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Quasistatic Frictional Wear in Electro-Elasto-Viscoplastic Materials with Unilateral Constraints*

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ABSTRACT: We investigate a mathematical model for wear-induced quasistatic frictional contact between a moving foundation and a piezoelectric body. Archard's law governs the evolution of the wear function. Taking damage effects into account, the electro-elasto-viscoplastic constitutive law is used. In order to account for foundation wear, the model takes into consideration both a regularized Coulomb's law of dry friction and a normal compliance condition with unilateral constraints. A parabolic inclusion with homogeneous Neumann boundary conditions describes the evolution of damage. We provide a variational formulation for the model, which is represented as a system that includes the wear field, damage field, electric potential field, and displacement field. Arguments based on differential equations, elliptic variational inequalities, parabolic inequalities, and the Banach fixed point theorem have been used to demonstrate the existence and uniqueness of a result.

Key Words: Electro-elastic-viscoplastic, wear, frictional contact, damage field normal compliance, fixed point, weak solution, unilateral constraint.

Contents

1. Introduction

In some materials, the intrinsic link between mechanical and electrical properties is described by the piezoelectric effect. This phenomenon shows a direct dependence on the deformation process and is characterized by the appearance of electric charges in crystals exposed to mechanical forces and surface traction. On the other hand, strain and stress are produced when an electric field is applied to these materials. These materials, known as piezoelectrics, are widely used as switches and actuators in a variety of technical domains, such as measurement systems, electroacoustics, and radioelectronics. [2], [4], and [5] provide comprehensive models for electro-elastic materials. [3] and [7] have examined static frictional contact issues in these materials. Additionally, [10] investigated slip-dependent frictional contact in piezoelectrics, whereas [11], [6], and [5] dealt with frictional issues for electro-viscoelastic materials with normal compliance. Numerous authors have taken into consideration examples and mechanical interpretations of elastic-viscoplastic materials that do not suffer material degradation from plastic deformations; for example, see [3,13] and the references therein. Because it directly affects the intended structure's or component's useful life, the damage topic is crucial to design engineering. A vast amount of engineering literature has been written about it. Mathematical studies have been conducted on models that consider the impact of internal material deterioration on the contact process. The virtual power theory was used in [7] to build general models for damage. One-dimensional problems can be mathematically analyzed in [8]. The damage function can only have values in the range of 0 and 1. When $\zeta = 1$ there is no

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damage in the material, when $\zeta = 0$ the material is completely damaged, when $0 < \zeta < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [1,9,14]. Wear of surfaces is a degradation phenomenon of the superfficial layer caused by many factors such as pressure, lubrication friction, and corrosion. The wear implies the evolution of contacting surfaces and these changes affect the contact process. Thus, due to its crucial role, there exists a large engineering and mathematical literature devoted to this topic. We resume to mention here the references [8,1,15], among others. The aim of this paper is to continue the study of problems begun in [12], [13]. The novelty of the present paper is describes the equilibrium of an electro-elesto-viscoplastic body in frictional contact with a moving foundation by taking into account the wear of the foundation. The paper is structured as follows. In Section 2, we present the notation and some preliminary material. In Section 3, we introduce the electro-elesto-viscoplastic contact model with sliding friction and wear, list the assumptions on the data and derive its variational formulation. The unique weak solvability of the contact problem is presented in Section 4. There, we state and prove our main existence and uniqueness result. The proof is based on arguments on time-dependent variational inequalities and fixed point argument is detailed in [9].

2. Notations and preliminaries

This section contains some preparatory material as well as the notation we will be using. The notation $\mathbb{N}$ is used throughout this paper to denote the set of positive integers, while $\mathbb{R}_+$ is used to denote the set of nonnegative real numbers, i.e., $\mathbb{R}_+ = [0, \infty)$. The space of second-order symmetric tensors on $\mathbb{R}^d$ is represented by $\mathbb{S}^d$ for $d \in \mathbb{N}$. Additionally, the norm and inner product on $\mathbb{R}^d$ and $\mathbb{S}^d$ are determined by

$$\mathbf{u}.\mathbf{v} = u_i v_i, \qquad \|\mathbf{u}\| = (u.u)^{\frac{1}{2}} \quad \text{for every } u, v \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\sigma}\| = (\sigma.\sigma)^{\frac{1}{2}} \quad \text{for every } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, for example, $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$. The notation is used here and below.

$$\nabla \mathbf{u} = (u_{i,j}), \ \varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \ \varepsilon_{ij}(\mathbf{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \ \forall \mathbf{u} \in H^1(\Omega)^d,$$
$$\operatorname{Div} \sigma\sigma = (\sigma_{ij,j}) \ \forall \sigma \in \mathcal{H}_1.$$

Now, let us consider the closed subspace of $H^1(\Omega)^d$, which is defined by

$$V = \left\{ v \in H^1(\Omega)^d \ / \ v = 0 \text{ on } \Gamma_1 \right\},$$

through space V, we take into account the inner product provided by over the space

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \ ,$$

and let the corresponding norm be $\|.\|_V$. For an element $v \in V$ we still write v for the trace of v on the boundary Γ. We denote by v_ν and v_τ the normal and the tangential component of v on Γ, respectively, defined by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$. By the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \quad \forall v \in V. \tag{2.1}$$

For a regular function $\sigma : \Omega \cup \Gamma \to S^d$ we denote by σ_ν and σ_τ the normal and the tangential components of the vector $\sigma\nu$ on Γ, respectively, and we recall that $\sigma_\nu = \sigma\nu.\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$. Further, *Green's formula* is as follows:

$$\int_\Omega \sigma.\varepsilon(v) \ dx + \int_\Omega \operatorname{Div} \sigma.v \ dx = \int_\Gamma \sigma\nu.v \ da \quad \forall v \in V. \tag{2.2}$$

Additionally, we present the spaces

$$W = \left\{ \psi \in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_a \right\},$$
$$\mathcal{W} = \{D = (D_i) /\ \ D_i \in L^2(\Omega);\ D_{i,i} \in L^2(\Omega)\}.$$

The inner products of the real Hilbert spaces W and $\mathcal{W}$ are provided by

$$(\psi, \varphi)_W = \int_\Omega \nabla\psi . \nabla\varphi \ dx,\ (D, F)_{\mathcal{W}} = \int_\Omega D.F \ dx + \int_\Omega \operatorname{div} D . \operatorname{div} F \ dx,$$

$\|.\|_W$ and $\|.\|_{\mathcal{W}}$ will be used to represent the corresponding norms.

$$\|\psi\|_W = \|\nabla\psi\|_H\,,\ \ \|D\|^2_{\mathcal{W}} = \|D\|^2_H + \|\operatorname{div} D\|^2_{L^2(\Omega)}\,. \tag{2.3}$$

Furthermore, the following Green's type formula applies when $D \in \mathcal{W}$ is a regular function

$$(D, \nabla\psi)_H + (\operatorname{div} D, \phi)_{L^2(\Omega)} = \int_\Gamma D.\nu \ \psi \ da \qquad \forall \psi \in H^1(\Omega). \tag{2.4}$$

Since $meas\,(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_H \geq C_F \ \|\psi\|_{H^1(\Omega)} \ \text{ for all } \psi \in W, \tag{2.5}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a.

