ring by [1, Theorem I.3], so $\mathbf{Z_3}$ is a symmetric ring.*
Let us consider $P = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then, $P^3 = P$ and so $P \in PT(R)$. Now, we consider the elements $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$ in R and a be any non-zero element in $\mathbf{Z_3}$.
Then $ABC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that $ACBP^{3-1} = ACBP^2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}^2$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This shows that R is a P-symmetric ring.

Remark 2.1 *It is obvious that every symmetric ring is p-symmetric for any $p \in PT(R)$, but the converse need not be true which can be shown as below.*
From Example 2.1., we can observe that the ring $R = M_2(\mathbf{Z_3})$ is P-symmetric but not a symmetric ring, because

$$ACB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^3 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 2.2 *Every e-symmetric ring [7] is also a p-symmetric ring, but every p-symmetric ring need not be e-symmetric. From Example 2.1 , we can observe that $R = M_2(\mathbf{Z_3})$ is a P-symmetric ring. But R is not a e-symmetric ring, as $P = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \notin E(R)$.*

Proposition 2.1 *Let R be a ring and $p \in PT(R)$. Then R is a p-symmetric ring if and only if $p^{n-1}Rp^{n-1}$ is a symmetric ring and p^{n-1} is left semicentral for $n \geq 2$.*

Proof: Let us assume that R is a p-symmetric ring. Let $x \in R$ and $y = (1-p^{n-1})xp^{n-1} + p^{n-1}$. Then, we have $p^{n-1}y = p^{n-1}(1-p^{n-1})xp^{n-1} + p^{n-1}p^{n-1} = p^{n-1}$. Similarly, it can be shown that $yp^{n-1} = y$; $y^2 = y$; $p^{n-1}yp^{n-1} = p^{n-1}$ and $(1-y)yp^{n-1} = 0$. Since R is a p-symmetric ring , so $(1-y)p^{n-1}yp^{n-1} = 0 \implies (1-y)p^{n-1} = 0 \implies p^{n-1} = yp^{n-1} = y$. Thus, $y = (1-p^{n-1})xp^{n-1} + p^{n-1} \implies (1-p^{n-1})xp^{n-1} = 0 \implies xp^{n-1} = p^{n-1}xp^{n-1}$. Hence, p^{n-1} is left semicentral.

Secondly, let $x, y, z \in p^{n-1}Rp^{n-1}$ such that $xyz = 0$. Since $p^{n-1}Rp^{n-1}$ is a subring of R and R is a p-symmetric ring, so we have $xzyp^{n-1} = 0$. This implies that $xzy = 0$, as $yp^{n-1} = y$. Thus, $p^{n-1}Rp^{n-1}$ is a symmetric ring.

Conversely, let us suppose that $p^{n-1}Rp^{n-1}$ is a symmetric ring and p^{n-1} is left semicentral for $n \geq 2$. Let $a, b, c \in R$ such that $abc = 0$. Then $p^{n-1}ap^{n-1}, p^{n-1}bp^{n-1}, p^{n-1}cp^{n-1} \in p^{n-1}Rp^{n-1}$. Since $p^{n-1}Rp^{n-1}$ is a symmetric ring, we have $(p^{n-1}ap^{n-1})(p^{n-1}bp^{n-1})(p^{n-1}cp^{n-1}) = 0$ implies $(p^{n-1}ap^{n-1})(p^{n-1}cp^{n-1})(p^{n-1}bp^{n-1}) = 0$. Therefore, $p^{n-1}ap^{n-1}bp^{n-1}cp^{n-1} = 0$ which implies $p^{n-1}ap^{n-1}cp^{n-1}bp^{n-1} = 0$. Since p^{n-1} is left semicentral, so we have $p^{n-1}ap^{n-1}cp^{n-1}bp^{n-1} = 0 \implies ap^{n-1}cp^{n-1}bp^{n-1} = 0 \implies acp^{n-1}bp^{n-1} = 0 \implies acbp^{n-1} = 0$. This shows that R is a p-symmetric ring.

