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Some Coupled fixed point theorems on G-metric spaces

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ABSTRACT:
In this paper, we establish some coupled fixed point theorems for the weakly increasing mappings f and g with respect to partial ordering relation in the framework of generalized metric spaces and illustrated the usability of result with the help of an example.

Key Words: G-metric space, coupled fixed point, weakly increasing mappings, partially ordered set.

Contents

1. Introduction

Banach contraction result [1] is one of fundamental result to find the solution of a nonlinear problem using fixed point approach. Several researchers like Ćirić [4,22,26], Kannan [7] and Chatterjea [3,12,14,15,16,17,18,19] generalized the contraction mapping and obtained very interesting results in fixed point theory. Dhage [5] was initiated the study of general metric spaces called $D-$ metric spaces. In 2005, Mustafa and Sims [8] give the notion of new structure of metric spaces called $G-$ metric spaces and derived some very interesting fixed point results in the setting of such spaces. Gnana Bhaskar and Lakshmikantham [6] obtained some coupled fixed point theorems for metric spaces having mixed monotone property. Thereafter, Choudhury and Maity [2] extend the results of Gnana Bhaskar and Lakshmikantham [6] in the setting of $G-$ metric spaces. Using the idea of Gnana Bhaskar and Lakshmikantham [6], Choudhury and Maity [2] and of Shatanawi [21,23,24,27], we obtain some couple fixed point theorems for weakly increasing mappings in the setting of partially ordered $G-$ metric spaces in this paper. Also, we provide an example to show the usability of the results obtained.

2. Preliminaries

To begin with, we give some basic definitions, notations and some results to be used in the sequel.

Definition 2.1 *[8,25] Let X be a non-empty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following*

1. $G(x, y, z) = 0$ *if* $x = y = z$,
2. $G(x, x, y) > 0$ *for all* $x, y \in X$, *with* $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $y \neq z$,
4. $G(x, y, z) = G(y, z, x) = G(z, x, y) = \cdots$ *(symmetry in all three variables),*
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangular inequality).*

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Then the function G is called a generalized metric or more specifically a $G-$metric on X and the pair (X,G) is a $G-$metric space.

Example 2.1 *If X is a non empty subset of R, then the function $G : X \times X \times X \to [0,\infty)$, given by $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x,y,z \in X$, is a $G-$metric on X.*

Example 2.2 *[20] Let $X = \{0,1,2\}$ and let $G : X \times X \times X \to [0,\infty)$ be the function given by the following table.*

(x,y,z)	G(x,y,z)
(0,0,0), (1,1,1), (2,2,2)	0
(0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0)	1
(1,2,2), (2,1,2), (2,2,1)	2
(0,0,2), (0,2,0), (2,0,0), (0,2,2), (2,0,2), (2,2,0)	3
(1,1,2), (1,2,1), (2,1,1), (0,1,2), (0,2,1), (1,0,2)	4
(1,2,0), (2,0,1), (2,1,0)	4

Then G is a $G-$ metric on X, but it is not symmetric because $G(1,1,2) = 4 \neq 2 = G(2,2,1)$.

Definition 2.2 *[8] Let (X,G) be a $G-$metric space, let $\{x_n\}$ be sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$ and we say that the sequence $\{x_n\}$ is $G-$convergent to x.*

Thus, if $x_n \to x$ in a $G-$metric space (X,G), then for any $\epsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$.

Definition 2.3 *[8] Let (X,G) be a $G-$metric space. The sequence $\{x_n\}$ is said to be $G-$ Cauchy if for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_l) < \in$ for all $n,m,l \geq N$.*

Lemma 2.1 *[8] Let (X,G) be a G- metric space, then the following are equivalent:*

(1) $\{x_n\}$ is G-convergent to x.

(2) $G(x_n,x_n,x) \to 0$, as $n \to \infty$.

(3) $G(x_n,x,x) \to 0$, as $n \to \infty$.

(4) $G(x_m,x_n,x) \to 0$, as $m,n \to \infty$.

Lemma 2.2 *[8,21] If (X,G) be a G- metric space, then the following are equivalent:*

(1) $\{x_n\}$ is G-Cauchy.

(2) for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_m) < \in$ for all $n,m \geq N$.

Lemma 2.3 *[8] If (X,G) be a G metric space, then $G(x,y,z) \leq 2G(x,y,z)$ for all $x,y \in X$.*

Lemma 2.4 *[8] If (X,G) be a G metric space, then The sequence $\{x_n\}$ is a $G-$Cauchy sequence if and only if for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_m) < \epsilon$ for all $m > n \geq N$.*

Definition 2.4 *[13,23,27] Let (X,G) and (X',G') be two $G-$metric spaces and $f : (X,G) \to (X',G')$ be a function, then f is said to be $G-$continous at a point $a \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G- convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Definition 2.5 *[2] A G metric space (X,G) is called symmetric G-metric space if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.*

Definition 2.6 *A $G-$metric space (X,G) is said to be $G-$complete (or complete $G-$metric space) if every $G-$Cauchy sequence in (X,G) is $G-$convergent in (X,G).*

Definition 2.7 (POSET) *A partial order is a binary relation $\leq$ over a set X satisfying the following properties. For all a, b and c in X,*

(1) Reflexivity : $a \leq a$ *(every element is related to itself)*

(2) Antisymmetry : *If* $a \leq b$ *and* $b \leq a$ *then* $a = b$

(3) Transitivity : *If* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$

A set X with a partial order that is $(X, \leq)$ is called a partially ordered set (also called a Poset).

