Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.77618

Extension of the k-Beta Function Using the Two-Parameter Generalized Mittag-Leffler Function

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ABSTRACT: This paper introduces a new extension of the $k-$Beta function through the two-parameter generalized ML function, highlighting a crucial link between these mathematical constructs. The k-Beta function, recognized for its diverse applications, is expanded to reflect the flexibility of the generalized ML function in fractional calculus and various fields. We discuss important identities and properties of this extended k-Beta function. Our results deepen the understanding of the relationship between these special functions and offer valuable resources for researchers in mathematics and related disciplines investigating fractional processes and their applications.

Keywords: k-Beta function, Mittag Lefler function.

Contents

Abbreviations Used: Mittag-Leffler (ML), Hypergeometric (HG), Confluent Hypergeometric (CHG).

1. Overview and Fundamental Concepts

Throughout the 20th century, special functions have captivated researchers, sparking multiple waves of interest. Defined often through improper integrals or infinite series, these functions serve as vital instruments across diverse fields, including scientific research, chemistry, computational physics, and techno-focused statistical study [1]–[4]. Their significance lies not only in pure mathematics but also in numerous applied contexts. Among these special functions [5], the beta function stands out due to

2020 *Mathematics Subject Classification:* 33B15, 33E12, 33E50, 33C15
Submitted July 02, 2025. Published March 28, 2026

its versatility and wide-ranging applications, impacting areas such as data evaluation, quantum physics, radio frequency field theory, statistical theory, and algebraic number theory.

The k-Beta function, an extension of the classical Beta function, has garnered considerable attention due to its versatility and applicability in various branches of mathematics and applied fields. Traditionally, the Beta function is utilized in contexts such as probability, statistics, and combinatorial mathematics. However, the k-Beta function expands upon this foundation, enabling a broader range of applications, particularly in the study of complex systems and phenomena that exhibit fractional dynamics.

In recent years, the exploration of generalized functions has provided new avenues for mathematical research, leading to a deeper understanding of various phenomena that do not conform to traditional exponential decay patterns. Among these generalized functions, the Mittag-Leffler function [6] has emerged as a powerful tool in fractional calculus and anomalous diffusion, capable of capturing the intricacies of non-exponential behaviors in diverse fields, including physics, biology, and finance. The two-parameter Generalized ML function, in particular, offers a rich framework for generalizing exponential functions and can effectively model processes characterized by variable rates of decay or growth.

By connecting the k-Beta function to the two-parameter Generalized ML function, this paper suggests a novel method for dealing with it. By forging this link, we aim to enrich the theoretical landscape of special functions and provide a comprehensive analysis of the new representations of the k-Beta function. Our exploration will reveal important identities and properties that enhance its applicability in fractional calculus.

1.1. Definition: The Classical Beta & Gamma Functions

The classical beta function is a first-kind Euler integral and is defined as follows:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t \tag{1}$$

This definition applies to any complex numbers m and n, provided that the real parts of both m and n are greater than zero (i.e., $\mathrm{Re}(m), \mathrm{Re}(n) > 0$).

The beta function is intricately linked to the gamma function, articulated in the equation:

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \tag{2}$$

One of the notable properties of the beta function is its symmetry, expressed as:

$$B(m,n) = B(n,m) \tag{3}$$

for all m and n.

$$B(m+n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} \tag{4}$$

The classical gamma function, denoted as $\Gamma(w)$, is a fundamental function in mathematics that extends the concept of factorials to non-integer values. It is defined for complex numbers w with a positive real part. The integral representation of the gamma function is:

$$\Gamma(w) = \int_0^\infty e^{-t}\,t^{w-1}\,\mathrm{d}t, \qquad \mathrm{Re}(w) > 0 \tag{5}$$

1.2. The Extended Form of Beta & Gamma Functions

The generalization of the beta function, introduced by Chaudhry and Zubair [7], is defined as follows:

$$B^b(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\exp\left(-\frac{b}{t(t-1)}\right)\mathrm{d}t \tag{6}$$

Here, the parameters m and n are arbitrary complex numbers with the conditions $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$, $\mathrm{Re}(b) > 0$. The equation (6) reduces to the classical beta function for $b = 0$, as demonstrated in equation (1) (see Chaudhary et al. [7] & [8]).

Chaudhary et al. [9] introduced a novel extension of HG and CHG functions using the extended beta function, which is defined as follows:

$$F^b(m,n;s;w) = \sum_{p=0}^{\infty}(m)_p \frac{B^b(m+p,s-n)}{B(m,s-n)} \frac{w^p}{p!} \tag{7}$$

Where $m,n,s \in \mathbb{R}$, $b > 0$, $|w| < 1$, $\mathrm{Re}(s) > \mathrm{Re}(n) > 0$.

And

$$\phi^b(n;s;w) = \sum_{p=0}^{\infty} \frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(n+p,s-n)}{B(n,s-n)} \frac{w^p}{p!} \tag{8}$$

Where $m,n,s \in \mathbb{R}$, $b \geq 0$, $\mathrm{Re}(s) > \mathrm{Re}(n) > 0$.

In 2018, Shadab et al. [10] introduced the extended form of the beta function by expressing it in relation to the ML function, as detailed below:

$$B^b_\alpha(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1} E_\alpha\left(-\frac{b}{t(t-1)}\right) \mathrm{d}t \tag{9}$$

Where $\mathrm{Re}(b) \geq 0$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$, $\alpha \in \mathbb{R}^+_0$.

And $E_\alpha(w)$ is the Mittag-Leffler function with one parameter and is defined as:

$$E_\alpha(w) = \sum_{p=0}^{\infty} \frac{w^p}{\Gamma(\alpha p + 1)}, \qquad w \in \mathbb{C},\ \alpha \in \mathbb{R}^+_0 \tag{10}$$

Chaudhary et al. [9] defined novel extended HG and CHG functions. Their approach involved utilizing an extended form of the beta function expressed in terms of the extended ML function:

$$F^b_\alpha(m,n;s;w) = \sum_{p=0}^{\infty}(m)_p \frac{B^b_\alpha(m+p,s-n)}{B(m,s-n)} \frac{w^p}{p!} \tag{11}$$

Where $\alpha \in \mathbb{R}^+_0$, $b \in \mathbb{R}^+_0$, $|w| < 1$, $\mathrm{Re}(s) > \mathrm{Re}(n) > 0$.