3. The model and variational formulation

In this section, we introduce the contact problem, list the assumptions on the data, and derive its variational formulation. We consider the following physical setting: A piezoelectric body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$, 3) with a Lipschitz continuous boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also subject to mechanical and electrical constraints on its boundary. To describe these constraints, we partition Γ into three measurable parts Γ_1, Γ_2 and Γ_3, as well as two other measurable parts, Γ_a and Γ_b, such that $meas\,\Gamma_1 > 0$, $meas\,\Gamma_a > 0$, and $\Gamma_3 \subseteq \Gamma_b$. We assume that the body is clamped on Γ_1 and surface traction of density f_2 act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. On Γ_3, the body is in frictional contact with a moving insulator obstacle, the so-called foundation. We denote by $\boldsymbol{v}^*$ the velocity of foundation, which is supposed to be a non-vanishing time-dependent function in the plane of Γ_3. The friction implies the wear of the foundation that we model with a surface variable, the wear function. Its evolution is governed by a simplified version of Archard's law, as describe in (see [12]). Moreover, we assume that the foundation is deformable and, therefore, its penetration is allowed. We model the contact with a normal compliance condition with unilateral constraint, which takes into account the wear of the foundation. We associate this condition to a sliding version of Coulomb's law of dry friction. We adopt the framework of the small strain theory and we assume that the contact process is quasistatic and it is studied in the interval of time $\mathbb{R}_+ = [0.\infty)$. To model the material's behavior, we employ an electro-elastic-viscoplastic constitutive law that incorporates damage effects. At this point, the classical formulation of the contact problem under consideration is as follows.

Problem $\mathcal{P}$ Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, an electric displacement field $\mathbf{D} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, a wear function

$w : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$, and a damage field $\zeta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned} &\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathbf{B}(\varepsilon(\boldsymbol{u}(t))) - \mathcal{E}^T\mathbf{E}(\varphi(t)) \\ &+ \int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s) + \mathcal{E}^T E(\varphi(s)), \varepsilon(\boldsymbol{u}(s)), \zeta(s)\right) ds \quad \text{in} \quad \Omega, \end{aligned} \tag{3.1}$$

$$\mathbf{D} = \mathcal{B}\mathbf{E}(\varphi) + \mathcal{E}\varepsilon(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{3.2}$$

$$\dot{\zeta} - k\Delta\zeta + \partial\varphi_K(\zeta) \ni \phi(\sigma - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{E}^T E(\varphi(t), \varepsilon(\boldsymbol{u}), \zeta) \quad \text{in } \Omega, \tag{3.3}$$

$$\mathrm{Div}\,\sigma + f_0 = 0 \quad \text{in } \Omega, \tag{3.4}$$

$$\mathrm{div}\,\mathbf{D} - q_0 = 0 \quad \text{in } \Omega, \tag{3.5}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_2, \tag{3.6}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2, \tag{3.7}$$

$$\left\{ \begin{array}{l} u_\nu(t) \leq \mathrm{g},\ \sigma_\nu(t) + p(u_\nu(t) - w(t)) \leq 0, \\ (u_\nu(t) - g)(\sigma_\nu(t) + p(u_\nu(t) - w(t))) = 0 \end{array} \right. \text{on } \Gamma_3, \tag{3.8}$$

$$-\sigma_\tau(t) = \mu p(u_\nu(t) - w(t))\,\boldsymbol{n}^*(t) \quad \text{on } \Gamma_3, \tag{3.9}$$

$$\dot{w}(t) = \alpha(t)\,p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_3, \tag{3.10}$$

$$\tfrac{\partial\zeta}{\partial\nu} = 0 \quad \text{on } \Gamma, \tag{3.11}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{3.12}$$

$$\mathbf{D}.\nu = q_2 \quad \text{on } \Gamma_b, \tag{3.13}$$

for all $t \in \mathbb{R}_+$ and, in addition,

$$w(0) = 0 \quad \text{on } \Gamma_3, \tag{3.14}$$

$$u(0) = u_0,\ \ \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{3.15}$$

Here and below, for simplicity, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. Moreover, the functions $\boldsymbol{n}^*$ and α are given by

$$\boldsymbol{n}^*(t) = -\frac{\upsilon^*(t)}{\|\upsilon^*(t)\|},\ \ \alpha(t) = h\,\|\upsilon^*(t)\|\ \ \forall t \in \mathbb{R}_+, \tag{3.16}$$

Where h represents the wear coefficient, and $\upsilon^*(t)$ is the velocity of the foundation at each moment t. We now provide a brief explanation of the equations and conditions in Problem $\mathcal{P}$. First, equation (3.1) and (3.2) represent the electro-elasto-viscoplastic constitutive law with damage, where u denotes the displacement field and σ, $\varepsilon(u)$ represent the stress and the linearized strain tensor, respectively. $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, where $\nabla\varphi = (\frac{\partial\varphi}{\partial x_i})$ is an electric potential, $\mathcal{E} = (e_{ijk})$ represents the piezoelectric operator, $\mathcal{E}^T$ is its transposed given by $\mathcal{E}^T = (e_{ijl})^T$, where $e^T_{ijl} = e_{lij}$ and $\mathcal{B}$ denotes the electric permittivity operator. Here $\mathcal{A}$ and $\mathbf{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function that describes the viscoplastic behavior of the material. We also consider that the function $\mathcal{G}$ depends on the internal state variable ζ describing the damage of the material caused by plastic deformations. The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (3.3), where ϕ is the mechanical source of the damage growth is assumed to be rather a general function of the strains and damage itself, $\partial\varphi_K$ is the sub-differential of the indicator function of the admissible damage functions set K. Next, equations (3.4) and (3.5) are the equilibrium equations for the stress and electric displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector-valued functions, respectively. Conditions (3.6) and (3.7) are the displacement and traction boundary conditions, respectively. Next, condition (3.8) represents the normal compliance contact condition with unilateral constraints in which $\mathrm{g} > 0$ and p is a positive Lipschitz continuous increasing function which vanishes for a negative argument, equation (3.9) represents the sliding version of Coulomb's law of dry friction, modified to take into account the wear of the foundation. The differential equation (3.10) describes the evolution of the wear function modeling how the wear depth changes over time (see [8,12]) for details

Equation (3.11) represents a homogeneous Fourier boundary condition where $\frac{\partial \zeta}{\partial \nu}$ represents the normal derivative of ζ. whereas (3.12) and (3.13) represent the electric boundary conditions. (3.14) represents the initial condition for the wear function, which shows that at the initial moment, the foundation is new. Finally, (3.15) represents the initial displacement field and the initial damage field. We note that considering an arbitrary contact surface Γ_3 and a thickness $\mathrm{g} = \mathrm{g}(\boldsymbol{x})$ depending on the spatial variable does not cause additional mathematical difficulties in the analysis of Problem $\mathcal{P}$. Nevertheless, we decided to assume that Γ_3 is plane and g is a constant since these assumptions arise in a large number of industrial process and lead to a simple geometry which helps the reader to better understand the wear phenomenon. In this study of mechanical Problem $\mathcal{P}$ we assume that The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l} (a)\ \text{There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \varepsilon_2)\| \le L_{\mathcal{A}}\ \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b)\ \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2), \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \ \boldsymbol{x} \in \Omega. \\ (c)\ \text{The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\ \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ (d)\ \text{The mapping } x \to \mathcal{A}(\boldsymbol{x}, 0) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{3.17}$$

The elasticity operator $\mathbf{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{array}{l} (a)\ \text{There exists a constant } L_{\mathbf{B}} > 0 \text{ such that} \\ \|\mathbf{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathbf{B}(\boldsymbol{x}, \varepsilon_2)\| \le L_{\mathbf{B}}\ \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b)\ \text{The mapping } \boldsymbol{x} \to \mathbf{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\ \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ (c)\ \text{The mapping } x \to \mathbf{B}(\boldsymbol{x}, 0) \text{ belongs to } \mathcal{H}. \end{array} \tag{3.18}$$