Example 2.3 *The set of natural numbers equipped with the relation of divisibility is a Poset.*

Definition 2.8 *For a, b elements of a partially ordered set X, if $a \leq b$ or $b \leq a$, then a and b are comparable.*

Definition 2.9 *[6] An element $(x, y) \in X \times X$; when X is any non empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 2.10 *Let (X, G) be a $G-$ metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two $G-$ convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $F(x_n, y_n)$ is $G-$ convergent to $F(x, y)$.*

Lemma 2.5 *[11,24] Let (X, G) be a G metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables. Every $G-$ metric on X will define a metric d_G on X by $d_G(x, y) = G(x, y, y) + G(y, x, x)$, for all $x, y \in X$. For a symmetric $G-$ metric space, $d_G(x, y) = 2G(x, y, y)$, for all $x, y \in X$.*

However, if G is not symmetric, then the following inequality holds $\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$, for all $x, y \in X$.

Definition 2.11 *[11] Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $f(x) \preceq g(f(x))$ and $g(x) \preceq f(g(x))$, for all $x \in X$.*

Two weakly increasing mappings need not be non-decreasing.

Example 2.4 *[11] Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be define by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Since $f(x) = x^2 \leq g(f(x)) = g(x)$. Then $f$and g are weakly increasing mappings. Note that f and g are not non-decreasing.*

The aim of this paper is to study a number of coupled fixed point results for two weakly increasing mappings f and g with respect to partial ordering relation $(\preceq)$ in a generalized metric space.

Now, we announce our first new result.

3. Main Results

Theorem 3.1 *Let $(X, \preceq)$ be a partially ordered set and there exists $G-$ metric in X such that (X, G) is $G-$ complete. Let $f, g : X \times X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose there exist non negative real numbers a, b and c with $2a + 4b + 4c < 1$ such that*

$$\begin{aligned} G(f(x,y), g(u,v), g(u,v)) \leq & a[G(x,u,u) + G(y,v,v)] \\ & + b[G(x, f(x,y), f(x,y)) + G(x, g(u,v), g(u,v)) \\ & + G(u, f(x,y), f(x,y)) + G(u, g(u,v), g(u,v))] \\ & + c[G(x, g(u,v), g(u,v)) + G(x, f(x,y), f(x,y)) \\ & + G(u, g(u,v), g(u,v)) + G(u, f(x,y), f(x,y))] \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}G(g(x,y),f(u,v),f(u,v)) \leq & a[G(x,u,u)+G(y,v,v)] \\ & + b[G(x,g(x,y),g(x,y))+G(x,f(u,v),f(u,v)) \\ & + G(u,g(x,y),g(x,y))+G(u,f(u,v),f(u,v))] \\ & + c[G(x,f(u,v),f(u,v))+G(x,g(x,y),g(x,y)) \\ & + G(u,f(u,v),f(u,v))+G(u,g(x,y),g(x,y))] \end{aligned} \tag{3.2}$$

for all $x,y,u,v \in X$. If f or g is continuous, then f and g have a common coupled fixed point in X that is $f(x,y)=x,\ f(y,x)=y$ and $g(x,y)=x,\ g(y,x)=y$

Proof: By inequality (3.2), we have

$$\begin{aligned}G(g(u,v),f(x,y),f(x,y)) \leq & a[G(u,x,x)+G(v,y,y)] \\ & + b[G(u,g(u,v),g(u,v))+G(u,f(x,y),f(x,y)) \\ & + G(x,g(u,v),g(u,v))+G(x,f(x,y),f(x,y))] \\ & + c[G(u,f(x,y),f(x,y))+G(u,g(u,v),g(u,v)) \\ & + G(x,f(x,y),f(x,y))+G(x,g(u,v),g(u,v))] \end{aligned} \tag{3.3}$$

If X is a symmetric $G-$ metric space, then by adding inequalities (3.1) and (3.3), we obtain

$$\begin{aligned}G(f(x,y),g(u,v),g(u,v)) + & G(g(u,v),f(x,y),f(x,y)) \\ & \leq a[G(x,u,u)+G(y,v,v)+G(u,x,x)+G(v,y,y)] \\ & + b[G(x,f(x,y),f(x,y))+G(x,g(u,v),g(u,v)) \\ & + G(u,f(x,y),f(x,y))+G(u,g(u,v),g(u,v)) \\ & + G(u,g(u,v),g(u,v))+G(u,f(x,y),f(x,y)) \\ & + G(x,g(u,v),g(u,v))+G(x,f(x,y),f(x,y))] \\ & + c[G(x,g(u,v),g(u,v))+G(x,f(x,y),f(x,y)) \\ & + G(u,g(u,v),g(u,v))+G(u,f(x,y),f(x,y)) \\ & + G(u,f(x,y),f(x,y))+G(u,g(u,v),g(u,v)) \\ & + G(x,f(x,y),f(x,y))+G(x,g(u,v),g(u,v))]. \end{aligned}$$

Thus

$$\begin{aligned}d_G(f(x,y),g(u,v)) \leq & a[d_G(x,u)+d_G(y,v)] \\ & + 2(b+c)G(x,f(x,y),f(x,y)) \\ & + 2(b+c)G(x,g(u,v),g(u,v)) \\ & + 2(b+c)G(u,f(x,y),f(x,y)) \\ & + 2(b+c)G(u,g(u,v),g(u,v)) \\ & \leq a[d_G(x,u)+d_G(y,v)] \\ & + (b+c)d_G(x,f(x,y))+(b+c)d_G(x,g(u,v)) \\ & + (b+c)d_G(u,f(x,y))+(b+c)d_G(u,g(u,v)). \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}d_G(f(x,y),g(u,v)) \leq & a[d_G(x,u)+d_G(y,v)] \\ & + b[d_G(x,f(x,y))+d_G(x,g(u,v)) \\ & + d_G(u,f(x,y))+d_G(u,g(u,v))] \\ & + c[d_G(x,f(x,y))+d_G(x,g(u,v)) \\ & + d_G(u,f(x,y))+d_G(u,g(u,v))]. \end{aligned} \tag{3.5}$$