□

Proposition 2.2 *Let R be a ring with unity 1 and $p \in PT(R)$. Then R is a p-symmetric ring if and only if $abc = 0$ implies $bacp^{n-1} = 0$ for all $a, b, c \in R$ and $n \geq 2$.*

Proof: Let us assume that R is a p-symmetric ring. Then by Proposition 2.1, p^{n-1} is left semicentral in R. Let $a, b, c \in R$ such that $abc = 0$. This gives, $1a(bc) = 0$. Since R is a p-symmetric ring, so we have $1bcap^{n-1} = 0 \implies bcap^{n-1} = 0$. Again by p-symmetricity of R, we have $b(ap^{n-1})cp^{n-1} = 0 \implies bacp^{n-1} = 0$, as p^{n-1} is semicentral in R.

Conversely, let us suppose that $abc = 0$ implies $bacp^{n-1} = 0$ for all $a, b, c \in R$ and $n \geq 2$. Let $x \in R$. Since $p \in PT(R)$, so $p^n = p$ for $n \geq 2$ and $(p^{n-1})^2 = p^n p^{n-2} = p^{n-1}$. So we get, $xp^{n-1}(1-p^{n-1})p^{n-1} = 0$. This implies that
$(1-p^{n-1})xp^{n-1}p^{n-1}p^{n-1} = 0 \implies (1-p^{n-1})Rp^{n-1} = 0$. Thus p^{n-1} is left semicentral.
Also, let $a, b, c \in p^{n-1}Rp^{n-1}$ such that $abc = 0$. Since $p^{n-1}Rp^{n-1}$ is a subring of R, so we have $bacp^{n-1} = 0 \implies bac = 0$. So $p^{n-1}Rp^{n-1}$ is a symmetric ring .
Hence by Proposition 2.1, we have R is a p-symmetric ring.

□

Similarly, we can establish the following results.

Proposition 2.3 *Let R be a ring with unity 1 and $p \in PT(R)$. Then R is a p-symmetric ring if and only if $abc = 0$ implies $cabp^{n-1} = 0$ for all $a, b, c \in R$ and $n \geq 2$.*

Proposition 2.4 *Let R be a ring with unity 1 and $p \in PT(R)$. Then R is a p-symmetric ring if and only if $abc = 0$ implies $cbap^{n-1} = 0$ for all $a, b, c \in R$ and $n \geq 2$.*

Proposition 2.5 *Let R be a ring and $p \in E(R)$. Then R is a symmetric ring if and only if R is both p-symmetric and $(1-p)$-symmetric ring.*

Proof: The necessary part is obvious.
For the sufficient part, let us assume that R is both p-symmetric and $(1-p)$-symmetric ring. Let $a, b, c \in R$ such that $abc = 0$. Then we have, $acb(1-p)^{n-1} = 0$, as R is a $(1-p)$-symmetric ring. This yields $acb(1-p^{n-1}) = 0$ as p is an idempotent. This implies $acb = acbp^{n-1}$. Again R is a p-symmetric ring, so we have $acbp^{n-1} = 0$. It follows that $acb = 0$. Thus R is a symmetric ring.

□

Some characteristics of p-symmetric triangular matrix rings are presented in the following results.

Proposition 2.6 *Let R be a ring, $p \in PT(R)$ and $P = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in PT(M_3(R))$. Then R is p-symmetric if and only if $M_3(R)$ is P-symmetric.*

Proof: Suppose that R is a p-symmetric ring. Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & g_1 \\ 0 & 0 & f_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & g_2 \\ 0 & 0 & f_2 \end{pmatrix}$ and $C = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & d_3 & g_3 \\ 0 & 0 & f_3 \end{pmatrix}$ are in $M_3(R)$ such that $ABC = 0$. This implies that $a_1a_2a_3 = 0$. Since R is a p-symmetric ring, so we have $a_1a_3a_2p^{n-1} = 0$. This gives $ACBP^{n-1} = 0$. Therefore, $M_3(R)$ is a P-symmetric ring.

Conversely, let $M_3(R)$ be a P-symmetric ring. Let $a, b, c \in R$ such that $abc = 0$. Then $A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are in $M_3(R)$ and $ABC = 0$. So by hypothesis, we have $ACBP^{n-1} = 0$. This implies that $acbp^{n-1} = 0$ and hence R is a p-symmetric ring.