The visco-plasticity operator $\mathcal{G}$: $\Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l} (a)\ \text{There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \|\mathcal{G}(\boldsymbol{x}, \sigma_1, \boldsymbol{\varepsilon}_1, \zeta_1) - \mathcal{G}(\boldsymbol{x}, \sigma_2, \boldsymbol{\varepsilon}_2, \zeta_2)\| \le \\ L_{\mathcal{G}}\ (\|\sigma_1 - \sigma_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\zeta_1 - \zeta_2\|) \\ \forall\ \sigma_1, \sigma_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and } \zeta_1, \zeta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (c)\ \text{The mapping } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x}, \sigma, \boldsymbol{\varepsilon}, \zeta) \text{ is Lebesgue measurable on } \Omega, \\ \forall \sigma,\ \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}. \\ (d)\ \text{The mapping } x \to \mathcal{G}(\boldsymbol{x}, 0, 0, 0) \ \in\ \mathcal{H}. \end{array}\right. \tag{3.19}$$

The piezoelectric operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l} (a)\ \ \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) = (e_{ijh}(\boldsymbol{x})\, \tau_{jh}), \quad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b)\ \ \mathcal{E} = (e_{ijh}) = (e_{ihj}) \in L^\infty(\Omega), \quad 1 \le i, j, h \le d, \\ (c)\ \ \mathcal{E}\boldsymbol{\sigma}.\tau = \boldsymbol{\sigma}.\mathcal{E}^T \tau \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d,\ \forall \tau \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{3.20}$$

The electric permittivity operator $\mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l} (a)\ \mathcal{B}(\boldsymbol{x}, \mathbf{E}) = (b_{ij}(x)\, \mathbf{E}_j)\ \forall\ \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b)\ \text{There holds } b_{ij} = b_{ji},\ 1 \le i,\ j \le d. \\ (c)\ \text{There exists a constant } m_{\mathcal{B}} > 0 \text{ such that} \\ \mathcal{B}(\boldsymbol{x}, \mathbf{E}).\mathbf{E} \ge m_{\mathcal{B}} |\mathbf{E}|^2\ \forall \mathbf{E} = (\mathbf{E}_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \end{array}\right. \tag{3.21}$$

The damage source function $\phi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l} (a)\ \text{There exists } L_\phi > 0 \text{ such that} \\ \|\phi(\boldsymbol{x}, \sigma_1, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\boldsymbol{x}, \sigma_2, \boldsymbol{\varepsilon}_2, \zeta_2)\| \le \\ L_\phi\ (\|\sigma_1 - \sigma_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\zeta_1 - \zeta_2\|) \\ \forall\ \sigma_1, \sigma_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and } \zeta_1, \zeta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (c)\ \text{The mapping } \boldsymbol{x} \to \phi(\boldsymbol{x}, \sigma, \boldsymbol{\varepsilon}, \zeta) \text{ is Lebesgue measurable on } \Omega, \\ \forall \sigma,\ \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}. \\ (d)\ \text{The mapping } x \to \phi(\boldsymbol{x}, 0, 0, 0) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{3.22}$$

The normal compliance function $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists } L_p > 0 \text{ such that} \\ \quad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \leq L_p |r_1 - r_2| \ \ \forall\, r_1, r_2 \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_3. \\ (b) \ (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0 \ \ \forall\, r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ (c) \text{ The mapping } \boldsymbol{x} \to p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \forall r \in \mathbb{R}. \\ (d) \ p(\boldsymbol{x}, r) = 0 \text{ for all } r \leq 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.23}$$

The densities of body forces, surface tractions, volume and surface electric charges satisfy

$$f_0 \in C(\mathbb{R}_+, H), \quad f_2 \in C\left(\mathbb{R}_+, L^2(\Gamma_2)^d\right), \tag{3.24}$$

$$q_0 \in C\left(\mathbb{R}_+, L^2(\Omega)\right), \quad q_2 \in C\left(\mathbb{R}_+, L^2(\Gamma_b)\right), \tag{3.25}$$

$$q_2 = 0 \ \text{ on } \Gamma_3 \ \ \forall t \in [0, T]. \tag{3.26}$$

Note that we need to impose assumption (3.26) for physical reasons, indeed the foundation is assumed to be an insulator and therefore the electric charges (which are prescribed on $\Gamma_3 \subset \Gamma_b$) have to vanish on the potential contact surface. The friction coefficient, the wear coefficient, and the foundation velocity verify

$$\boldsymbol{\mu} \in L^\infty(\Gamma_3), \ \boldsymbol{\mu}(\boldsymbol{x}) \geq 0 \ \text{ a.e. on } \Gamma_3, \tag{3.27}$$

$$h \in L^\infty(\Gamma_3), \ h(\boldsymbol{x}) \geq 0 \ \text{ a.e. on } \Gamma_3, \tag{3.28}$$

$$\boldsymbol{v}^* \in C\left(\mathbb{R}_+, \mathbb{R}^d\right), \text{ there exists } \boldsymbol{v} > 0 \text{ such that } \|\boldsymbol{v}^*(t)\| \geq v, \ \forall t \in \mathbb{R}_+. \tag{3.29}$$

Note that assumption (3.29) is compatible with the physical setting described above since, at each time moment, the velocity of the foundation is assumed to be large enough. In addition (3.16), (3.28) and (3.29) imply that

$$\boldsymbol{n}^* \in C\left(\mathbb{R}_+, \mathbb{R}^d\right), \ \alpha \in C(\mathbb{R}_+, L^\infty(\Gamma_3)), \tag{3.30}$$

moreover,

$$\alpha(t) \geq 0 \ \text{ a.e on } \Gamma_3, \quad \text{for all } t \in \mathbb{R}_+. \tag{3.31}$$

The initial displacement and damage field satisfy

$$\mathbf{u}_0 \in V. \tag{3.32}$$

Next, we introduce the set of admissible displacement fields defined by

$$U = \{v \in V : \ v_\nu \leq \mathrm{g} \text{ on } \Gamma_3\}. \tag{3.33}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varsigma) = k \int_\Omega \nabla\xi . \nabla\varsigma \, dx. \tag{3.34}$$

Using Riesz's representation theorem, we define the functions $f : \mathbb{R}_+ \to V$ and $q : \mathbb{R}_+ \to W$ by

$$(f(t), v)_V = \int_\Omega f_0(t).v \, dx + \int_{\Gamma_2} f_2(t).v \, da, \tag{3.35}$$

$$(q(t), \beta)_W = \int_\Omega q_0(t)\beta \, dx - \int_{\Gamma_b} q_2(t)\beta \, da. \tag{3.36}$$

$\forall v \in V$, $\beta \in W$, a.e. $t \in \mathbb{R}_+$, and note that conditions (3.24) and (3.25) imply that

$$f \in C(\mathbb{R}_+, V), \tag{3.37}$$

$$q \in C(\mathbb{R}_+, W). \tag{3.38}$$

Also, define $j : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by the formula

$$j(w, u, v) = \int_{\Gamma_3} (p(u_\nu(t) - w(t)) v_\nu + \mu p(u_\nu(t) - w(t)) \boldsymbol{n}^*(t) . v_\tau) da \tag{3.39}$$

and note that the integral is well-defined due to the assumptions (3.23) and (3.29)-(3.31).