Take $d_G = d$ in (3.5), we have

$$\begin{aligned} d(f(x,y),g(u,v)) \leq\ & a[d(x,u)+d(y,v)] \\ & + b[d(x,f(x,y))+d(x,g(u,v)) \\ & + d(u,f(x,y))+d(u,g(u,v))] \\ & + c[d(x,f(x,y))+d(x,g(u,v)) \\ & + d(u,f(x,y))+d(u,g(u,v))], \end{aligned} \tag{3.6}$$

for all $x,y \in X$ with $0 \leq 2a+4b+4c < 1$. In this case for given $x_0 \in X$, choose $x_1 \in X$ such that $x_1 = f(x_0,y_0)$ and $x_2 = g(x_1,y_1)$, $x_3 = f(x_2,y_2)$. Since f and g are weakly increasing with respect to $\preceq$, we have $x_1 = f(x_0,y_0) \preceq g(f(x_0,y_0),f(y_0,x_0)) = g(x_1,y_1) = x_2 \preceq f(g(x_1,y_1),g(y_1,x_1)) = f(x_2,y_2) = x_3 \preceq \ldots$.

Now we construct sequences $\{x_n\}$ and $\{y_n\}$ such that $x_{2k+1} = f(x_{2k},y_{2k})$, $y_{2k+1} = f(y_{2k},x_{2k})$, $x_{2k+2} = g(x_{2k+1},y_{2k+1})$, $y_{2k+2} = g(y_{2k+1},x_{2k+1})$. Using (3.6), we have

$$\begin{aligned} d(x_{2k+1},x_{2k+2}) =& d(f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1})) \\ & \leq a[d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})] \\ & + b[d(x_{2k},f(x_{2k},y_{2k}))+d(x_{2k},g(x_{2k+1},y_{2k+1})) \\ & + d(x_{2k+1},f(x_{2k},y_{2k}))+d(x_{2k+1},g(x_{2k+1},y_{2k+1}))] \\ & + c[d(x_{2k},g(x_{2k+1},y_{2k+1}))+d(x_{2k},f(x_{2k},y_{2k})) \\ & + d(x_{2k+1},g(x_{2k+1},y_{2k+1}))+d(x_{2k+1},f(x_{2k},y_{2k}))] \\ =& a[d(x_{2k},x_{2k+1})+d(y_{2k},y_{2k+1})] \\ & + b[d(x_{2k},x_{2k+1})+d(x_{2k},x_{2k+2}) \\ & + d(x_{2k+1},x_{2k+1})+d(x_{2k+1},x_{2k+2})] \\ & + c[d(x_{2k},x_{2k+2})+d(x_{2k},x_{2k+1}) \\ & + d(x_{2k+1},x_{2k+2})+d(x_{2k+1},x_{2k+1})] \\ \leq& a[d(x_{2k},x_{2k+1})]+a[d(y_{2k},y_{2k+1})] \\ & + b[d(x_{2k},x_{2k+1})+d(x_{2k},x_{2k+1}) \\ & + d(x_{2k+1},x_{2k+2})+d(x_{2k+1},x_{2k+2})] \\ & + c[d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2}) \\ & + d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})] \\ =& (a+2b+2c)d(x_{2k},x_{2k+1}) \\ & + (2b+2c)d(x_{2k+1},x_{2k+2})+ad(y_{2k},y_{2k+1}). \end{aligned}$$

Therefore, we have

$$(1-2b-2c)d(x_{2k+1},x_{2k+2}) \leq (a+2b+2c)d(x_{2k},x_{2k+1})+ad(y_{2k},y_{2k+1}).$$

This implies that

$$d(x_{2k+1},x_{2k+2}) \leq \frac{a+2b+2c}{1-2b-2c}d(x_{2k},x_{2k+1})+\frac{a}{1-2b-2c}d(y_{2k},y_{2k+1}) \tag{3.7}$$

Proceeding similarly it is easy to prove that

$$d(y_{2k+1},y_{2k+2}) \leq \frac{a+2b+2c}{1-2b-2c}d(y_{2k},y_{2k+1})+\frac{a}{1-2b-2c}d(x_{2k},x_{2k+1}) \tag{3.8}$$

Adding (3.7) and (3.8), we get

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) + & d(y_{2k+1}, y_{2k+2}) \\ & \leq \frac{2a+2b+2c}{1-2b-2c} d(x_{2k}, x_{2k+1}) + \frac{2a+2b+2c}{1-2b-2c} d(y_{2k}, y_{2k+1}) \\ & = \frac{2a+2b+2c}{1-2b-2c} [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \\ & = q[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})], \end{aligned}$$

where $q = \frac{2a+2b+2c}{1-2b-2c} < 1$. Also, we have

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) + & d(y_{2k+2}, y_{2k+3}) \\ & \leq q[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \\ & \leq q^2[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})]. \end{aligned}$$

Continuing in this manner, we have

$$\begin{aligned} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) & \leq q[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \\ & \leq q^2[d(x_{n-2}, x_{n-1}) + d(y_{n-2}, y_{n-1})] \\ & \leq ... \leq q^n[d(x_0, x_1) + d(y_0, y_1)]. \end{aligned}$$