□

Proposition 2.7 *Let R be a ring and $P = \begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix} \in PT(M_2(R))$ for each $r \in R$. Then $M_2(R)$ is a P-symmetric ring if and only if R is a symmetric ring.*

Proof: Let us assume that $M_2(R)$ is a P-symmetric ring . Let $a, b, c \in R$ such that $abc = 0$. So we have, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(R)$. Since $M_2(R)$ is P-symmetric, so we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} acb & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (-1)^n & (-1)^{n-1}r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} acb(-1)^n & acb(-1)^{n-1}r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have $acb(-1)^n = 0$ which implies $acb = 0$. Thus, R is a symmetric ring.

Conversely, let R be a symmetric ring. Let $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$, $C = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix} \in M_2(R)$ such that $ABC = 0$. Then, we have $a_1a_2a_3 = 0$. Since R is symmetric, so we have $a_1a_3a_2 = 0$.
Now, $ACBP^{n-1} = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^{n-1}$
$= \begin{pmatrix} a_1a_3a_2(-1)^n & a_1a_3a_2(-1)^{n-1}r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, as $a_1a_3a_2 = 0$.
Hence, $M_2(R)$ is a P-symmetric ring .

□

Proposition 2.8 *Let R be a ring, $p \in PT(R)$ and $P = \begin{pmatrix} p & p \\ 0 & 0 \end{pmatrix} \in PT(M_2(R))$. Then $M_2(R)$ is a P-symmetric ring if and only if R is a p-symmetric ring.*

Proof: The proof is similar to the proof of Proposition 2.6.

□

We now characterize p-symmetric rings with the help of left min-p-abel rings.

Proposition 2.9 *Let R be a ring and $p^{n-1} \in MP_l(R)$. Then R is a left min-p-abel ring if and only if $p^{n-1} = abp^{n-1}$ implies $p^{n-1} = bap^{n-1}$ for any $a, b \in R$.*

Proof: Let us assume that R is a left min-p-abel ring and $p^{n-1} \in MP_l(R)$. Then p^{n-1} is left semicentral. If $p^{n-1} = abp^{n-1}$ for any $a, b \in R$, then $p^{n-1} = ap^{n-1}bp^{n-1}$. Therefore, we have $p^{n-1} = p^{n-1}p^{n-1} = ap^{n-1}bp^{n-1}ap^{n-1}bp^{n-1}$. Thus, $bp^{n-1}ap^{n-1} \neq 0$. So, we have $Rbp^{n-1}ap^{n-1} = Rp^{n-1}$, $p^{n-1} = cbp^{n-1}ap^{n-1}$ for some $c \in R$. Therefore, $p^{n-1}bp^{n-1} = cbp^{n-1}ap^{n-1}bp^{n-1} = cbp^{n-1}p^{n-1} = cbp^{n-1}$ and $p^{n-1} = cbp^{n-1}ap^{n-1} = p^{n-1}bp^{n-1}ap^{n-1} = bp^{n-1}ap^{n-1} = bap^{n-1}$.

Conversely, let $p^{n-1} = abp^{n-1}$ implies $p^{n-1} = bap^{n-1}$ for any $a, b \in R$ and $p^{n-1} \in MP_l(R)$. Let $h = (1 - p^{n-1})ap^{n-1}$ for any $a \in R$. Let us assume that $h \neq 0$. Now, we have $Rh = Rp^{n-1}$. Then $p^{n-1} = ch$ for some $c \in R$, $h = hp^{n-1} = hch$ and $p^{n-1} = ch = chp^{n-1}$. By hypothesis, we have $p^{n-1} = hcp^{n-1}$. Then, we get $h = hp^{n-1} = h^2cp^{n-1} = 0$, which is a contradiction. Thus $h = (1 - p^{n-1})ap^{n-1} = 0$ for any $a \in R$. Hence R is a left min-p-abel ring.