Using the above notation and Green's formula, we drive the following variational formulation of mechanical Problem $\mathcal{P}$

Problem $\mathcal{P}_V$. Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$, an electric potential $\varphi : \mathbb{R}_+ \to W$, a stress field $\sigma : \mathbb{R}_+ \to \mathcal{H}$, an electric displacement field $\mathbf{D} : \mathbb{R}_+ \to \mathcal{W}$, a damage field $\zeta : \mathbb{R}_+ \to H^1(\Omega)$ and a wear function $w : \mathbb{R}_+ \to L^2(\Gamma_3)$ such that

$$\begin{aligned} &\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathbf{B}(\varepsilon(\boldsymbol{u}(t))) - \mathcal{E}^T \mathbf{E}(\varphi(t)) \\ &+ \int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s) + \mathcal{E}^T E(\varphi(s)), \varepsilon(\boldsymbol{u}(s)), \zeta(s)\right) ds \quad \text{a.e. } t \in \mathbb{R}_+, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &(\sigma(t), \varepsilon(\mathbf{v}(t)) - \varepsilon(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + j(w(t), \dot{u}(t), \mathbf{v}(t) - \dot{u}(t)) \\ &\qquad \geq (f(t), \mathbf{v}(t) - \dot{\boldsymbol{u}}(t))_V \quad \forall \mathbf{v} \in U, t \in \mathbb{R}_+, \end{aligned} \tag{3.41}$$

$$\begin{aligned} &\zeta(t) \in K, \ \left(\dot{\zeta}(t), \xi - \zeta(t)\right)_{L^2(\Omega)} + a(\zeta(t), \xi - \zeta(t)) \\ &\geq \phi(\sigma(t) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)), \varepsilon(\boldsymbol{u}(t)), \zeta(t), \xi - \zeta(t))_{L^2(\Omega)} \\ &\forall \xi \in K, \ a.e. \ t \in \mathbb{R}_+, \end{aligned} \tag{3.42}$$

$$(\mathcal{B}\nabla\varphi(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u(t)), \nabla\psi)_H = (q(t), \psi)_W \quad \forall \psi \in W, \ t \in \mathbb{R}_+, \tag{3.43}$$

$$w(t) = \int_0^t \alpha(s) p(u_\nu(s) - w(s)) ds \quad \mathbb{R}_+, \tag{3.44}$$

$$u(0) = u_0, \ \zeta(0) = \zeta_0, \ w(0) = w_0. \tag{3.45}$$

In this section with some additional comments on our contact model. Assume that (3.1)–(3.15) has a classical solution. Then, since α and p are positive functions, it follows from (3.16) that $\dot{w}(t) \geq 0$ for all t, i.e. the wear is increasing, at each point of the contact surface. Moreover, if at a moment t_0 we have $w(t_0) = \text{g}$, then, using Equation (3.16) and the properties (3.23) of the function p, it can be easily proved that $w(t_0) = \text{g}$ for all $t \geq t_0$. This behavior shows that the wear of the foundation is limited by the constraint $w(t) \leq \text{g}$, which fits with the assumption that the rigid layer of the foundation does not wear.

4. An existence and uniqueness result

Our main existence and uniqueness result in the study of Problem $\mathcal{P}_V$ is the following.

Theorem 4.1 *Assume that (3.17)-(3.32) hold. Then there exists a constant μ_0 which depends only on Ω, Γ_1, Γ_3 and $\mathcal{A}$ such that, if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0, \tag{4.1}$$

then Problem $\mathcal{P}_V$ has a unique solution. Moreover, the solution has the regularity

$$u \in C^1(\mathbb{R}_+, U), \tag{4.2}$$

$$\varphi \in C(\mathbb{R}_+, W), \tag{4.3}$$

$$\sigma \in C(\mathbb{R}_+, \mathcal{H}_1), \tag{4.4}$$

$$\zeta \in W^{1,2}\left(\mathbb{R}_+, L^2(\Omega)\right) \cap L^2\left(\mathbb{R}_+, H^1(\Omega)\right), \tag{4.5}$$

$$D \in C(\mathbb{R}_+, \mathcal{W}), \tag{4.6}$$

$$w \in C^1\left(\mathbb{R}_+, L^2(\Gamma_3)\right), \tag{4.7}$$

and, in addition,

$$w(t) \geq 0 \quad a,e \text{ on } \Gamma_3, \text{ for all } t \in \mathbb{R}_+. \tag{4.8}$$

We conclude that the weak solution $(u, \sigma, \varphi, \zeta, D, w)$ of the piezoelectric contact Problem $\mathcal{P}$ has the regularity (4.2)-(4.7).

The proof of theorem 4.1 will be carried out in several steps. We assume in the rest of this section that (3.17)-(3.32) hold, and we consider that C is a generic positive constant which depends on Ω, Γ_1, Γ_3 and may change from place to place. In the first step, we consider a given wear function $w \in C\left(\mathbb{R}_+, L^2(\Gamma_3)\right)$, Let $\eta \in C(\mathbb{R}_+, \mathcal{H})$ and we contract the following intermediate variational problem.

Problem $\mathcal{P}_{V_\eta}$. Find $\upsilon_\eta : \mathbb{R}_+ \to V$ such that

$$\sigma_\eta(t) = \mathcal{A}(\varepsilon(\upsilon_\eta(t))) + \eta(t), \tag{4.9}$$

$$\begin{aligned}(\mathcal{A}(\varepsilon(\upsilon_\eta(t))), \varepsilon(\mathbf{v}) - \varepsilon(\boldsymbol{\upsilon}_\eta(t)))_{\mathcal{H}} + j(w(t), \upsilon_\eta, \mathbf{v} - \boldsymbol{\upsilon}_\eta(t)) \\ \geq (f(t) - \eta(t), \mathbf{v} - \boldsymbol{\upsilon}_\eta(t))_V \quad \forall \mathbf{v} \in U, t \in \mathbb{R}_+\end{aligned} \tag{4.10}$$

We consider

$$(f(t) - \eta(t), \mathbf{v})_V = (f_\eta(t), \mathbf{v})_V \quad \forall \mathbf{v} \in U, t \in \mathbb{R}_+ \tag{4.11}$$

We use assumption (3.37) to deduce that $f_\eta(t) \in C(\mathbb{R}_+, V)$.

Lemma 4.1 *There exists a constant $\mu_0 > 0$ which depends on Ω, Γ_1, Γ_3, $\mathcal{A}$ and p such that Problem $\mathcal{P}_{V_\eta}$ has a unique solution $\boldsymbol{\upsilon}_\eta \in C(\mathbb{R}_+, U)$.*

Proof: Let $t \in \mathbb{R}_+$ and consider the operator $A : V \to V$ defined by

$$(A\upsilon_\eta(t), \mathbf{v} - \upsilon_\eta(t)) = (\mathcal{A}(\varepsilon(\upsilon_\eta(t))), \mathbf{v} - \upsilon_\eta(t)) + j(w, \mathbf{v}, \upsilon - \boldsymbol{\upsilon}_\eta(t)) \quad \forall \mathbf{v} \in V. \tag{4.12}$$

we use assumptions (3.17)(a), (3.23)(a), (3.27) and inequality (2.1) to see that the operator A is Lipschitz continuous, i.e. it verifies the inequality

$$|A\upsilon_{\eta 1} - A\upsilon_{\eta 2}|_V \leq \left(L_{\mathcal{A}} + c_0^2 L_p \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right)\right) \|\upsilon_{\eta 1} - \upsilon_{\eta 2}\|_V, \tag{4.13}$$

for all $\upsilon_{\eta 1}, \upsilon_{\eta 2} \in V$. Next, we introduce the constant μ_0 defined by

$$\mu_0 = \frac{m_{\mathcal{A}}}{c_0^2 L_p}, \tag{4.14}$$

and note that it depends only on $\Omega, \Gamma_1, \Gamma_3, \mathcal{A}$ and p. Assume that (4.1) holds. Then, we obtain

$$c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} \leq m_{\mathcal{A}}. \tag{4.15}$$