If $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = d_n$ then $d_n \leq q d_{n-1} \leq q^2 d_{n-2} \leq ... \leq q^n d_{n_0}$. For $m > n$, we have

$$\begin{aligned} d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ & \leq q^n d(x_0, x_1) + q^{n+1} d(x_0, x_1) + ... + q^{m-1} d(x_0, x_1) \\ & \leq (q^n + q^{n+1} + ... + q^{m-1}) d(x_0, x_1) \\ & \leq q^n (1 + q + q^2 + ... + q^{m-n-1}) d(x_0, x_1) \\ & \leq \frac{q^n}{1-q} d(x_0, x_1). \end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned} d(y_n, y_m) & \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m) \\ & \leq q^n d(y_0, y_1) + q^{n+1} d(y_0, y_1) + ... + q^{m-1} d(y_0, y_1) \\ & \leq (q^n + q^{n+1} + ... + q^{m-1}) d(y_0, y_1) \\ & \leq q^n (1 + q + q^2 + ... + q^{m-n-1}) d(y_0, y_1) \\ & \leq \frac{q^n}{1-q} d(y_0, y_1) \end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10), we get

$$d(x_n, x_m) + d(y_n, y_m) \leq \frac{q^n}{1-q} [d(x_0, x_1) + d(y_0, y_1)]. \tag{3.11}$$

As $n \to \infty$, sequences $\{x_n\}$ and $\{y_n\}$ are $G-$ Cauchy sequences in X. As X is complete $G-$ metric space, so there exists $x, y \in X$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$ as $n \to \infty$.

Now we will prove that $f(x,y)=x$ and $f(y,x)=y$. From (3.6), we have

$$\begin{aligned} d(x,f(x,y)) &\leq d(x,x_{n+1})+d(x_{n+1},f(x,y)) \\ &= d(x,x_{n+1})+d(f(x_n,y_n),f(x,y)) \\ &\leq d(x,x_{n+1})+a[d(x_n,x)+d(y_n,y)] \\ &+ b[d(x_n,f(x_n,y_n))+d(x_n,f(x,y)) \\ &+ d(x,f(x_n,y_n))+d(x,f(x,y))] \\ &+ c[d(x_n,f(x,y))+d(x_n,f(x_n,y_n)) \\ &+ d(x,f(x,y))+d(x,f(x_n,y_n))] \\ &= d(x,x_{n+1})+a[d(x_n,x)+d(y_n,y)] \\ &+ (b+c)d(x_n,f(x_n,y_n))+(b+c)d(x_n,f(x,y)) \\ &+ (b+c)d(x,f(x,y))+(b+c)d(x,f(x_n,y_n)). \end{aligned}$$

Taking limit $n\to\infty$, we get $d(x,f(x,y))=0$. Therefore $f(x,y)=x$. Similarly, we can prove that $f(y,x)=y$. Also, we can prove that $g(x,y)=x$ and $g(y,x)=y$. Hence (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the coupled fixed point, if possible let (p,q) be the another common coupled fixed point of f and g. Then by using the inequality (3.6), we have

$$\begin{aligned} d(x,p) =& d(f(x,y),g(p,q)) \\ &\leq a[d(x,p)+d(y,q)] \\ &+ b[d(p,g(p,q))+d(p,f(x,y))+d(x,g(p,q))+d(x,f(x,y))] \\ &+ c[d(p,f(x,y))+d(p,g(p,q))+d(x,f(x,y))+d(x,g(p,q))] \\ &= a[d(x,p)+d(y,q)] \\ &+ b[d(p,p)+d(x,p)+d(p,x)+d(x,x)] \\ &+ c[d(p,x)+d(p,p)+d(x,x)+d(x,p)] \\ &= ad(x,p)+ad(y,q)+b[d(p,x)+d(x,p)]+c[d(p,x)+d(x,p)] \\ &= ad(x,p)+ad(y,q)+2bd(x,p)+2cd(x,p). \end{aligned}$$

Therefore $d(x,p)\leq \frac{a}{1-a-3b-3c}d(y,q)$. Similarly, $d(y,q)\leq \frac{a}{1-a-3b-3c}d(x,p)$. Adding last two inequalities, we get $d(x,p)+d(y,q)\leq \frac{a}{1-a-3b-3c}[d(x,p)+d(y,q)]$. This implies $d(x,p)+d(y,q)=0$. Hence $(x,y)=(p,q)$. Thus f and g have unique coupled common fixed point.

Now if X is not a symmetric $G-$ metric space then by the definition of metric (X,d_G) and inequalities (3.1) and (3.3), we obtain

$$\begin{aligned} d_G(f(x,y),g(u,v)) &\leq a[d_G(x,u)+d_G(y,v)] \\ &+ \frac{4}{3}b[d_G(x,f(x,y))+d_G(x,g(u,v)) \\ &+ d_G(u,f(x,y))+d_G(u,g(u,v))] \\ &+ \frac{4}{3}c[d_G(x,f(x,y))+d_G(x,g(u,v)) \\ &+ d_G(u,f(x,y))+d_G(u,g(u,v))] \end{aligned}$$

for all $x,y,u,v\in X$. Here, the contractivity factor $2a+\frac{8}{3}b+\frac{8}{3}c$ may not be less than 1. Therefore $G-$ metric gives no information. In this case for given $x_0\in X$, choose $x_1\in X$ such that $x_1=f(x_0,y_0)$, $x_2=g(x_1,y_1)$ and $x_3=f(x_2,y_2)\cdots$. Since f and g are weakly increasing with respect to $\preceq$, we have

$$x_1=f(x_0,y_0)\preceq g(f(x_0,y_0),f(y_0,x_0))=g(x_1,y_1)=x_2\preceq f(g(x_1,y_1),g(y_1,x_1))=f(x_2,y_2)=x_3\preceq ... \tag{3.12}$$