□

Proposition 2.10 *Let R be a ring and $p \in PT(R)$. Then R is a left min-p-abel if and only if R is a p-symmetric for any $p^{n-1} \in MP_l(R)$.*

Proof: Let us assume that R is a left min-p-abel ring. Let, $a, b, c \in R$ such that $abc = 0$. If $acbp^{n-1} \neq 0$, then $Rp^{n-1} = Racbp^{n-1}$ and $p^{n-1} = dacbp^{n-1}$ for some $d \in R$. Since R is a left min-p-abel ring, so by Proposition 2.9 we have, $p^{n-1} = bdacp^{n-1} = cbdap^{n-1}$. Then, we get $p^{n-1} = dacbp^{n-1} = dap^{n-1}cbp^{n-1} = dabdacp^{n-1}cbp^{n-1} = dabp^{n-1}dacp^{n-1}cbp^{n-1} = dabcbdap^{n-1}dacp^{n-1}cbp^{n-1} = 0$, as $abc = 0$, which is a contradiction. Therefore, we must have $acbp^{n-1} = 0$. This shows that R is a p-symmetric ring.

Conversely, let R is a p-symmetric ring and $p^{n-1} \in MP_l(R)$. Then by Proposition 2.1, we have p^{n-1} is left semicentral in R. Thus, R is a left min-p-abel ring.

□

As a consequence of Proposition 2.10 and [[12], Theorem 1.2] we have the following corollary.

Corollary 2.1 *Let R be a ring and $p \in PT(R)$. Then R is a left quasi-duo ring if and only if R is a $MELT$ ring and R is a p-symmetric ring for each $p^{n-1} \in MP_l(R)$.*

Proposition 2.11 *Let R be a ring and $p \in PT(R)$. Then R is an p-abel ring if and only if for any $a, b \in R$, $p^{n-1} = ab$ implies $p^{n-1} = bap^{n-1}$.*

Proof: We assume that R is a p-abel ring and $p^{n-1} = ab$ for any $a, b \in R$. Let $g = ba$, then $g^2 = b(ab)a = bp^{n-1}a = bap^{n-1} = gp^{n-1}$. Now, $(g^2)^2 = gp^{n-1}gp^{n-1} = g^2p^{n-1}p^{n-1} = gp^{n-1} = g^2$, so $g^2 \in E(R) \subseteq PT(R)$. Therefore, $p^{n-1} = ababp^{n-1} = agbp^{n-1} = agp^{n-1}b = ag^2b = g^2ab = gp^{n-1} = bap^{n-1}$.

Conversely, let $p^{n-1} = ab$ implies $p^{n-1} = bap^{n-1}$ for any $a, b \in R$. Now, let us assume that $g = (1 - p^{n-1})ap^{n-1} + p^{n-1}$. Then $gp^{n-1} = g$ and $g \in E(R) \subseteq PT(R)$. So by hypothesis, $g = p^{n-1}gg = p^{n-1}$, it follows that $(1 - p^{n-1})ap^{n-1} = 0$ for each $a \in R$. Thus, R is a p-abel ring.

□

3. Strongly p-Symmetric Rings

In this section, we introduce the notion of strongly p-symmetric rings and establish a characterization of such rings in terms of strongly left min-p-abel rings. We begin with the following definition.

Definition 3.1 *Let R be a ring and $p \in PT(R)$. Then R is called strongly p-symmetric if and only if $abc = 0$ implies $acp^{n-1}b = 0$, for all $a, b, c \in R$ and $n \geq 2$.*

Example 3.1 *Let us consider the ring $R = M_2(\mathbf{Z_3})$. We know that $\mathbf{Z_3}$ is a reduced ring. Since every reduced ring is also a symmetric ring by [1, Theorem I.3], so $\mathbf{Z_3}$ is a symmetric ring.*
Let us consider $P = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then, $P^3 = P$ and so $P \in PT(R)$. Now, we consider the elements $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ and $C = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$ in R and a be any non-zero element in $\mathbf{Z_3}$.
Then $ABC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that $ACP^{3-1}B = ACP^2B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This shows that R is a strongly P-symmetric ring.