We use again assumptions (3.17)(b), (3.23)(b), (3.27) and inequalities (2.1) and (4.15) to deduce that the operator A is strongly monotone, i.e it satisfies the inequality

$$(A\upsilon_{\eta 1} - A\upsilon_{\eta 2}, \upsilon_{\eta 1} - \upsilon_{\eta 2})_V \geq \left(m_{\mathcal{A}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}\right) \|\upsilon_{\eta 1} - \upsilon_{\eta 2}\|_V^2, \tag{4.16}$$

for all $\upsilon_{\eta 1}$, $\upsilon_{\eta 2} \in V$. Using these ingredients, by classical results for elliptic variational inequalities, (*see* for example [6]), we deduce that there exists a unique element $\upsilon_\eta \in V$ such that

$$(A\upsilon_\eta, \mathbf{v} - \upsilon_\eta)_V \geq (f_\eta(t), \mathbf{v} - \upsilon_\eta)_V \qquad \forall \mathbf{v} \in V. \tag{4.17}$$

Then, it follows from (4.17), (4.12) that the element $\upsilon_\eta(t) \in V$ is the unique element which solves the variational inequality (4.12). We now prove the continuity of the function $t \to \upsilon_\eta(t) : \mathbb{R}_+ \to V$. To this end, let $t_1, t_2 \in \mathbb{R}_+$ and denote $\boldsymbol{\upsilon}_\eta(t_i) = \boldsymbol{\upsilon}_i$, $w_i = w(t_i)$, $f_{\eta i} = f(t_i)$, $\boldsymbol{n}_i^\star = \boldsymbol{n}^\star(t_i)$, for $i = 1, 2$. We use standard arguments in (4.9)-(4.10) to find that

$$\begin{aligned}
&(\mathcal{A}\varepsilon(\boldsymbol{\upsilon}_1) - \mathcal{A}\varepsilon(\boldsymbol{\upsilon}_2), \varepsilon(\boldsymbol{\upsilon}_1) - \varepsilon(\boldsymbol{\upsilon}_2)) \\
&\leq (f_1 - f_2, \boldsymbol{\upsilon}_1 - \boldsymbol{\upsilon}_2)_V + \int_{\Gamma_3} [p(\upsilon_{1\nu} - w_1) - p(\upsilon_{2\nu} - w_2)](w_2 - w_1)\, da \\
&+ \int_{\Gamma_3} [p(\upsilon_{1\nu} - w_1) - p(\upsilon_{2\nu} - w_2)][(\upsilon_{2\nu} - w_2) - (\upsilon_{1\nu} - w_1)]\, da \\
&+ \int_{\Gamma_3} \mu [p(\upsilon_{1\nu} - w_1)\boldsymbol{n}_1^* - p(\boldsymbol{\upsilon}_{2\nu} - w_2)\boldsymbol{n}_1^*] . (\boldsymbol{\upsilon}_{2\tau} - \boldsymbol{\upsilon}_{1\tau})\, da \\
&+ \int_{\Gamma_3} \mu [p(\upsilon_{2\nu} - w_2)\boldsymbol{n}_1^* - p(\upsilon_{2\nu} - w_2)\boldsymbol{n}_2^*] . (\boldsymbol{\upsilon}_{2\tau} - \boldsymbol{\upsilon}_{1\tau})\, da.
\end{aligned}$$

Therefore, (3.17), (3.23), (3.27) and inequality (2.1) yield

$$\left(m_{\mathcal{A}} - C_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}\right) \|\upsilon_1 - \upsilon_2\|_V^2$$
$$\leq \left(c_0 L_p \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) \|w_1 - w_2\|_{L^2(\Gamma_3)} + \|f_1 - f_2\|_V\right.$$
$$\left.+ c_0 p(g) \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{n}_1^* - \boldsymbol{n}_2^*\|\right) \|\upsilon_1 - \upsilon_2\|_V + L_p \|w_1 - w_2\|^2_{L^2(\Gamma_3)}.$$

We use (4.15) and the elementary inequality

$$x, y, z \geq 0 \text{ and } x^2 \leq yx + z \Longrightarrow x^2 \leq y^2 + 2z$$

to deduce that

$$\|\upsilon_1 - \upsilon_2\|_X^2 \leq$$
$$a\left(\|w_1 - w_2\|_{L^2(\Gamma_3)} + \|f_1 - f_2\|_V + \|\boldsymbol{n}_1^* - \boldsymbol{n}_2^*\|\right)^2 + b\|w_1 - w_2\|^2_{L^2(\Gamma_3)}, \tag{4.18}$$

where a and b denote two positive constants which do not depend on t_1 and t_2. This inequality combined with (3.30), (3.31) and the regularity $f_\eta \in C(\mathbb{R}_+, V)$, $w \in C\left(\mathbb{R}_+, L^2(\Gamma_3)\right)$ show that $\upsilon_\eta \in C(\mathbb{R}_+, V)$. Also, we denote by $u_\eta : \mathbb{R}_+ \to V$ the function defined by

$$u_\eta(t) = \int_0^t \upsilon_\eta(s)\, ds + u_0, \quad \forall t \in \mathbb{R}_+ \tag{4.19}$$

From Lemma 4.1 we deduce that $u_\eta \in C^1(\mathbb{R}_+, U)$. □

In the second step we use the displacement field u_η obtained in Lemma 4.1 **Problem** Q_{V_η} : Find the electric potential field $\varphi_\eta : \mathbb{R}_+ \to W$ such that

$$(\mathcal{B}\nabla\varphi_\eta(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_\eta(t)), \nabla\psi)_H = (q(t), \psi)_W \quad \forall\psi \in W,\ t \in \mathbb{R}_+. \tag{4.20}$$

Lemma 4.2 *Q_{V_η} has a unique solution φ_η which satisfies the regularity (4.3). moreover, if φ_i represents the solution of Problem QV_η corresponding to u_i, $i = 1, 2$, then there exists $C > 0$ such that*

$$\|\varphi_1 - \varphi_2\|_W \leq C\|u_1 - u_2\|_V. \tag{4.21}$$

Proof: We define a bilinear form $b : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \psi) = (\mathcal{B}\nabla\varphi(t), \nabla\psi)_H \quad \forall\varphi, \psi \in W. \tag{4.22}$$

We use (2.5) and (3.21) to show that the bilinear form b is continuous, symmetric, and coercive on W. Moreover using Riesz Representation Theorem we may define an element $L_\eta : \mathbb{R}_+ \to W$ such that

$$(L_\eta(t), \psi)_W = (\mathcal{E}\varepsilon(u_\eta(t)), \nabla\psi)_H + (q(t), \psi)_W \quad \forall\psi \in W,\ t \in \mathbb{R}_+,$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element such that

$$b(\varphi_\eta(t), \psi) = (L_\eta(t), \psi)_W \quad \forall\psi \in W. \tag{4.23}$$

We conclude that $\varphi_\eta(t)$ is a solution of QV_η . Let $t_1, t_2 \in \mathbb{R}_+$, it follows from (2.3), (2.5), (3.20), (3.21) and (4.20) that

$$|\varphi_\eta(t_1) - \varphi_\eta(t_2)|_W \leq C\left(|u_\eta(t_1) - u_\eta(t_2)|_V + |q_1 - q_2|_W\right).$$

the previous inequality and the regularity of u_η and q imply that $\varphi_\eta \in C(\mathbb{R}_+, W)$. Finally, inequality (4.21) is obtained by arguments similar to those used in the proof of the previous inequality, which concludes the proof. □

In the third step, we let $\theta \in L^2\left(\mathbb{R}_+; L^2(\Omega)\right)$, be given and consider the following variational problem for the damage field.