Now we construct sequences $\{x_n\}$ and $\{y_n\}$ such that $x_{2k+1} = f(x_{2k}, y_{2k})$, $y_{2k+1} = f(y_{2k}, x_{2k})$, $x_{2k+2} = g(x_{2k+1}, y_{2k+1}), y_{2k+2} = g(y_{2k+1}, x_{2k+1})$. From (3.1) we have

$$\begin{aligned}
G(x_{2k+1}, x_{2k+2}, x_{2k+2}) =& G(f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1})) \\
&\leq a[G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(y_{2k}, y_{2k+1}, y_{2k+1})] \\
&+ b[G(x_{2k}, f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k})) \\
&+ G(y_{2k}, g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1})) \\
&+ G(x_{2k+1}, f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k})) \\
&+ G(y_{2k+1}, g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1}))] \\
&+ c[G(x_{2k}, g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1})) \\
&+ G(y_{2k}, f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k})) \\
&+ G(x_{2k+1}, g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1})) \\
&+ G(y_{2k+1}, f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}))] \\
&= a[G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(y_{2k}, y_{2k+1}, y_{2k+1})] \\
&+ b[G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(x_{2k}, x_{2k+2}, x_{2k+2}) \\
&+ G(x_{2k+1}, x_{2k+1}, x_{2k+1}) + G(x_{2k+1}, x_{2k+2}, x_{2k+2})] \\
&+ c[G(x_{2k}, x_{2k+2}, x_{2k+2}) + G(x_{2k}, x_{2k+1}, x_{2k+1}) \\
&+ G(x_{2k+1}, x_{2k+2}, x_{2k+2}) + G(x_{2k+1}, x_{2k+1}, x_{2k+1})] \\
&= a[G(x_{2k}, x_{2k+1}, x_{2k+1})] + a[G(y_{2k}, y_{2k+1}, y_{2k+1})] \\
&+ b[G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(x_{2k}, x_{2k+1}, x_{2k+1}) \\
&+ G(x_{2k+1}, x_{2k+2}, x_{2k+2}) + G(x_{2k+1}, x_{2k+2}, x_{2k+2})] \\
&+ c[G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(x_{2k+1}, x_{2k+2}, x_{2k+2}) \\
&+ G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(x_{2k+1}, x_{2k+2}, x_{2k+2})] \\
&= (a + 2b + 2c)G(x_{2k}, x_{2k+1}, x_{2k+1}) \\
&+ (2b + 2c)G(x_{2k+1}, x_{2k+2}, x_{2k+2}) \\
&+ aG(y_{2k}, y_{2k+1}, y_{2k+1}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
G(x_{2k+1}, x_{2k+2}, x_{2k+2}) \leq & \frac{a + 2b + 2c}{1 - 2b - 2c} G(x_{2k}, x_{2k+1}, x_{2k+1}) \\
&+ \frac{a}{1 - 2b - 2c} G(y_{2k}, y_{2k+1}, y_{2k+1}). \qquad (3.13)
\end{aligned}$$

Proceeding similarly it is easy to see that

$$\begin{aligned}
G(y_{2k+1}, y_{2k+2}, y_{2k+2}) \leq & \frac{a + 2b + 2c}{1 - 2b - 2c} G(y_{2k}, y_{2k+1}, y_{2k+1}) \\
&+ \frac{a}{1 - 2b - 2c} G(x_{2k}, x_{2k+1}, x_{2k+1}). \qquad (3.14)
\end{aligned}$$

Adding (3.13) and (3.14), we get

$$\begin{aligned} G(x_{2k+1}, x_{2k+2}, x_{2k+2}) + G(y_{2k+1}, y_{2k+2}, y_{2k+2}) \\ \leq \frac{2a+2b+2c}{1-2b-2c} G(x_{2k}, x_{2k+1}, x_{2k+1}) \\ + \frac{2a+2b+2c}{1-2b-2c} G(y_{2k}, y_{2k+1}, y_{2k+1}) \\ = \frac{2a+2b+2c}{1-2b-2c} [G(x_{2k}, x_{2k+1}, x_{2k+1}) \\ + G(y_{2k}, y_{2k+1}, y_{2k+1}] \\ = h[G(x_{2k}, x_{2k+1}, x_{2k+1}) \\ + G(y_{2k}, y_{2k+1}, y_{2k+1})], \end{aligned}$$

where $h = \frac{2a+2b+2c}{1-2b-2c} < 1$ Also, it is easy to see that

$$\begin{aligned} G(x_{2k+2}, x_{2k+3}, x_{2k+3}) + G(y_{2k+2}, y_{2k+3}, y_{2k+3}) \\ \leq h[G(x_{2k+1}, x_{2k+2}, x_{2k+2}) + G(y_{2k+1}, y_{2k+2}, y_{2k+2})] \\ \leq h^2[G(x_{2k}, x_{2k+1}, x_{2k+1}) + G(y_{2k}, y_{2k+1}, y_{2k+1})]. \end{aligned}$$

Continuing in this way, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\ \leq h[G(x_{n-1}, x_n, x_n) + G(y_{n-1}, y_n, y_n)] \\ \leq h^2[G(x_{n-2}, x_{n-1}, x_{n-1}) + G(y_{n-2}, y_{n-1}, y_{n-1})] \\ \leq ... \leq h^n[G(x_0, x_1, x_1) + G(y_0, y_1, y_1)]. \end{aligned}$$