And

$$\phi^b(n;s;w) = \sum_{p=0}^{\infty} \frac{B^b_\alpha(n+p,s-n)}{B(n,s-n)} \frac{w^p}{p!} \tag{12}$$

Where $\alpha \in \mathbb{R}^+_0$, $b \in \mathbb{R}^+_0$, $|w| < 1$, $\mathrm{Re}(s) > \mathrm{Re}(n) > 0$.

Chaudhary and Zubair [7] defined the extended form of the gamma function as:

$$\Gamma^b(w) = \int_0^\infty t^{w-1} \exp\left(-t - \frac{b}{t}\right) \mathrm{d}t, \qquad \mathrm{Re}(b) > 0 \tag{13}$$

1.3. The Definition of k-Beta & k-Gamma Functions

The k-beta and k-gamma functions were introduced by Akhtar et al. [11]. The k-beta function, $B_k(m,n)$, is the extension of the classical beta function and is defined as

$$B_k(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} \mathrm{d}t \tag{14}$$

where m, n, and k are real or complex numbers.

Also, we have

$$B_k(m,n) = \frac{\Gamma_k(m)\,\Gamma_k(n)}{\Gamma_k(m+n)}, \qquad \mathrm{Re}(m) > 0,\ \mathrm{Re}(n) > 0 \tag{15}$$

and

$$B_k(m,n) = \frac{1}{k} B\Big(\frac{m}{k}, \frac{n}{k}\Big) \tag{16}$$

The integral representation of the k-gamma function is given by:

$$\Gamma_k(w) = \int_0^\infty t^{w-1} \exp\Big(-\frac{t^k}{k}\Big)\,\mathrm{d}t, \qquad \mathrm{Re}(w) > 0 \tag{17}$$

Also,

$$\Gamma_k(w+k) = w\,\Gamma_k(w) \tag{18}$$

This k-gamma function has a relationship with the k-Pochhammer symbol as:

$$(w)_{n,k} = \frac{\Gamma_k(w+nk)}{\Gamma_k(w)} \tag{19}$$

1.4. The Extended k-Gamma & Extended k-Beta Function

Definition 1.1 (Extended k-Gamma Function [12]) *The extended k-gamma function is defined as*

$$\Gamma_k^b(w) = \int_0^\infty t^{w-1} \exp\Big(-\frac{t^k}{k} - \frac{b^k}{k\,t^k}\Big)\,\mathrm{d}t, \qquad \mathrm{Re}(w) > 0,\ b \geq 0,\ k > 0. \tag{20}$$

Remark 1.1 If $b = 0$, the function $\Gamma_k^b(w)$ transforms into the gamma function $\Gamma_k(w)$. When $k = 1$, $\Gamma_k^b(w)$ approaches the extended gamma function $\Gamma_1^b(w)$ as defined in equation (14). When both $b = 0$ and $k = 1$, $\Gamma_k^b(w)$ turns into the classical gamma function $\Gamma(w)$.

Definition 1.2 (Extended k-Beta Function) *The extended k-beta function, denoted as $B_k^b(m,n)$, is defined by*

$$B_k^b(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} \exp\Big(-\frac{b^k}{k\,t(1-t)}\Big)\,\mathrm{d}t \tag{21}$$

where $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$, $k > 0$, *and* $\mathrm{Re}(b) > 0$.

Remark 1.2

(i) For $k = 1$, the above equation (21) becomes valid for the extended beta function.

(ii) For $b = 0$, the above equation (21) becomes valid for the k-beta function.

(iii) For $k = 1$ and $b = 0$, the above equation (21) becomes valid for the classical beta function.

Remark 1.3 Mubeen et al. [12] provide the trigonometric expression for the extended k-beta function:

$$B_k^b(m,n) = \frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2m}{k}-1}(\sin\theta)^{\frac{2n}{k}-1} \exp\Big(-\frac{b^k}{k\cos^2\theta\sin^2\theta}\Big)\,\mathrm{d}\theta. \tag{22}$$

Remark 1.4 Mubeen et al. [12] furnished the integral representation of the extended form of the k-beta function as follows:

$$\int_0^\infty b^{r-1} B_k^b(m,n)\,\mathrm{d}b = \Gamma_k(r)\,B_k(m+r, n+r). \tag{23}$$

1.5. The Mittag-Leffler (ML) Function

Definition 1.3 (Mittag-Leffler Function) *The ML function [13] is defined as*

$$E_a(\Omega) = \sum_{m=0}^{\infty} \frac{\Omega^m}{\Gamma(am+1)}, \tag{24}$$

where $a > 0$ is a parameter, Ω is a complex variable, and Γ is the gamma function.

1.6. One & Two Parameter Mittag-Leffler Functions

Definition 1.4 (One-Parameter Mittag-Leffler Function)

$$E_a(\Omega) = \sum_{m=0}^{\infty} \frac{\Omega^m}{\Gamma(am+1)}. \tag{25}$$

Definition 1.5 (Two-Parameter Mittag-Leffler Function [14])

$$E_{(a,b)}(\Omega) = \sum_{m=0}^{\infty} \frac{\Omega^m}{\Gamma(am+b)}, \tag{26}$$

where $a, b \in \mathbb{C}$ with $a, b > 0$ and $\mathrm{Re}(b) > 0$ and Ω is a complex parameter .