Problem $\mathcal{P}_{V_\theta}$: Find the damage field $\zeta_\theta : \mathbb{R}_+ \to H^1(\Omega)$ such that

$$\begin{aligned} &\zeta_\theta(t) \in K, \ \left(\dot{\zeta}_\theta(t), \xi - \zeta_\theta(t)\right)_{L^2(\Omega)} + a\left(\zeta_\theta(t), \xi - \zeta_\theta(t)\right) \\ &\geq \left(\theta(t), \xi - \zeta_\theta(t)\right)_{L^2(\Omega)} \ \forall \xi \in K, \ \ a.e. \ t \in \mathbb{R}_+, \end{aligned} \tag{4.24}$$

$$\zeta_\theta(0) = \zeta_0. \tag{4.25}$$

In the study of the Problem $\mathcal{P}_{V_\theta}$ we have the following result.

Lemma 4.3 *Problem $\mathcal{P}_{V_\theta}$ has a unique solution ζ_θ which satisfying*

$$\zeta_\theta \in H^1\left(\mathbb{R}_+, L^2(\Omega)\right) \cap L^2\left(\mathbb{R}_+, H^1(\Omega)\right). \tag{4.26}$$

Proof: We use a classical existence and uniqueness results in parabolic inequalities (see, e.g. [14], p.47). □

In the fourth step we use the displacement field u_η obtained in Lemma 4.1 and ζ_θ obtained in Lemma 4.3 to construct the following Cauchy problem for the stress field. **Problem** $\mathcal{P}_{V_{\eta\theta}}$. Find a stress field $\sigma_{\theta\eta} : \mathbb{R}_+ \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\theta}(t) = \mathbf{B}\left(\varepsilon(\boldsymbol{u}_\eta(t))\right) + \int_0^t \mathcal{G}\left(\sigma_{\eta\theta}(s)), \varepsilon(\boldsymbol{u}_\eta(s)), \zeta_\theta(s)\right) ds \ \ \text{a.e. } t \in \mathbb{R}_+. \tag{4.27}$$

In the study of Problem $\mathcal{P}_{V_{\eta\theta}}$ we have the following result.

Lemma 4.4 *There exists a unique solution of Problem $\mathcal{P}_{V_{\eta\theta}}$ and it satisfies $\boldsymbol{\sigma}_{\eta\theta}(t) \in W^{1,2}(\mathbb{R}_+, \mathcal{H})$. Moreover, if σ_i, u_i and ζ_i represent the solutions of Problem $\mathcal{P}_{V_{\eta_i\theta_i}}$, $\mathcal{P}_{V_{\eta_i}}$ and $\mathcal{P}_{V_{\theta i}}$, respectively, for $(\eta_i, \theta_i) \in L^2\left(\mathbb{R}_+; \mathcal{H} \times L^2(\Omega)\right)$, $i = 1, 2$, then there exists $C > 0$ such that*

$$\begin{aligned} &\|\sigma_1 - \sigma_2\|^2_{\mathcal{H}} \leq C(\|u_1 - u_2\|^2_V + \int_0^t \|u_1(s) - u_2(s)\|^2_V \, ds \\ &+ \int_0^t \|\zeta_1(s) - \zeta_2(s)\|^2_{L^2(\Omega)} \, ds \ \ ) \ \forall t \leq \mathbb{R}_+. \end{aligned} \tag{4.28}$$

Proof: Let $\Lambda_{\eta\theta} : L^2(\mathbb{R}_+; \mathcal{H}) \to L^2(\mathbb{R}_+; \mathcal{H})$ be the operator given by

$$\Lambda_{\eta\theta}\sigma(t) = \mathbf{B}\left(\varepsilon(\boldsymbol{u}_\eta(t))\right) + \int_0^t \mathcal{G}\left(\sigma_{\eta\theta}(s)), \varepsilon(\boldsymbol{u}_\eta(s)), \zeta_\theta(s)\right) ds \ , \tag{4.29}$$

for all $\sigma \in L^2(\mathbb{R}_+; \mathcal{H})$ and $t \in \mathbb{R}_+$. For σ_1, $\sigma_2 \in L^2(\mathbb{R}_+; \mathcal{H})$ we use (4.29) and (3.19) to obtain for all $t \in \mathbb{R}_+$

$$\|\Lambda_{\eta\theta}\sigma_1(t) - \Lambda_{\eta\theta}\sigma_2(t)\|_{\mathcal{H}} \leq L_{\mathcal{G}} \int_0^t \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \, ds.$$

It follows from this inequality that for n large enough, a power $\Lambda^n_{\eta,\theta}$ of the operator $\Lambda_{\eta,\theta}$ is a contraction on the Banach space $L^2(\mathbb{R}_+; \mathcal{H})$ and, therefore, there exists a unique element $\sigma_{\eta\theta} \in L^2(\mathbb{R}_+; \mathcal{H})$ such that $\Lambda_{\eta,\theta}\sigma_{\eta\theta} = \sigma_{\eta\theta}$. Moreover, $\sigma_{\eta\theta}$ is the unique solution of Problem $\mathcal{P}_{V_{\eta\theta}}$ and, using (4.27), the regularity of u_η, the regularity of ζ_θ and the properties of the operators $\mathbf{B}$ and $\mathcal{G}$, it follows that $\sigma_{\eta\theta} \in W^{1,2}(\mathbb{R}_+; \mathcal{H})$.

Consider now (η_1, θ_1), $(\eta_2, \theta_2) \in L^2(\mathbb{R}_+; \mathcal{H} \times L^2(\Omega))$ and for $i = 1, 2$, denote $u_{\eta_i} = u_i$, $\sigma_{\eta_i \theta i} = \sigma_i$ and $\zeta_{\theta_i} = \zeta_i$. we have

$$\boldsymbol{\sigma}_i(t) = \mathbf{B}(\varepsilon(\boldsymbol{u}_i(t))) + \int_0^t \mathcal{G}(\sigma_i(s)), \varepsilon(\boldsymbol{u}_i(s)), \zeta_i(s))\, ds \quad \text{a.e. } \forall t \in \mathbb{R}_+,$$

and, using the properties (3.18) and (3.19) we find

$$\begin{aligned} &\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}}^2 \leq C(\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2\, ds \\ &+ \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2\, ds + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^2(\Omega)}^2\, ds)\ \forall t \in \mathbb{R}_+. \end{aligned} \tag{4.30}$$

using now a Gronwall argument in the previous inequality we deduce (4.27), which concludes the proof of Lemma 4.4. □