If $G(x_n, x_{n+1}, x_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) = G_n$ then $G_n \leq hG_{n-1} \leq h^2 G_{n-2} \leq ... \leq h^n G_{n_0}$.

$$\begin{aligned} G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ + ... + G(x_{m-1}, x_m, x_m) \\ \leq h^n G(x_0, x_1, x_1) + h^{n+1} G(x_0, x_1, x_1) \\ + ... + h^{m-1} G(x_0, x_1, x_1) \\ \leq (h^n + h^{n+1} + ... + h^{m-1}) G(x_0, x_1, x_1) \\ \leq h^n (1 + h + h^2 + ... + h^{m-n-1}) G(x_0, x_1, x_1) \\ \leq \frac{h^n}{1-h} G(x_0, x_1, x_1) \end{aligned} \tag{3.15}$$

Similarly, we get

$$\begin{aligned} G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ + ... + G(y_{m-1}, y_m, y_m) \\ \leq h^n G(y_0, y_1, y_1) + h^{n+1} G(y_0, y_1, y_1) \\ + ... + h^{m-1} G(y_0, y_1, y_1) \\ = (h^n + h^{n+1} + ... + h^{m-1}) G(y_0, y_1, y_1) \\ = h^n (1 + h + h^2 + ... + h^{m-n-1}) G(y_0, y_1, y_1) \\ = \frac{h^n}{1-h} G(y_0, y_1, y_1) \end{aligned} \tag{3.16}$$

Adding (3.15) and (3.16), we get

$$\begin{aligned} G(x_n, x_m, x_m) + G(y_n, y_m, y_m) &\leq \frac{h^n}{1-h}[G(x_0, x_1, x_1) + G(y_0, y_1, y_1)] \\ &= \frac{h^n}{1-h} G_0. \end{aligned} \tag{3.17}$$

Letting $n \to \infty$, $\{x_n\}$ and $\{y_n\}$ are $G-$ Cauchy sequences in a complete $G-$ metric space (X, G), so there exists $x, y \in X$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$ as $n \to \infty$. Now we will prove that $f(x, y) = x$ and $f(y, x) = y$. From (3.2), we have

$$\begin{aligned} G(x, f(x,y), f(x,y)) &\leq G(x, x_{n+1}, x_{n+1}) + G(x_{n+1}, f(x,y), f(x,y)) \\ &= G(x, x_{n+1}, x_{n+1}) + G(f(x_n, y_n), f(x,y), f(x,y)) \\ &\leq G(x, x_{n+1}, x_{n+1}) + a[G(x_n, x, x) + G(y_n, y, y)] \\ &+ b[G(x_n, f(x_n, y_n), f(x_n, y_n)) + G(x_n, f(x,y), f(x,y)) \\ &+ G(x, f(x_n, y_n), f(x_n, y_n)) + G(x, f(x,y), f(x,y))] \\ &+ c[G(x_n, f(x,y), f(x,y)) + G(x_n, f(x_n, y_n), f(x_n, y_n)) \\ &+ G(x, f(x,y), f(x,y)) + G(x, f(x_n, y_n), f(x_n, y_n))] \\ &= G(x, x_{n+1}, x_{n+1}) + a[G(x_n, x, x) + G(y_n, y, y)] \\ &+ (b+c)G(x_n, f(x_n, y_n), f(x_n, y_n)) \\ &+ (b+c)G(x_n, f(x,y), f(x,y)) \\ &+ (b+c)G(x, f(x,y), f(x,y)) \\ &+ (b+c)G(x, f(x_n, y_n), f(x_n, y_n)). \end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are convergent to x and y. Therefore by taking limit as $n \to \infty$, we get $G(x, f(x,y), f(x,y)) = 0$, that is $f(x,y) = x$. Similarly, we can prove that $f(y,x) = y$. Also, it is easy to see that $g(x,y) = x$ and $g(y,x) = y$. Hence (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the coupled fixed point, if possible let (p,q) be another common coupled fixed point of f and g. Then by using the inequality (3.1), we have

$$\begin{aligned} G(p,x,x) =& G(g(p,q), f(x,y), f(x,y)) \\ \leq& a[G(p,x,x) + G(q,y,y)] \\ &+ b[G(p, g(p,q), g(p,q)) + G(p, f(x,y), f(x,y)) \\ &+ G(x, g(p,q), g(p,q)) + G(x, f(x,y), f(x,y))] \\ &+ c[G(p, f(x,y), f(x,y)) + G(p, g(p,q), g(p,q)) \\ &+ G(x, f(x,y), f(x,y)) + G(x, g(p,q), g(p,q))] \\ &\leq a[G(p,x,x) + G(q,y,y)] \\ &+ b[G(p,p,p) + G(p,x,x) + G(x,p,p) + G(x,x,x)] \\ &+ c[G(p,x,x) + G(p,p,p) + G(x,x,x) + G(x,p,p)] \\ \leq& aG(p,x,x) + aG(q,y,y) + b[G(p,x,x) + G(x,p,p)] \\ &+ c[G(p,x,x) + G(x,p,p)] \\ =& aG(p,x,x) + aG(q,y,y) + 3bG(p,x,x) + 3cG(p,x,x) \end{aligned}$$