2. A New Extension of Extended k-Beta Function

Definition 2.1 (Extended k-Beta Function via Two-Parameter ML Function) *A novel extension of the extended k-beta function in terms of the two-parameter generalized Mittag-Leffler function is*

$$^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} E_{\alpha,\beta}\left[-\frac{b^k}{k\, t^{\delta}(1-t)^{\theta}}\right] \mathrm{d}t, \tag{27}$$

where $k \in \mathbb{R}^+$, $b \in \mathbb{R}_0^+$, $\alpha, \beta \in \mathbb{R}_0^+$, $\delta, \theta \in \mathbb{R}^+$, $\mathrm{Re}(b) \geq 0$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$.

Definition 2.2 (Two-Parameter Generalized ML Function)

$$E_{\alpha,\beta}(W) = \sum_{p=0}^{\infty} \frac{W^p}{\Gamma(\alpha p+\beta)}, \qquad \alpha, \beta \in \mathbb{R}_0^+,\ W \in \mathbb{C}. \tag{28}$$

Remark 2.1

1. Setting $\alpha = 1$, $\beta = 1$, $\delta = 1$, and $\theta = 1$ in equation (27) gives
$$^{ML}B^{b,1,1}_{k,1,1}(m,n) = B^b_k(m,n). \tag{29}$$
2. If $\alpha = \beta = \delta = \theta = 1$ and $b = 1$, then
$$^{ML}B^{1,1,1}_{k,1,1}(m,n) = B_k(m,n). \tag{30}$$
3. Setting $k = 1$ gives
$$^{ML}B^{b,\delta,\theta}_{1,\alpha,\beta}(m,n) = {}^{ML}B^{b,\delta,\theta}_{\alpha,\beta}(m,n). \tag{31}$$
4. For $k = 1$, $\beta = 1$, $\delta = 1$, $\theta = 1$,
$$^{ML}B^{b,\delta,\theta}_{1,\alpha,1}(m,n) = B^b_{\alpha}(m,n). \tag{32}$$
5. For $k = 1$, $\alpha = \beta = \delta = \theta = 1$,
$$^{ML}B^{b,1,1}_{1,1,1}(m,n) = B^b(m,n). \tag{33}$$
6. $$^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\,{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}\left(\frac{m}{k},\frac{n}{k}\right). \tag{34}$$
7. $$^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(n,m). \tag{35}$$

3. Properties of the Extended k-Beta Function in terms of Two Parameter ML Function

In this section, we derive several crucial relationships related to the extended form of the k-beta function.

Theorem 3.1 *The extended form of the k-beta function in terms of the two-parameter ML function holds the following relation:*

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = {}^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m+k,n) + {}^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n+k) \tag{36}$$

where $k \in \mathbb{R}^+$, $b \in \mathbb{R}_0^+$, $\alpha, \beta \in \mathbb{R}_0^+$, $\delta, \theta \in \mathbb{R}^+$, $\mathrm{Re}(b) \geq 0$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$.

Proof: By the definition of the extended k-beta function in terms of two-parameter ML function, we have

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right]\mathrm{d}t.$$

Then

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n+k) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n+k}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right]\mathrm{d}t \tag{37}$$

and

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m+k,n) = \frac{1}{k}\int_0^1 t^{\frac{m+k}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right]\mathrm{d}t. \tag{38}$$

Adding these gives

$$= \frac{1}{k}\int_0^1 \left[t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}(1-t) + t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}t\right]E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right]\mathrm{d}t$$

which simplifies to

$$= {}^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n).$$

Hence proved. □

Theorem 3.2 *The following integral representations are valid for the extended k-beta function with two-parameter ML function:*

2.1

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\phi)^{\frac{2m}{k}-1}(\sin\phi)^{\frac{2n}{k}-1}E_{\alpha,\beta}\left[-s(\sec^2\phi)^\delta(\csc^2\phi)^\theta\right]\mathrm{d}\phi \tag{39}$$

2.2

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{1}{k}\int_0^\infty \frac{C^{\frac{m}{k}-1}}{(1+C)^{\frac{m+n}{k}}}E_{\alpha,\beta}\left[-\frac{s(1+C)^{\delta+\theta}}{C^\delta}\right]\mathrm{d}C, \tag{40}$$

2.3

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{2^{1-\frac{m}{k}-\frac{n}{k}}}{k}\int_{-1}^{+1}(1+C)^{\frac{m}{k}-1}(1-C)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{2^{\delta+\theta}s}{(1+C)^\delta(1-C)^\theta}\right]\mathrm{d}C. \tag{41}$$

Proof 2.1. By the definition of extended k-beta with the two-parameter ML function, we have

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right]dt. \tag{42}$$

Let $t=\cos^2\phi$, then $dt = -2\cos\phi\sin\phi\,d\phi$. Also, when $t=0$, $\phi=\frac{\pi}{2}$ and when $t=1$, $\phi=0$.

Thus,

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = -\frac{1}{k}\int_{\frac{\pi}{2}}^{0} (\cos^2\phi)^{\frac{m}{k}-1}(\sin^2\phi)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k(\cos^2\phi)^{\delta}(\sin^2\phi)^{\theta}}\right](2\cos\phi\sin\phi)\,d\phi. \tag{43} $$

$$ = \frac{2}{k}\int_{0}^{\frac{\pi}{2}} (\cos\phi)^{\frac{2m}{k}-1}(\sin\phi)^{\frac{2n}{k}-1}E_{\alpha,\beta}\left[-s(\sec^2\phi)^{\delta}(\csc^2\phi)^{\theta}\right]d\phi, \tag{44} $$

where $s = \frac{b^k}{k}$. □

Proof 2.2. By the definition of the extended k-beta two-parameter ML function, we have

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt. \tag{45} $$

Let $t = \frac{C}{1+C}$, then $dt = \frac{1}{(1+C)^2}\,dC$. Also, when $t=0$, $C=0$ and when $t=1$, $C=\infty$. Thus,