Remark 3.1 *It is observed that p-symmetric rings need not be strongly p-symmetric rings, which can be shown as below.*
From Example 2.1., the ring $R = M_2(\mathbf{Z_3})$ is P-symmetric but not a strongly P-symmetric ring, because
$ACP^{3-1}B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4a^3 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Remark 3.2 *In the above Definitions 2.1 and 3.1, it is observed that for $p = 1$, R is a symmetric ring if and only if R is a (strongly) p-symmetric ring.*

Proposition 3.1 *Let R be a ring such that $p \in PT(R)$. Then R is a strongly p-symmetric ring if and only if $p^{n-1}Rp^{n-1}$ is a symmetric ring and $p^{n-1} \in Z(R)$.*

Proof: Let us assume that R is a strongly p-symmetric ring. For each $a \in R$, let us consider $x = p^{n-1} + p^{n-1}a(1 - p^{n-1})$. Then $p^{n-1}x = p^{n-1}p^{n-1} + p^{n-1}p^{n-1}a(1 - p^{n-1}) = x$, as $p^{n-1}p^{n-1} = p^{n-1}$. Similarly, $xp^{n-1} = p^{n-1}$. Also, $x(1 - x)p^{n-1} = 0$ and since R is strongly p-symmetric, so we have, $xp^{n-1}p^{n-1}(1 - x) = 0 \implies p^{n-1}(1 - x) = 0 \implies p^{n-1} = p^{n-1}x = x$. This implies that $p^{n-1}a(1 - p^{n-1}) = 0 \implies p^{n-1}a = p^{n-1}ap^{n-1}$ for each $a \in R$. Now let, $y = p^{n-1} + (1 - p^{n-1})ap^{n-1}$. Then $yp^{n-1} = y$ and $p^{n-1}y = p^{n-1}$. Also $(1 - p^{n-1})p^{n-1}y = 0$, since R is strongly p-symmetric, so we have, $(1 - p^{n-1})yp^{n-1}p^{n-1} = 0 \implies (1 - p^{n-1})yp^{n-1} = 0 \implies (1 - p^{n-1})y = 0$. Therefore $p^{n-1}y = y = p^{n-1}$. So we have $(1 - p^{n-1})ap^{n-1} = 0 \implies ap^{n-1} = p^{n-1}ap^{n-1}$ for each $a \in R$. Thus we have $p^{n-1} \in Z(R)$.

Again, let $x, y, z \in p^{n-1}Rp^{n-1}$ such that $xyz = 0$. Since $p^{n-1}Rp^{n-1}$ is a subring of strongly p-symmetric ring R , so we have $xzp^{n-1}y = 0$. This implies that $xzy = 0$, as $p^{n-1}y = y$. Thus, $p^{n-1}Rp^{n-1}$ is a symmetric ring.

Conversely, let us assume that $p^{n-1}Rp^{n-1}$ is a symmetric ring and $p^{n-1} \in Z(R)$. Also, let $a, b, c \in R$ such that $abc = 0$.Since $p^{n-1}Rp^{n-1}$ is a symmetric ring, we have
$(p^{n-1}ap^{n-1})(p^{n-1}bp^{n-1})(p^{n-1}cp^{n-1}) = 0$ implies
$(p^{n-1}ap^{n-1})(p^{n-1}cp^{n-1})(p^{n-1}bp^{n-1}) = 0$.
Therefore, $p^{n-1}ap^{n-1}bp^{n-1}cp^{n-1} = 0$ which implies $p^{n-1}ap^{n-1}cp^{n-1}bp^{n-1} = 0$.
As $p^{n-1} \in Z(R)$, it follows that $p^{n-1}a = ap^{n-1}$, $p^{n-1}b = bp^{n-1}$ and $p^{n-1}c = cp^{n-1}$ for each $a, b, c \in R$. This implies that $acp^{n-1}b = 0$. Thus R is a strongly p-symmetric ring.

□

We have the following corollary as a consequence of Propositions 2.1 and 3.1.

Corollary 3.1 *Let R be a ring and $p \in PT(R)$. Then R is a strongly p-symmetric ring if and only if R is a p-symmetric ring and $p^{n-1} \in Z(R)$.*

Proposition 3.2 *Let R be a ring and $p \in PT(R)$. Then R is a symmetric ring if and only if R is a strongly p-symmetric and $(1-p)R(1-p)$ is a symmetric ring.*

Proof:

The necessary part is obvious.