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$, the operator $\mathbf{B}$ and the function ϕ, for $\forall t \in \mathbb{R}_+$, we consider the element

$$\Lambda(\eta, \theta)(t) = (\Lambda_1(\eta, \theta)(t),\ \Lambda_2(\eta, \theta)(t)) \in \mathcal{H} \times L^2(\Omega), \tag{4.31}$$

defined by the equalities

$$\begin{aligned} &\Lambda_1(\eta, \theta)(t) = \mathbf{B}(\varepsilon(\boldsymbol{u}_\eta(t))) - \mathcal{E}^T \mathbf{E}(\varphi_\eta(t)) \\ &+ \int_0^t \mathcal{G}(\sigma_{\eta\theta}(s), \varepsilon(\boldsymbol{u}_\eta(s)), \zeta_\theta(s))\, ds \quad \text{a.e. } t \in \mathbb{R}_+, \end{aligned} \tag{4.32}$$

$$\Lambda_2(\eta, \theta)(t) = \phi(\sigma_{\eta,\theta}(t), \varepsilon(\boldsymbol{u}_\eta(t)), \zeta_\theta(t)). \tag{4.33}$$

Here, for every $(\eta, \theta) \in L^2(\mathbb{R}_+; \mathcal{H} \times L^2(\Omega))$, u_η, φ_η, ζ_θ and $\sigma_{\eta\theta}$ represent the displacement field, an electric potential field, the damage field, and the stress field obtained in lemmas 4.1, 4.2, 4.3 and 4.4 respectively. We have the following result.

Lemma 4.5 *The operator Λ has a unique fixed point $(\eta^*, \theta^*) \in L^2(\mathbb{R}_+; \mathcal{H} \times L^2(\Omega))$ such that*

$$\Lambda(\eta^*, \theta^*) = (\eta^*, \theta^*).$$

Proof: Let now (η_1, θ_1), $(\eta_2, \theta_2) \in L^2(\mathbb{R}_+; \mathcal{H} \times L^2(\Omega))$. We use the notation $u_{\eta i} = u_i$, $\dot{u}_{\eta i} = v_{\eta i} = v_i, \varphi_{\eta_i} = \varphi_i$, $\zeta_{\theta i} = \zeta_i$ and $\sigma_{\eta_i \theta_i} = \sigma_i$ for $i = 1, 2$. Using (2.3),(3.18)-(3.19), and (4.21), we have

$$\begin{aligned} &\|\Lambda_1(\eta_1, \theta_1)(t) - \Lambda_1(\eta_2, \theta_2)(t)\|_{\mathcal{H} \times L^2(\Omega)}^2 \leq \|\mathbf{B}\varepsilon(u_1)(t) - \mathbf{B}\varepsilon(u_2)(t)\|_{\mathcal{H}} \\ &+ \int_0^t \|\mathcal{G}(\sigma_1(s), \varepsilon(\boldsymbol{u}_1(s)), \zeta_1(s)) - \mathcal{G}(\sigma_2(s), \varepsilon(\boldsymbol{u}_2(s)), \zeta_2(s))\|_{\mathcal{H}}^2\, ds \\ &+ \left\|\mathcal{E}^T \nabla\varphi_1(t) - \mathcal{E}^T \nabla\varphi_2(t))\right\|_H \\ &\leq C\left(\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2\, ds \right. \\ &\left. + \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2\, ds + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^2(\Omega)}^2\, ds \right). \end{aligned}$$

We use (4.28) to obtain

$$\begin{aligned} &\|\Lambda_1(\eta_1, \theta_1)(t) - \Lambda_1(\eta_2, \theta_2)(t)\|_{\mathcal{H} \times L^2(\Omega)}^2 \\ &\leq C(\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2\, ds + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^2(\Omega)}^2\, ds). \end{aligned} \tag{4.34}$$

By a similar argument, for (4.28), (4.33) and (3.22) it follows that

$$\begin{aligned} & \left\|\Lambda_2\left(\eta_1,\theta_1\right)(t)-\Lambda_2\left(\eta_2,\theta_2\right)(t)\right\|_{\mathcal{H}\times L^2(\Omega)}^2 \leq \\ & \leq C\left\|u_1(t)-u_2(t)\right\|_V^2+\left\|\sigma_1(t)-\sigma_2(t)\right\|_{\mathcal{H}}^2+\left\|\zeta_1(t)-\zeta_2(t)\right\|_{L^2(\Omega)}^2 . \\ & \leq C\left(\left\|u_1(t)-u_2(t)\right\|_V^2+\int\limits_0^t\left\|u_1(s)-u_2(s)\right\|_V^2 ds\right. \\ & \left.+\left\|\zeta_1(t)-\zeta_2(t)\right\|_{L^2(\Omega)}^2+\int_0^t\left\|\zeta_1(s)-\zeta_2(s)\right\|_{L^2(\Omega)}^2 ds\right). \end{aligned} \tag{4.35}$$

Using (4.34), (4.35) to obtain

$$\begin{aligned} & \left\|\Lambda\left(\eta_1,\theta_1\right)(t)-\Lambda\left(\eta_2,\theta_2\right)(t)\right\|_{\mathcal{H}\times L^2(\Omega)}^2 \\ & \leq C\left(\left\|u_1(t)-u_2(t)\right\|_V^2+\int\limits_0^t\left\|u_1(s)-u_2(s)\right\|_V^2 ds\right. \\ & \left.+\left\|\zeta_1(t)-\zeta_2(t)\right\|_{L^2(\Omega)}^2+\int\limits_0^t\left\|\zeta_1(s)-\zeta_2(s)\right\|_{L^2(\Omega)}^2 ds\right). \end{aligned} \tag{4.36}$$

Moreover, from (4.10) we obtain that

$$\begin{aligned} & \left(\mathcal{A}\left(\varepsilon\left(v_1\right)\right)-\mathcal{A}\left(\varepsilon\left(v_2\right)\right),\varepsilon\left(v_1\right)-\varepsilon\left(\boldsymbol{v}_2\right)\right)_{\mathcal{H}}+j\left(w,v_1,v_1-\boldsymbol{v}_2\right)-j\left(w,v_2,v_1-\boldsymbol{v}_2\right) \\ & +\left(\eta_1-\eta_2,v_1-\boldsymbol{v}_2\right)_{\mathcal{H}\times V} \;=0 \qquad \forall t\in\mathbb{R}_+. \end{aligned}$$

We use again assumptions (3.17)(b), (3.23)(b), (3.27) and inequalities (2.1) and (4.15) to find

$$\left(m_{\mathcal{A}}-c_0^2 L_p\|\mu\|_{L^\infty(\Gamma_3)}\right)\left\|v_1-v\right\|_V^2 \leq -\left\|\eta_1-\eta_2\right\|_{\mathcal{H}}^2,$$

which, by the hypothesis (4.1), implies

$$\left\|v_1-v\right\|_V^2 \leq C\left\|\eta_1-\eta_2\right\|_{\mathcal{H}}^2,$$

we integrate this inequality for time, we obtain

$$\int\limits_0^t\left\|v_1(s)-v(s)\right\|_V^2 ds \leq C\int\limits_0^t\left\|\eta_1(s)-\eta_2(s)\right\|_{\mathcal{H}}^2 ds. \tag{4.37}$$

Since

$$\begin{aligned} & u_i(t)=\int\limits_0^t v_i(s)\,ds+u_0 \quad \forall t\in\mathbb{R}_+, \\ & u_1(0)=u_2(0)=u_0 \end{aligned}$$

we have

$$\left\|u_1(t)-u_2(t)\right\|_V^2=C\int\limits_0^t\left\|v_1(s)-v_2(s)\right\| ds,$$