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_0^{\infty}\left(\frac{C}{1+C}\right)^{\frac{m}{k}-1}\left(1-\frac{C}{1+C}\right)^{\frac{n}{k}-1}\frac{1}{(1+C)^2}E_{\alpha,\beta}\left[-\frac{b^k}{k\left(\frac{C}{1+C}\right)^{\delta}\left(1-\frac{C}{1+C}\right)^{\theta}}\right]dC. $$

$$ = \frac{1}{k}\int_0^{\infty}\frac{C^{\frac{m}{k}-1}}{(1+C)^{\frac{m+n}{k}}}E_{\alpha,\beta}\left[-\frac{b^k(1+C)^{\delta+\theta}}{k\,C^{\delta}}\right]dC. \tag{46} $$

Therefore,

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_0^{\infty}\frac{C^{\frac{m}{k}-1}}{(1+C)^{\frac{m+n}{k}}}E_{\alpha,\beta}\left[-\frac{s(1+C)^{\delta+\theta}}{C^{\delta}}\right]dC, $$

where $s = \frac{b^k}{k}$. □

Proof 2.3. By the definition of the extended k-beta two-parameter ML function, we have

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt. \tag{47} $$

Let $t = \frac{1+C}{2}$, then $dt = \frac{1}{2}dC$. Also, when $t=0$, $C=-1$ and when $t=1$, $C=1$. Thus,

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_{-1}^{1}\left(\frac{1+C}{2}\right)^{\frac{m}{k}-1}\left(1-\frac{1+C}{2}\right)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\left(\frac{1+C}{2}\right)^{\delta}\left(1-\frac{1+C}{2}\right)^{\theta}}\right]\frac{1}{2}\,dC. $$

$$ = \frac{2^{1-\frac{m}{k}-\frac{n}{k}}}{k}\int_{-1}^{1}(1+C)^{\frac{m}{k}-1}(1-C)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{s\,2^{\delta+\theta}}{(1+C)^{\delta}(1-C)^{\theta}}\right]dC. \qquad \square $$

where $s = \frac{b^k}{k}$.

Remark 3.1 If $\alpha=\beta=\delta=\theta=1$, then Theorem 2 reduces to the extended form of the k-beta function.

Remark 3.2

If $\delta=\theta$, then the above theorem (2.3) reduces to the following relation:

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{2^{1-\frac{m}{k}-\frac{n}{k}}}{k}\int_{-1}^{+1}(1+C)^{\frac{m}{k}-1}(1-C)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{s4^{\delta}}{(1-C^2)^{\delta}}\right]\mathrm{d}C. \tag{48} $$

Theorem 3.3 *Let $k \in \mathbb{R}^+$, $b \in \mathbb{R}_0^+$. Then the extended k-beta two-parameter ML function holds the relation*

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \sum_{p=0}^{\infty} {^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}}(m+pk, n+k). \tag{49}$$

Proof: We have the extended k-beta two-parameter ML functions defined as

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} E_{\alpha,\beta}\left[-\frac{b^k}{kt^\delta(1-t)^\theta}\right] \mathrm{d}t.$$

$$= \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}}(1-t)^{-1} E_{\alpha,\beta}\left[-\frac{b^k}{kt^\delta(1-t)^\theta}\right] \mathrm{d}t.$$

Then,

$$= \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}} \sum_{p=0}^{\infty} t^p E_{\alpha,\beta}\left[-\frac{b^k}{kt^\delta(1-t)^\theta}\right] \mathrm{d}t. \tag{50}$$

Through summation and rearrangement, we obtain

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \sum_{p=0}^{\infty}\frac{1}{k}\int_0^1 t^{\frac{m}{k}+p-1}(1-t)^{\frac{n}{k}+1-1} E_{\alpha,\beta}\left[-\frac{b^k}{kt^\delta(1-t)^\theta}\right] \mathrm{d}t. \tag{51}$$

$$= \sum_{p=0}^{\infty}\frac{1}{k}\int_0^1 t^{\frac{m+kp}{k}-1}(1-t)^{\frac{n+k}{k}-1} E_{\alpha,\beta}\left[-\frac{b^k}{kt^\delta(1-t)^\theta}\right] \mathrm{d}t.$$

Thus,

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \sum_{p=0}^{\infty} {^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}}(m+pk, n+k) \qquad \text{(Proved).}$$

□

Theorem 3.4 *The given summation formula is true for the extended k-beta function expressed in terms of the two-parameter ML function:*

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \sum_{p=0}^{r}\binom{r}{p} {^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}}(m+pk, n+rk-p), \qquad p \in \mathbb{N}_0. \tag{52}$$

Proof: We have

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right] dt.$$

$$= \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}\big(t+(1-t)\big) E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right] dt.$$

$$= \frac{1}{k}\int_0^1 t^{\left(\frac{m}{k}+1-1\right)}(1-t)^{\frac{n}{k}-1} E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right] dt + \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\left(\frac{n}{k}+1-1\right)} E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^\delta(1-t)^\theta}\right] dt. \tag{53}$$

Thus,

$$^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(m,n) = {^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}}(m+k,n) + {^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}}(m,n+k). \tag{54}$$

Repeating the same argument for the above, we have:

$$
\begin{aligned}
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) &= \frac{1}{k}\int_0^1 t^{\left(\frac{m}{k}+1-1\right)}(1-t)^{\frac{n}{k}-1}\big(t+(1-t)\big)E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt \\
&+ \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\left(\frac{n}{k}+1-1\right)}\big(t+(1-t)\big)E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt.
\end{aligned}
\tag{55}
$$

Which expands to

$$
\begin{aligned}
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) &= \frac{1}{k}\int_0^1 t^{\left(\frac{m}{k}+2-1\right)}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt \\
&+ \frac{1}{k}\int_0^1 t^{\left(\frac{m}{k}+1-1\right)}(1-t)^{\left(\frac{n}{k}+1-1\right)}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt \\
&+ \frac{1}{k}\int_0^1 t^{\left(\frac{m}{k}+1-1\right)}(1-t)^{\left(\frac{n}{k}+1-1\right)}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt \\
&+ \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\left(\frac{n}{k}+2-1\right)}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt.
\end{aligned}
\tag{56}
$$

Thus,

$$
\begin{aligned}
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) &= {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+2k,n) + {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+k,n+k) \\
&+ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+k,n+k) + {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n+2k).
\end{aligned}
\tag{57}
$$