For the sufficient part, let us assume that R is a strongly p-symmetric and $(1-p)R(1-p)$ is a symmetric ring. Since R is a strongly p-symmetric ring, so by Proposition 3.1 we have, $p^{n-1}Rp^{n-1}$ is a symmetric ring and $p^{n-1} \in Z(R)$. This implies that $p^{n-1}Rp^{n-1} \cong R/(1-p^{n-1})R(1-p^{n-1})$ and $(1-p^{n-1})R(1-p^{n-1}) \cong R/p^{n-1}Rp^{n-1}$. This implies $R/(1-p^{n-1})R(1-p^{n-1})$ and $R/p^{n-1}Rp^{n-1}$ are symmetric rings, as $p^{n-1}Rp^{n-1}$ and $(1-p^{n-1})R(1-p^{n-1})$ are symmetric rings. Thus, $R/((1-p^{n-1})R(1-p^{n-1}) \cap p^{n-1}Rp^{n-1}))$ is a symmetric ring. But $((1-p^{n-1})R(1-p^{n-1}) \cap p^{n-1}Rp^{n-1})) = 0$. Hence R is a symmetric ring.

□

Lemma 3.1 *Let R be a ring and $p^{n-1} \in MP_l(R)$. If p^{n-1} is right semicentral then p^{n-1} left semicentral.*

Proof: We assume that $a \in R$. If $(1-p^{n-1})ap^{n-1} \neq 0$, then $Rp^{n-1} = R(1-p^{n-1})ap^{n-1}$. Let us write $p^{n-1} = c(1-p^{n-1})ap^{n-1}$ for some $c \in R$. Since p^{n-1} is right semicentral, so we have $p^{n-1} = p^{n-1}c(1-p^{n-1})ap^{n-1} = p^{n-1}cp^{n-1}(1-p^{n-1})ap^{n-1} = 0$, which a contradiction. Hence $(1-p^{n-1})ap^{n-1} = 0$ for each $a \in R$. This shows that p^{n-1} is left semicentral.

□

Proposition 3.3 *Let R be a ring and $p \in PT(R)$. Then R is a strongly left min-p-abel ring if and only if R is a strongly p-symmetric ring for any $p^{n-1} \in MP_l(R)$.*

Proof: We assume that R is a strongly left min-p-abel ring. Then by Lemma 3.1, R is a left min-p-abel ring. By Proposition 2.10, R is a p-symmetric ring for each $p^{n-1} \in MP_l(R)$. Again by Lemma 3.1, each element of $MP_l(R)$ is central. Hence by Corollary 3.0.1, R is a strongly p-symmetric ring.

Conversely, let R be a strongly p-symmetric for each $p^{n-1} \in MP_l(R)$. Then by Proposition 3.1, $p^{n-1} \in Z(R)$ for each $p^{n-1} \in MP_l(R)$. This implies that R is a strongly left min-p-abel ring.

□

4. p-Reduced Rings

In this section, we define right (left) p-reduced rings and study their relationships with p-symmetric rings and left min-p-abel rings. We begin with the following definition.

Definition 4.1 *Let R be a ring and $p \in PT(R)$. Then,*

(i) R is called right p-reduced if $N(R)p^{n-1} = 0$.

(ii) R is called left p-reduced if $p^{n-1}N(R) = 0$.

Example 4.1 *Let F be any field. Let us consider the matrix ring $R = M_3(F)$. Then $N(R) = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$, as $\begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent.*

Let $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then for any odd positive integer n, we have $P^n = P$, so $P \in PT(R)$. Then

$$N(R)P^{n-1} = N(R)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Thus, R is a right P-reduced ring. But R is not a left P-reduced ring as,

$$P^{n-1}N(R) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{n-1} N(R) = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Remark 4.1 *We can also construct a left p-reduced ring which is not a right p-reduced. In Example 4.1, if we consider* $P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in PT(R)$*, then R is a left P-reduced ring, but R is not a right P-reduced ring.*