Therefore, we get

$$G(p,x,x) \leq \frac{a}{1-a-3b-3c} G(q,y,y) \tag{3.18}$$

Similarly,

$$G(q,y,y) \leq \frac{a}{1-a-3b-3c} G(p,x,x) \tag{3.19}$$

Adding (3.18) and (3.19), we get

$$G(p,x,x)+G(q,y,y)\leq \frac{a}{1-a-3b-3c}[G(p,x,x)+G(q,y,y)]$$

This implies that $G(p,x,x)+G(q,y,y)=0$. Hence $(x,y)=(p,q)$. Thus f and g have unique coupled common fixed point. This completes the proof. □

Theorem 3.2 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists $G-$ metric in X such that (X,G) is $G-$ complete. Let $f,g : X\times X\to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose there exist non negative real numbers a,b and c with $2a+4b+4c<1$ such that*

$$\begin{aligned}G(f(x,y),g(u,v),g(u,v)) \leq & a[G(x,u,u)+G(y,v,v)]\\ & + b[G(x,f(x,y),f(x,y))+G(x,g(u,v),g(u,v))\\ & + G(u,f(x,y),f(x,y))+G(u,g(u,v),g(u,v))]\\ & + c[G(x,g(u,v),g(u,v))+G(x,f(x,y),f(x,y))\\ & + G(u,g(u,v),g(u,v))+G(u,f(x,y),f(x,y))]\end{aligned}$$

and

$$\begin{aligned}G(g(x,y),f(u,v),f(u,v)) \leq & a[G(x,u,u)+G(y,v,v)]\\ & + b[G(x,g(x,y),g(x,y))+G(x,f(u,v),f(u,v))\\ & + G(u,g(x,y),g(x,y))+G(u,f(u,v),f(u,v))]\\ & + c[G(x,f(u,v),f(u,v))+G(x,g(x,y),g(x,y))\\ & + G(u,f(u,v),f(u,v))+G(u,g(x,y),g(x,y))]\end{aligned}$$

for all $x,y,u,v\in X$. Assume that X has the following property:
(P) If $\{x_n\}$ and $\{y_n\}$ are an increasing sequences converges to x and y respectively in X; then $x_n\preceq x$ and $y_n\preceq y$ for all $n\in N$ then f and g have a common coupled fixed point $(x,y)\in X\times X$ that is $f(x,y)=x$, $f(y,x)=y$ and $g(x,y)=x$, $g(y,x)=y$.

Proof: As in the proof of Theorem 3.1, it is easy to construct increasing sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{2n+1}=f(x_{2n},y_{2n})$, $y_{2n+1}=f(y_{2n},x_{2n})$, $x_{2n+2}=g(x_{2n+1},y_{2n+1}), y_{2n+2}=g(y_{2n+1},x_{2n+1})$ and show that $\{x_n\}$ and $\{y_n\}$ are $G-$ Cauchy. Since X is $G-$ complete, there are $x,y\in X$ such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y in X. Thus $\{x_{2n}\}$, $\{x_{2n+1}\}$, $f(x_{2n},y_{2n})$, $g(x_{2n+1},y_{2n+1})$ are converging to x and $\{y_{2n}\}$, $\{y_{2n+1}\}$, $f(y_{2n},x_{2n})$, $g(y_{2n+1},x_{2n+1})$ are converging to y. Since X satisfies the property (P), we get that $x_n\preceq x$ and $y_n\preceq y$ for all $n\in\mathbb{N}$. Thus $\{x_{2n}\}$ and x are comparative. Here by inequality (3.1), we have

$$\begin{aligned}G(f(x_{2n},y_{2n}),g(x,y),g(x,y)) \leq & a[G(x_{2n},x,x)+G(y_{2n},y,y)]+b[G(x_{2n},f(x_{2n},y_{2n}),f(x_{2n},y_{2n}))\\ & + G(x_{2n},g(x,y),g(x,y))+G(x,f(x_{2n},y_{2n}),f(x_{2n},y_{2n}))\\ & + G(x,g(x,y),g(x,y))]+c[G(x_{2n},g(x,y),g(x,y))\\ & + G(x_{2n},f(x_{2n},y_{2n}),f(x_{2n},y_{2n}))+G(x,g(x,y),g(x,y))\\ & + G(x,f(x_{2n},y_{2n}),f(x_{2n},y_{2n}))]\end{aligned} \tag{3.20}$$

On letting $n\to\infty$, we get

$$G(x,g(x,y),g(x,y))\leq (2b+2c)G(x,g(x,y),g(x,y)). \tag{3.21}$$

Since, $2(b+c)<1$, we get $G(x,g(x,y),g(x,y))=0$. Hence $g(x,y)=x$. Similarly, it is easy to show that $g(y,x)=y$, $f(x,y)=x$ and $f(y,x)=y$. Hence the result. □

Corollary 3.1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists $G-$ metric in X such that (X, G) is $G-$ complete. Let $f : X \times X \to X$ be a continuous mapping such that $f(x) \preceq f(f(x))$, for all $x \in X$. Suppose there exist non negative real numbers a, b and c with $2a + 4b + 4c < 1$ such that*

$$\begin{aligned} G(f(x,y), f(u,v), f(u,v)) \leq & a[G(x,u,u) + G(y,v,v)] \\ & + b[G(x, f(x,y), f(x,y)) + G(x, f(u,v), f(u,v)) \\ & + G(u, f(x,y), f(x,y)) + G(u, f(u,v), f(u,v))] \\ & + c[G(x, f(u,v), f(u,v)) + G(x, f(x,y), f(x,y)) \\ & + G(u, f(u,v), f(u,v)) + G(u, f(x,y), f(x,y))] \end{aligned}$$

for all $x, y, u, v \in X$. Then f has a unique coupled fixed point $(x, y) \in X \times X$ that is $f(x, y) = x$, $f(y, x) = y$.