This leads us to:

$$
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+2k,n) + 2\,{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+k,n+k) + {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n+2k).
\tag{58}
$$

Thus, the desired result is obtained through the application of mathematical induction on this ongoing process:

$$
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \sum_{p=0}^{r}\binom{r}{p}\,{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+pk,\ n+rk-p). \quad \text{Proved.}
$$

□

Theorem 3.5 *The following relation holds:*

$$
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,k-n) = \sum_{p=0}^{\infty}\frac{(n/k)_p}{p!}\,{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+pk,k),
\tag{59}
$$

where $(n/k)_p$ *denotes the Pochhammer symbol.*

Proof: We have

$$
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt.
$$

Then,

$$
{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,k-n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{k-n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt.
\tag{60}
$$

That is,

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,k-n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{-\frac{n}{k}} E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right] dt. \tag{61} $$

From the generalized binomial theorem,

$$ (1-t)^{-z} = \sum_{p=0}^{\infty} \frac{(z)_p}{p!}\, t^p, \qquad |t|<1, \tag{62} $$

we obtain

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,k-n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}\sum_{p=0}^{\infty}\frac{(n/k)_p}{p!}\, t^p E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right] dt. \tag{63} $$

Changing the order of summation and integration gives

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,k-n) = \sum_{p=0}^{\infty}\frac{(n/k)_p}{p!}\,\frac{1}{k}\int_0^1 t^{\frac{m+pk}{k}-1}(1-t)^{\frac{k}{k}-1} E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right] dt. \tag{64} $$

Thus,

$$ {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,k-n) = \sum_{p=0}^{\infty}\frac{(n/k)_p}{p!}\; {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+pk,k), $$

which completes the proof. □

Theorem 3.6 *The extended k-beta two-parameter ML function satisfies the Mellin transform*

$$ M\left\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\right\} = k^{\left(\frac{s}{k}-2\right)}\frac{\pi}{\sin\left(\frac{\pi s}{k}\right)\Gamma\left(\beta-\frac{s}{k}\alpha\right)} B\left(\frac{m+\delta s}{k}, \frac{n+\theta s}{k}\right), \tag{65} $$

where

$$ \mathrm{Re}(\alpha)>0,\quad \mathrm{Re}(m)>0,\quad \mathrm{Re}(\delta)>0,\quad \mathrm{Re}(\alpha-s\delta)>0,\quad \mathrm{Re}(m+s\delta)>0,\quad \mathrm{Re}(b)\geq 0. $$

Proof: Applying the Mellin transform yields

$$ M\left\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\right\} = \int_0^{\infty} b^{s-1}\,{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\,\mathrm{d}b. \tag{66} $$

Substituting the definition,

$$ M\left\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\right\} = \int_0^{\infty} b^{s-1}\left(\frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{kt^{\delta}(1-t)^{\theta}}\right]\mathrm{d}t\right)\mathrm{d}b. \tag{67} $$

Interchanging the order of integration and letting

$$ w = \frac{b^k}{kt^{\delta}(1-t)^{\theta}}, $$

$$ dw = \frac{kb^{k-1}}{kt^{\delta}(1-t)^{\theta}}db. \tag{68} $$

$$ b^{1-k}t^{\delta}(1-t)^{\theta}dw = db. \tag{69} $$

Also,

$$ wkt^{\delta}(1-t)^{\theta} = b^k, $$

$$ w^{\frac{1}{k}}k^{\frac{1}{k}}t^{\frac{\delta}{k}}(1-t)^{\frac{\theta}{k}} = b. \tag{70} $$

$$w^{\frac{s-k}{k}}k^{\frac{s-k}{k}}t^{\frac{\delta(s-k)}{k}}(1-t)^{\frac{\theta(s-k)}{k}}=b^{(s-k)}. \tag{71}$$

We have:

$$M\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\}=\frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}t^{\delta}(1-t)^{\theta}\left(\int_0^\infty b^{1-k}b^{s-1}E_{\alpha,\beta}[-w]dw\right)dt. \tag{72}$$

$$=\frac{1}{k}\int_0^1 t^{\frac{m}{k}+\delta-1}(1-t)^{\frac{n}{k}+\theta-1}\left(\int_0^\infty b^{s-k}E_{\alpha,\beta}[-w]dw\right)dt.$$

Now, by using the formula:

$$\int_0^\infty w^{m-1}E^{\gamma}_{(\alpha,\beta)}[-wu]dw=\frac{\Gamma(m)\Gamma(\gamma-m)}{\Gamma(\gamma)u^m\Gamma(\beta-m\alpha),} \tag{73}$$

We have:

$$M\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\}=\frac{k^{\frac{s-k}{k}}}{k}\int_0^1 t^{\left(\frac{m+s\delta}{k}-1\right)}(1-t)^{\left(\frac{n+s\theta}{k}-1\right)}\frac{\Gamma\left(\frac{s}{k}\right)\Gamma\left(1-\frac{s}{k}\right)}{\Gamma\left(\beta-\frac{s}{k}\alpha\right)}dt. \tag{74}$$

Using Euler's reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}, \tag{75}$$

We arrive at

$$M\left\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\right\}=k^{\left(\frac{s}{k}-2\right)}\frac{\pi}{\sin\left(\frac{\pi s}{k}\right)\Gamma\left(\beta-\frac{s}{k}\alpha\right)}\int_0^1 t^{\frac{m+\delta s}{k}-1}(1-t)^{\frac{n+\theta s}{k}-1}\mathrm{d}t. \tag{76}$$

Finally, recognizing the beta function, we obtain

$$M\left\{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\right\}=k^{\left(\frac{s}{k}-2\right)}\frac{\pi}{\sin\left(\frac{\pi s}{k}\right)\Gamma\left(\beta-\frac{s}{k}\alpha\right)}B\left(\frac{m+\delta s}{k},\frac{n+\theta s}{k}\right),$$

which completes the proof. □

Theorem 3.7 *The following relation holds:*

$${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)=\sum_{p=0}^{\infty}\frac{(-b^k)^p}{k^{p+1}\,\Gamma(\alpha p+\beta)}\,B(m-k\delta p,\ n-k\theta p)\,. \tag{77}$$