Proposition 4.1 *Let R be a ring and $p \in PT(R)$. Then R is right p-reduced if and only if p^{n-1} is left semicentral in R and $p^{n-1}Rp^{n-1}$ is reduced.*

Proof: Let us assume that R is a right p-reduced ring for any $p \in PT(R)$. Then, we have $(1-p^{n-1})p^{n-1} = 0$. Now, for each $x \in R$, $(1-p^{n-1})xp^{n-1} \in N(R)$ and $(1-p^{n-1})xp^{n-1} \in N(R)p^{n-1} = 0$, as R is a p-reduced ring. So, we have $(1-p^{n-1})xp^{n-1} = 0 \implies xp^{n-1} = p^{n-1}xp^{n-1}$. Thus, p^{n-1} is left semicentral in R. Again, we have $N(p^{n-1}Rp^{n-1}) \subseteq N(R)p^{n-1} = 0$. This implies that $p^{n-1}Rp^{n-1}$ is reduced ring.

Conversely, let p^{n-1} be left semicentral in R and $p^{n-1}Rp^{n-1}$ be a reduced ring. Then $N(R)p^{n-1} = p^{n-1}N(R)p^{n-1} = N(p^{n-1}Rp^{n-1}) = 0$. This implies that R is a p-reduced ring.

□

Similarly, we can establish the following result.

Proposition 4.2 *Let R be a ring and $p \in PT(R)$. Then R is left p-reduced if and only if p^{n-1} is right semicentral in R and $p^{n-1}Rp^{n-1}$ is reduced.*

Proposition 4.3 *Right p-reduced rings are p-symmetric rings.*

Proof: Let R be a right p-reduced ring. Then by Theorem 4.1, we get p^{n-1} is left semicentral in R and $p^{n-1}Rp^{n-1}$ is reduced. Since reduced rings are symmetric by [[1], Theorem I.3], so we have p^{n-1} is left semicentral in R and $p^{n-1}Rp^{n-1}$ is a symmetric ring . Thus, by Proposition 2.1 we have, R is a p-symmetric ring.

□

Remark 4.2 *Right p-reduced rings need not be strongly p-symmetric by Example 4.1 and Proposition 3.1.*

Proposition 4.4 *Let R be a ring and $p \in PT(R)$. Then R is a left min-p-abel ring if and only if R is a right p-reduced ring for each $p^{n-1} \in MP_l(R)$.*

Proof: Let us assume that R is a left min-p-abel ring for each $p^{n-1} \in MP_l(R)$. If $N(R)p^{n-1} \neq 0$, then there exists $a \in N(R)$ such that $ap^{n-1} \neq 0$ which implies there exists $b \in R$ such that $p^{n-1} = bap^{n-1}$, as $Rp^{n-1} = Rap^{n-1}$. Now, by Proposition 2.9 we have, $p^{n-1} = abp^{n-1}$. This implies $p^{n-1} = bap^{n-1} = ba^2bp^{n-1} = b^2a^2p^{n-1} = ... = b^m a^m p^{n-1} = ...$ for each $m \geq 1$. Since $a \in N(R)$, so $a^m = 0$ for some $m \geq 1$. Thus, we have $p^{n-1} = 0$, which is a contradiction. Hence $N(R)p^{n-1} = 0$ and so R is a right p-reduced ring.

The sufficient part follows directly from Proposition 4.1 .

□

Similarly, we can establish the following result.

Proposition 4.5 *Let R be a ring and $p \in PT(R)$. Then R is a strongly left min-p-abel if and only if R is a left p-reduced for each $p^{n-1} \in MP_l(R)$.*

Corollary 4.1 *Let R be a ring and $p, q \in PT(R)$. If R is right p-reduced and $Rp^{n-1} \cong Rq^{n-1}$ as left R-modules, then R is a right q-reduced.*

Proof: Let us assume that R is right p-reduced ring. Then by Proposition 4.1 we have, p^{n-1} is left semicentral and $N(R)p^{n-1} = 0$. Since $Rp^{n-1} \cong Rq^{n-1}$, $p^{n-1}q^{n-1} = q^{n-1}$. Which gives $N(R)q^{n-1} = 0$. Thus R is a q-reduced ring. □

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