On the other hand, from (4.24) we have

$$\begin{aligned} & \left(\dot{\zeta}_1-\dot{\zeta}_2,\zeta_1-\zeta_2\right)_{L^2(\Omega)}+a\left(\zeta_1-\zeta_2,\zeta_1-\zeta_2\right) \\ & \leq\left(\theta_1-\theta_2,\zeta_1-\zeta_2\right)_{L^2(\Omega)} \quad a.e.\ t\in\mathbb{R}_+. \end{aligned}$$

Integrating the previous inequality on $\mathbb{R}_+$, after some manipulation we obtain

$$\frac{1}{2}\left\|\zeta_1-\zeta_2\right\|_{L^2(\Omega)}^2 \leq \int\limits_0^t\left\|\zeta_1(s)-\zeta_2(s)\right\|_{L^2(\Omega)}^2 ds+\int\limits_0^t\left\|\theta_1(s)-\theta_2(s)\right\|_{L^2(\Omega)}^2 ds.$$

Applying Gronwall's inequality to the previous inequality, we find

$$\|\zeta_1-\zeta_2\|^2_{L^2(\Omega)}\le C\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds, \tag{4.38}$$

we consider the previous inequality and (4.36) to obtain

$$\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\le C\left(\int_0^t \|\upsilon_1(s)-\upsilon_2(s)\|^2_V\,ds\right.$$
$$\left.+\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\zeta_1(s)-\zeta_2(s)\|^2_{L^2(\Omega)}\,ds\right).$$

It follows now from the previous inequality, the estimates (4.37) and (4.38) that

$$\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\le C\int_0^t \|(\eta_1,\theta_1)(s)-(\eta_2,\theta_2)(s)\|^2_{L^2(\Omega)}\,ds.$$

Reiterating this inequality m times leads to

$$\|\Lambda^m(\eta_1,\theta_1)(t)-\Lambda^m(\eta_2,\theta_2)(t)\|^2_{L^2(\mathbb{R}_+,\mathcal{H}\times L^2(\Omega))}\le \frac{C^mT^m}{m!}\|(\eta_1,\theta_1)-(\eta_2,\theta_2)\|^2_{L^2(\mathbb{R}_+,\mathcal{H}\times L^2(\Omega))}.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $L^2\left(\mathbb{R}_+,\mathcal{H}\times L^2(\Omega)\right)$, and so Λ has a unique fixed point. □

We assume in what follows that (4.1), (4.14) hold and we consider the operator $\mathcal{L}:C\left(\mathbb{R}_+,L^2(\Gamma_3)\right)\to C\left(\mathbb{R}_+,L^2(\Gamma_3)\right)$ defined by

$$\mathcal{L}w(t)=\int_0^t \alpha(s)\,p\,(u_\nu(s)-w(s))\,ds, \tag{4.39}$$

for all $w\in C\left(\mathbb{R}_+,L^2(\Gamma_3)\right)$, where $\boldsymbol{u}_\eta$ is the unique solution of Problem $\mathcal{P}_{V_\eta}$. We have the following fixed point result, which represents the second step in the proof of Theorem 4.1.

Lemma 4.6 *The operator $\mathcal{L}$ has a unique fixed point $w^*\in C\left(\mathbb{R}_+,L^2(\Gamma_3)\right)$ such that $\mathcal{L}w^*=w^*$.*

Proof: Let $w_1,w_2\in C\left(\mathbb{R}_+,L^2(\Gamma_3)\right)$. For simplicity we denote by $\boldsymbol{u}_i$, $i=1,2$ the solutions of Problems $\mathcal{P}_{V_\eta}$, i.e. $\boldsymbol{u}_i=\boldsymbol{u}_{i\eta}$. Let $n\in\mathbb{N}$ and let $t\in[0,n]$. Taking into account (4.39), (3.16) and (3.23) we deduce that

$$\|\mathcal{L}w_1-\mathcal{L}w_2\|_{L^2(\Gamma_3)}$$
$$\le v_n^*\left(c_0\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds+\int_0^t\|w_1(s)-w_2(s)\|_{L^2(\Gamma_3)}\,ds\right), \tag{4.40}$$

where

$$\boldsymbol{v}_n^*=L_p\|k\|_{L^\infty(\Gamma_3)}\max_{r\in[0,n]}\|\boldsymbol{v}^*(r)\|.$$

On the other hand, using arguments similar to those used in the proof of (4.18)

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V\le\sqrt{a+b}\,\|w_1(t)-w_2(t)\|_{L^2(\Gamma_3)} \tag{4.41}$$

which implies that

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V\le\sqrt{a+b}\,\|w_1(t)-w_2(t)\|_{L^2(\Gamma_3)}. \tag{4.42}$$

We now combine the inequalities (4.42) and (4.41) to deduce that

$$\|\mathcal{L}w_1-\mathcal{L}w_2\|_{L^2(\Gamma_3)}\le\boldsymbol{v}_n^*\left(c_0\sqrt{a+b}+1\right)\int_0^t\|w_1(s)-w_2(s)\|_{L^2(\Gamma_3)}\,ds. \tag{4.43}$$

Lemma 4.6 is now a direct consequence of Theorem 4.1. □

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof:
Existence. Let $\eta^*, \theta^* \in L^2\left(\mathbb{R}_+, \mathcal{H}\times L^2(\Omega)\right)$ and $w^* \in C\left(\mathbb{R}_+, L^2(\Gamma_3)\right)$ be the fixed point of the operators Λ and $\mathcal{L}$ defined by (4.31)-(4.33) and (4.39) respectively and let $(\boldsymbol{u}_{\eta^*}(t), \varphi_{\eta^*}(t), \zeta_{\theta^*}(t))$ be the solution of Problems $\mathcal{P}_{V_\eta}$, Q_{V_η}, $\mathcal{P}_{V_\theta}$ for $\eta = \eta^*$ that is $u^* = \boldsymbol{u}_{\eta^*}$, $\varphi^* = \varphi_{\eta^*}$, $\zeta^* = \zeta_{\theta^*}$ and

$$\begin{aligned}\boldsymbol{\sigma}^*(t) &= \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}^*(t)) - \mathcal{E}^T\mathbf{E}(\varphi^*(t)) + \sigma_{\eta^*\theta^*},\\ \mathbf{D}(t) &= \mathcal{B}\mathbf{E}(\varphi^*(t)) + \mathcal{E}\varepsilon(\boldsymbol{u}^*(t)).\end{aligned}$$

It results from (4.31)-(4.33), for $\Lambda_1(\eta^*, \theta^*) = \eta^*$ and $\Lambda_2(\eta^*, \theta^*) = \theta^*$ that $(\boldsymbol{u}^*, \sigma^*, \varphi^*, \zeta^*, w^*)$ a solution of $\mathcal{P}_V$. The regularities (4.2)-(4.7) follows from Lemmas 1, 2, 3, 4, 6
Uniqueness. The uniqueness of the solution follows from the unique solvability of the Problems $\mathcal{P}V_\eta$, $\mathcal{P}V_{\eta\theta}$, $\mathcal{P}V_\theta$ and QV_η combined with the uniqueness of the fixed point of the operators Λ, $\mathcal{L}$ defined by (4.31)-(4.33) and (4.39) respectively. □

5. Conclusion

This study develops an analytical model for quasistatic frictional contact with wear in damaged electro-elasto-viscoplastic materials, integrating unilateral constraints. The variational formulation and existence-uniqueness proofs enable reliable simulations of multi-physics interactions. By unifying wear, damage, and electro-mechanical coupling, the work advances predictive capabilities for material behavior in engineering applications, while its mathematical framework extends to broader challenges in contact mechanics.

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