Proof: From the extended k-beta function defined in section (1.4)

$${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)=\frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]\mathrm{d}t.$$

Using

$$E_{\alpha,\beta}(\Omega)=\sum_{p=0}^{\infty}\frac{\Omega^p}{\Gamma(\alpha p+\beta)},$$

we obtain

$${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)=\frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}\sum_{p=0}^{\infty}\frac{(-b^k)^p}{k^p t^{\delta p}(1-t)^{\theta p}\Gamma(\alpha p+\beta)}\,\mathrm{d}t.$$

Exchanging summation and integration gives

$${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)=\sum_{p=0}^{\infty}\frac{(-b^k)^p}{k^{p+1}\Gamma(\alpha p+\beta)}\int_0^1 t^{\frac{m}{k}-\delta p-1}(1-t)^{\frac{n}{k}-\theta p-1}\mathrm{d}t.$$

$${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)=\sum_{p=0}^{\infty}\frac{(-b^k)^p}{k^{p+1}\,\Gamma(\alpha p+\beta)}\,B(m-k\delta p,\ n-k\theta p)\,.$$

This completes the proof. □

4. Beta Distribution

We define the beta distribution associated with ${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)$ by the density function expressed as

$$f(t)=\begin{cases}\dfrac{t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\dfrac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]}{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)}, & 0<t<1,\\ 0, & \text{otherwise.}\end{cases} \tag{78}$$

For $d\in\mathbb{R}$, the d^{th} moment of the random variable x is given by

$$E(x^d)=\frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+dk,n)}{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)}. \tag{79}$$

Where $m,n\in\mathbb{R}$, $b>0$, $\alpha,\beta\in\mathbb{R}^+$, $\delta,\theta\in\mathbb{R}^+$.
When $d=1$, we have the distribution's mean, which is a special case of (79) and defined as

$$\rho=E(x)=\frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+k,n)}{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)}. \tag{80}$$

And the distribution's variance is given by:

$$\sigma^2=E(x^2)-\{E(x)\}^2 \tag{81}$$

$$\operatorname{Var}(x)=\frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+2k,n)\,{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)-\left({}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+k,n)\right)^2}{\left({}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)\right)^2}. \tag{82}$$

And the distribution's moment-generating function is described as

$$M(t)=\sum_{p=0}^{\infty}\frac{t^p}{p!}E(x^p)=\frac{1}{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)}\sum_{p=0}^{\infty}{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m+pk,n)\frac{t^p}{p!}. \tag{83}$$

And the cumulative distribution is defined as

$$F(x)=\frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta,x}(m,n)}{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(m,n)}, \tag{84}$$

where

$${}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta,x}(m,n)=\frac{1}{k}\int_0^x t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]\mathrm{d}t. \tag{85}$$

is an incomplete extended k-beta function in terms of ML function and $m,n\in\mathbb{R}$, $b>0$, $\alpha,\beta\in\mathbb{R}^+$, $\delta,\theta\in\mathbb{R}^+$.

5. Gauss Hypergeometric and Confluent Hypergeometric Functions

The classical Gauss HG function [15] is defined as:

$$F(m,n;s;z)=\sum_{p=0}^{\infty}\frac{(m)_p(n)_p}{(s)_p}\frac{z^p}{p!}, \tag{86}$$

Where $(m)_p$ is the Pochhammer symbol defined for $m\in\mathbb{C}$:

$$(m)_p = \frac{\Gamma(m+p)}{\Gamma(p)} \tag{87}$$

The Pochhammer-k symbol is defined as:

$$(m)_{(p,k)} = \begin{cases} m(m+k)(m+2k)\cdots(m+(p-1)k) & \text{if } p \in \mathbb{N} \\ 1 & \text{if } p = 0 \end{cases} \tag{88}$$

And also:

$$(m)_{(p,k)} = \frac{\Gamma_k(m+pk)}{\Gamma_k(p)} \tag{89}$$

Furthermore, we have:

$$(m)_{p,k} = k^p \cdot (m/k)_p \quad \text{where } m \in \mathbb{C}, k \in \mathbb{R}^+. \tag{90}$$

The CHG function is defined as:

$$\phi(n;s;z) = \sum_{p=0}^{\infty} \frac{(n)_p}{(s)_p} \frac{z^p}{p!} \tag{91}$$

5.1. Extended Hypergeometric (HG) & Extended Confluent Hypergeometric (CHG) Functions

The extended k-beta two-parameter ML function can be used to further extend the Gauss HG function and the extended CHG function [9] as follows:

$$^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;z) = \sum_{p=0}^{\infty} \frac{(m)_{p,k}\, {}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(n+pk, s-n)}{B_k(n,s-n)} \frac{z^p}{p!} \tag{92}$$

Where $k \in \mathbb{R}^+, m,n,s \in \mathbb{R}, b > 0, b \in \mathbb{R}_0^+, \alpha,\beta \in \mathbb{R}^+, \delta,\theta \in \mathbb{R}^+, |z| < 1, \mathrm{Re}(s) > \mathrm{Re}(n) > 0$.
And the extended CHG [9] & [16] function is defined as:

$$^{ML}\phi^{b,\delta,\theta}_{k,\alpha,\beta}(n;s;z) = \sum_{p=0}^{\infty} \frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(n+pk, s-n)}{B_k(n,s-n)} \frac{z^p}{p!} \tag{93}$$

Where $k \in \mathbb{R}^+, m,n,s \in \mathbb{R}, b \geq 0, b \in \mathbb{R}_0^+, \alpha,\beta \in \mathbb{R}^+, \delta,\theta \in \mathbb{R}^+, \mathrm{Re}(s) > \mathrm{Re}(n) > 0$.