Proof: It follows from Theorem 3.1 by taking $g = f$. □

Corollary 3.2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists $G-$ metric in X such that (X, G) is $G-$ complete. Let $f : X \times X \to X$ be a continuous mapping such that $f(x) \preceq f(f(x))$, for all $x \in X$. Suppose there exist non negative real numbers a, b and c with $2a + 4b + 4c < 1$ such that*

$$\begin{aligned} G(f(x,y), f(u,v), f(u,v)) \leq & a[G(x,u,u) + G(y,v,v)] \\ & + b[G(x, f(x,y), f(x,y)) + G(x, f(u,v), f(u,v)) \\ & + G(u, f(x,y), f(x,y)) + G(u, f(u,v), f(u,v))] \\ & + c[G(x, f(u,v), f(u,v)) + G(x, f(x,y), f(x,y)) \\ & + G(u, f(u,v), f(u,v)) + G(u, f(x,y), f(x,y))] \end{aligned}$$

for all $x, y, u, v \in X$. Assume that X has the following property:
(P) If $\{x_n\}$ and $\{y_n\}$ are an increasing sequences converges to x and y respectively in X; then $x_n \preceq x$ and $y_n \preceq y$ for all $n \in N$. Then f has a coupled fixed point $(x, y) \in X \times X$ that is $f(x, y) = x$, $f(y, x) = y$.

Proof: It follows from Theorem 3.2 by taking $g = f$. □

4. Examples

Example 4.1 *Let $X = \mathbb{R}$ be the set of real numbers and $(\mathbb{R}, \leq)$ be a poset and let $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Then (X, G) is a $G-$ metric space. Let $f(x, y) = g(x, y) = \frac{2x-2y+8}{8}$ for all $x, y \in X$. Here f, g are weakly increasing mappings that is $f(x) \leq g(f(x))$. Consider*

$$\begin{aligned} G(f(x,y), g(u,v), g(u,v)) &= |f(x,y) - g(u,v)| + |g(u,v) - g(u,v)| \\ &\quad + |g(u,v) - f(x,y)| \\ &= 2|f(x,y) - g(u,v)| \\ &= 2|\frac{2x - 2y + 8}{8} - \frac{2u - 2v + 8}{8}| \\ &= \frac{1}{4}|2x - 2u - 2y + 2v| \\ &= \frac{1}{4}[2|x - u| + 2|y - v|] \\ &= \frac{1}{4}[G(x,u,u) + G(y,v,v)] \\ &\leq \frac{1}{3}[G(x,u,u) + G(y,v,v)] \end{aligned}$$

Hence inequality (3.1) is satisfied with $a = \frac{1}{3}$ *and* $b = c = 0$. *Note that Corollary 3.1 is also satisfied. Therefore* f, g *have a unique coupled fixed point* $(x, y) = (1, 1)$.

Example 4.2 *Let* $X = \mathbb{R}$ *be the set of real numbers with the usual partial order* $\leq$, *and define the G-metric* $G : X \times X \times X \to [0, \infty)$ *by*

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$. *Then* (X, G) *is a G-metric space.*

Define the mappings $f, g : X \times X \to X$ *as*

$$f(x, y) = \frac{x + 3y + 4}{8}, \quad g(x, y) = \frac{3x + y + 4}{8}, \quad \text{for all } x, y \in X.$$

We observe that f *and* g *are weakly increasing. Indeed, for* $x \leq u$ *and* $y \leq v$, *we have*

$$f(x, y) = \frac{x + 3y + 4}{8} \leq \frac{u + 3v + 4}{8} = f(u, v), \quad g(x, y) = \frac{3x + y + 4}{8} \leq \frac{3u + v + 4}{8} = g(u, v).$$

Now, we verify the contractive condition:

$$\begin{aligned} G(f(x, y), g(u, v), g(u, v)) &= 2\,|f(x, y) - g(u, v)| \\ &= 2\left|\frac{x + 3y + 4}{8} - \frac{3u + v + 4}{8}\right| \\ &= \frac{1}{4}|x + 3y - 3u - v| \\ &\leq \frac{1}{4}(|x - u| + 3|y - v|) \\ &= \frac{1}{4}\left[|x - u| + |y - v| + |y - v| + |y - v|\right] \\ &= \frac{1}{4}\left[G(x, u, u) + 2|y - v|\right] \leq \frac{1}{3}[G(x, u, u) + G(y, v, v)]. \end{aligned}$$

Hence, the inequality (3.1) holds with $a = \frac{1}{3}$, $b = c = 0$, *and the mappings satisfy the conditions of Theorem 3.1 and Corollary 3.1.*

To find the coupled fixed point, solve the equations:

$$f(x, y) = x, \quad f(y, x) = y.$$

That is,

$$x = \frac{x + 3y + 4}{8} \Rightarrow 8x = x + 3y + 4 \Rightarrow 7x - 3y = 4 \quad (1)$$

$$y = \frac{y + 3x + 4}{8} \Rightarrow 8y = y + 3x + 4 \Rightarrow 7y - 3x = 4 \quad (2)$$

Solving equations (1) and (2) simultaneously:

$$\begin{aligned} 49x - 21y = 28 &\quad \text{(multiply (1) by 7)} \\ 21y - 9x = 12 &\quad \text{(multiply (2) by 3)} \\ \hline 40x = 40 &\Rightarrow x = 1 \end{aligned}$$

$$\text{Substitute in (1): } 7(1) - 3y = 4 \Rightarrow y = 1.$$

Therefore, $(x, y) = (1, 1)$ *is the unique coupled fixed point of* f *and* g.

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