5.2. The extended (HG) and extended (CHG) Functions in Their Integral Representation

Theorem 5.1

$$^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;z) = \frac{1}{B_k(n,s-n)} \cdot \frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}(1-kzt)^{-m/k} E_{\alpha,\beta}\left[-\frac{b^k}{k\, t^{\delta}(1-t)^{\theta}}\right] \mathrm{d}t. \tag{94}$$

Where $b \in \mathbb{R}_0^+, \alpha,\beta,\delta,\theta \in \mathbb{R}^+, |arg(1-z)| < \pi, \mathrm{Re}(s) > \mathrm{Re}(n) > 0$.
And

$$^{ML}\phi^{b,\delta,\theta}_{k,\alpha,\beta}(n;s;z) = \frac{1}{B_k(n,s-n)} \cdot \frac{1}{k}\int_0^1 t^{(n/k-1)}(1-t)^{(s-n)/k-1} e^{zt} E_{\alpha,\beta}\left[-\frac{b^k}{kt^{\delta}(1-t)^{\theta}}\right] dt \tag{95}$$

Where $b \in \mathbb{R}_0^+, \alpha,\beta,\delta,\theta \in \mathbb{R}^+, \mathrm{Re}(s) > \mathrm{Re}(n) > 0$.
Proof: We have

$$^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;z) = \sum_{p=0}^{\infty} \frac{(m)^{ML}_{p,k} B^{b,\delta,\theta}_{k,\alpha,\beta}(n+pk, s-n)}{B_k(n,s-n)} \frac{z^p}{p!}$$

$$^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z)=\sum_{p=0}^{\infty}(m)_{p,k}\frac{\frac{1}{k}\int_0^1 t^{\frac{n+pk}{k}-1}(1-t)^{\frac{s-n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt}{B_k(n,s-n)}\frac{z^p}{p!}. \tag{96}$$

Now by switching the order of the summation and integral we have,

$$=\frac{1}{B_k(n,s-n)}\cdot\frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]\sum_{p=0}^{\infty}(m)_{p,k}\frac{(zt)^p}{p!}\,dt \tag{97}$$

Also, by the binomial theorem:

$$\sum_{p=0}^{\infty}(W)_{p,k}\frac{(Xt)^p}{p!}=(1-kXt)^{-W/k} \tag{98}$$

Then,

$$^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z)=\frac{1}{B_k(n,s-n)}\cdot\frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}(1-ztk)^{-\frac{m}{k}}$$
$$\times E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt. \qquad \text{(Proved)}$$

Similarly, for the extended CHG function:

$$^{ML}\phi_{k,\alpha,\beta}^{b,\delta,\theta}(n;s;z)=\sum_{p=0}^{\infty}\frac{^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(n+pk,\,s-n)}{B_k(n,\,s-n)}\frac{z^p}{p!}. \tag{99}$$

$$^{ML}\phi_{k,\alpha,\beta}^{b,\delta,\theta}(n;s;z)=\frac{1}{B_k(n,s-n)}\frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}E_{\alpha,\beta}\left[-\frac{b^k}{kt^{\delta}(1-t)^{\theta}}\right]dt\sum_{p=0}^{\infty}\frac{(zt)^p}{p!}$$

$$^{ML}\phi_{k,\alpha,\beta}^{b,\delta,\theta}(n;s;z)=\frac{1}{B_k(n,s-n)}\cdot\frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}e^{zt}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt. \qquad \text{(Proved)}$$

5.3. The r-th Derivative

Theorem 5.2 *The r-th derivative of the extended HG and CHG functions is given by*

$$\frac{d^r}{dz^r}\left[{}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z)\right]=\frac{(m)_{r,k}\,(n)_{r,k}}{(s)_{r,k}}\;{}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m+rk,\,n+rk;\,s+rk;\,z), \tag{100}$$

and

$$\frac{d^r}{dz^r}\left[{}^{ML}\phi_{k,\alpha,\beta}^{b,\delta,\theta}(n;s;z)\right]=\frac{(n)_{r,k}}{(s)_{r,k}}\;{}^{ML}\phi_{k,\alpha,\beta}^{b,\delta,\theta}(n+rk;\,s+rk;\,z). \tag{101}$$

Proof: From the definition of the extended HG function (92),

$$^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z)=\sum_{p=0}^{\infty}(m)_{p,k}\frac{^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(n+pk,\,s-n)}{B_k(n,\,s-n)}\frac{z^p}{p!}.$$

Where $k\in\mathbb{R}^+, m,n,s\in\mathbb{R}, b>0, b\in\mathbb{R}_0^+, \alpha,\beta\in\mathbb{R}^+, \delta,\theta\in\mathbb{R}^+, |z|<1, \mathrm{Re}(s)>\mathrm{Re}(n)>0$.

Differentiating with respect to z, we obtain

$$\frac{d}{dz}{}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z)=\sum_{p=1}^{\infty}(m)_{p,k}\frac{^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(n+pk,\,s-n)}{B_k(n,\,s-n)}\frac{z^{p-1}}{(p-1)!}. \tag{102}$$

Replacing p by $p+1$ gives

$$\frac{d}{dz}\,{}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;z)=\sum_{p=0}^{\infty}(m)_{p+1,k}\,\frac{{}^{ML}B^{b,\delta,\theta}_{k,\alpha,\beta}(n+pk+k,\ s-n)}{B_k(n,\ s-n)}\,\frac{z^p}{p!}. \tag{103}$$

Now by applying the relation

$$B_k(n,s-n)=\frac{(s)_{(1,k)}}{(n)_{(1,k)}}B(n+k,s-n) \tag{104}$$

and

$$(w+k)_{(p,k)}=w(w+1)_{(p)}$$

we get

$$\frac{d}{dz}[{}^{ML}F^{(b,\delta,\theta)}_{(k,\alpha,\beta)}(m,n;s;z)]=\frac{(m)_{1,k}(n)_{1,k}}{(s)_{1,k}}\sum_{p=0}^{\infty}(m+k)_p\frac{{}^{ML}B^{b,\delta,\theta)}_{k,\alpha,\beta}(n+pk+k,s-n)}{B_k(n,s-n)}z^p/p! \tag{105}$$

we obtain

$$\frac{d}{dz}\,{}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;z)=\frac{(m)_{1,k}(n)_{1,k}}{(s)_{1,k}}\ {}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m+k,n+k;s+k;z). \tag{106}$$

Repeating this process r times yields,

$$\frac{d^r}{dz^r}\,{}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;z)=\frac{(m)_{r,k}(n)_{r,k}}{(s)_{r,k}}\ {}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m+rk,n+rk;s+rk;z),$$

which proves the first identity. The second one can be obtained by differentiating equation (93) r times w. r.to z in the same way.

$$\frac{d^r}{dz^r}\left[{}^{ML}\phi^{b,\delta,\theta}_{k,\alpha,\beta}(n;s;z)\right]=\frac{(n)_{r,k}}{(s)_{r,k}}\ {}^{ML}\phi^{b,\delta,\theta}_{k,\alpha,\beta}(n+rk;\ s+rk;\ z).\qquad\text{(Proved)}$$

□

Theorem 5.3 *The extended hypergeometric and extended confluent hypergeometric functions satisfy the following transformation and summation formulas:*

$${}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,n;s;zk)=(1-kz)^{-m/k}\ {}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}\left(m,s-n;s;-\frac{zk}{1-kz}\right) \tag{107}$$

$${}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}\left(m,n;s;1-\frac{1}{kz}\right)=(kz)^{m/k}\ {}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,s-n;s;1-kz) \tag{108}$$

$${}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}\left(m,n;s;\frac{zk}{1+zk}\right)=(1+kz)^{m/k}\ {}^{ML}F^{b,\delta,\theta}_{k,\alpha,\beta}(m,s-n;s;-zk) \tag{109}$$

$${}^{ML}\phi^{b,\delta,\theta}_{k,\alpha,\beta}(n;s;z)=e^{zk}\ {}^{ML}\phi^{b,\delta,\theta}_{k,\alpha,\beta}(s-n;s;-zk) \tag{110}$$

where

$$k>0,\qquad b\in\mathbb{R}^+_0,\qquad \alpha,\beta,\delta,\theta\in\mathbb{R}^+,\qquad |z|<1,\qquad \Re(s)>0.$$

Proof: From the integral representation (94):

$$ {}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z) = \frac{1}{B_k(n,s-n)}\cdot\frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}(1-kzt)^{-m/k}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt. \quad (111)$$

Replacing $t\mapsto 1-t$, assuming $\delta=\theta$, and using

$$ [1-kz(1-t)]^{-m/k} = (1-kz)^{-m/k}\left(1+\frac{kz}{1-kz}\,t\right)^{-m/k} \quad (112)$$

we obtain

$$ {}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z) = \frac{(1-kz)^{-m/k}}{B_k(n,s-n)}\cdot\frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{s-n}{k}-1}\left(1-\frac{-kz}{1-kz}\,t\right)^{-m/k}E_{\alpha,\beta}\left[-\frac{b^k}{k\,t^{\delta}(1-t)^{\theta}}\right]dt. \quad (113)$$

Recognizing again the integral form in (94), we obtain (107).
Using substitution $kz = 1-\frac{1}{kz}$ in equation (107) proves (108), and substituting $kz=\frac{kz}{1+kz}$ in equation (107) proves (109).
To get the proof of (110), we can do the same procedure as we have done for (107). (Proved)

□

6. Generating Function for ${}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m,n;s;z)$

Theorem 6.1 *For $k>0$,*

$$ \sum_{q=0}^{\infty}(m)_{q,k}\,{}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m+qk,n;s;z)\frac{D^q}{q!} = (1-kD)^{-m/k}\,{}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}\left(m,n;s;\frac{z}{1-kD}\right), \quad (114)$$

where $|D|<1,\ |z|<1,\ b\in\mathbb{R}_0^+,\ \alpha,\beta,\delta,\theta\in\mathbb{R}^+$.

Proof: Starting with

$$ \sum_{q=0}^{\infty}(m)_{q,k}\,{}^{ML}F_{k,\alpha,\beta}^{b,\delta,\theta}(m+qk,n;s;z)\frac{D^q}{q!}, $$

and inserting the definition,

$$ = \sum_{q=0}^{\infty}(m)_{q,k}\left[\sum_{p=0}^{\infty}(m+qk)_{p,k}\frac{{}^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(n+pk,s-n)}{B_k(n,s-n)}\frac{z^p}{p!}\right]\frac{D^q}{q!}. $$

Interchanging the order of summation gives

$$ = \sum_{p=0}^{\infty}(m)_{p,k}\frac{{}^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(n+pk,s-n)}{B_k(n,s-n)}\frac{z^p}{p!}\sum_{q=0}^{\infty}(m+pk)_{q,k}\frac{D^q}{q!}. $$

Using

$$ \sum_{q=0}^{\infty}(m+pk)_{q,k}\frac{D^q}{q!} = (1-kD)^{-(m+pk)/k}, $$

we obtain

$$ = (1-kD)^{-m/k}\sum_{p=0}^{\infty}(m)_{p,k}\frac{{}^{ML}B_{k,\alpha,\beta}^{b,\delta,\theta}(n+pk,s-n)}{B_k(n,s-n)}\frac{\left(\frac{z}{1-kD}\right)^p}{p!}. $$

Recognising the extended hypergeometric function on the right–hand side completes the proof. □

7. Conclusion

In this paper, a new extension of the k-beta function has been introduced by means of the generalized Mittag–Leffler function. Several analytical properties of this extended k-beta function have been investigated, including various integral representations. Furthermore, extended hypergeometric and confluent hypergeometric functions associated with this extended k-beta function have been defined and studied.

In addition, a number of important results such as differentiation formulas, integral transforms, Mellin transforms, together with transformation and summation formulas, have been established. It is expected that the results obtained in this work will provide a useful contribution to the ongoing research in fractional calculus and its applications.

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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