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Jordan Endo-Biderivations on Prime Rings

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ABSTRACT: For a ring $\mathcal{R}$, set $\mathcal{S} = \mathcal{R} \times \mathcal{R}$. Our goal in this essay is to offer the concepts of Endo-Biderivation, Jordan Endo-Biderivation and Quasi Endo-Biderivation on $\mathcal{S}$: A bi-additive mapping $\xi : \mathcal{S} \to \mathcal{S}$ is referred to as Endo-Biderivation on $\mathcal{S}$ if $\xi(\alpha\mathfrak{b}, \mathfrak{c}\lceil) = \xi(\alpha, \mathfrak{c})(\mathfrak{b}, \lceil) + (\alpha, \mathfrak{c})\xi(\mathfrak{b}, \lceil)$ fulfilled for any $\alpha, \mathfrak{b}, \mathfrak{c}, \lceil \in \mathcal{R}$. A mapping ξ is referred to as Jordan Endo-Biderivation on $\mathcal{S}$ if $\forall\ \alpha\ , \mathfrak{c} \in \mathcal{R}$, then $\xi(\alpha^2, \mathfrak{c}^2) = \xi(\alpha, \mathfrak{c})(\alpha, \mathfrak{c}) + (\alpha, \mathfrak{c})\xi(\alpha, \mathfrak{c})$. It's shown that in this occurrence that $\mathcal{S}$ is prime-ring (P.R) of characteristic $\neq 2$, & $\xi : \mathcal{S} \longrightarrow \mathcal{S}$ is a Jordan Endo-Biderivation, then ξ is Endo-Biderivation.

Key Words: Prime rings, Semi prime ring (S.P.R), Bi-additive mapping.

Contents

1. Introduction

Throughout, $\mathcal{R}$ considered here as an associative ring and $\mathcal{S} = \mathcal{R} \times \mathcal{R}$. We will indicate with an $\mathcal{Z}(\mathcal{R})$ the center to $\mathcal{R}.Let n \in \mathbb{Z}^+$, such as $\mathbb{Z}$ is integer, if $nu \neq 0$ for every non zero $u \in \mathcal{R}$, Afterward, $\mathcal{R}$ is referred to as n-torsion free [7]. For $u, \dagger \in \mathcal{R}$, by convention, the symbol $[u, \dagger]$ denotes the commutatoru $\dagger - \dagger u$ and use the basic identities $[u\dagger, z] = [u, z]y + u[y, z]$ and $[u, \dagger z] = [u, \dagger]z + \dagger[u, z]$ [4]. A ring $\mathcal{R}$ is known prime if whenever $u\mathcal{R}\dagger = 0$ suggests $u = 0$ or $\dagger = 0$. In case that whenever $u\mathcal{R}$ u=0 suggests $u = 0$, then $\mathcal{R}$ is referred to as S.P.R [5].

The idea of Symmetric Biderivation mapping was submitted by Maksa [8]. A mapping $\mathfrak{B} : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ will know as Symmetric if $\mathfrak{B}(u, \dagger) = \mathfrak{B}(\dagger, u)$, $\forall\ u,\ y \in R$. A Symmetric Biadditive (i.e. additive in both arguments) mapping $\mathfrak{B} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is known Symmetric Biderivation if $\mathfrak{B}(u\dagger, z)$ $= \mathfrak{B}(u, z)\dagger = u\ \mathfrak{B}(\dagger_x, z)$ holds for all $u, \dagger \in R$. Several authors have shown Significantly interest for Symmetric Biderivations and related mappings [8,9,10,1,6,3,2].

Labor goal is to proffer the notions of Endo-Biderivation, Quasi Endo-Biderivation and Jordan Endo-Biderivation. Furthermore, we considered under which suitable conditions that make every Jordan Endo-Biderivation is Endo-Biderivation.

2. Preliminary Results

We'll frequently use This is well-known result.

Lemma 2.1 *[1] If $\mathcal{R}$ is a S.P.R with Chara $(\mathcal{R}) \neq 2$, and $\alpha, \mathfrak{b} \in \mathcal{R}$. After that, all of them are comparable:*

1. $\alpha\omega\mathfrak{b} = 0,\ \forall\omega \in \mathcal{R}$.
2. $\mathfrak{b}\omega\alpha = 0, ; \forall\ \omega \in \mathcal{R}$.
3. $\alpha\omega\mathfrak{b} + \mathfrak{b}\omega\alpha = 0, ; \forall\ \omega \in \mathcal{R}$.

Furthermore, $\alpha\mathfrak{b} = \mathfrak{b}\alpha = 0$ whenever any one of these conditions is fulfilled.

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3. Endo-Biderivations and Jordan Endo-Biderivations

We start this section by present the concept of Endo-Biderivation.

Definition 3.1 *$\mathcal{S}=\mathcal{R}\times\mathcal{R}$. A Bi-additive mapping $\xi:\mathcal{S}\longrightarrow\mathcal{S}$ is referred to as Endo-Biderivation on $\mathcal{S}$ if:*

$$\xi(\alpha\mathfrak{b},\mathfrak{c}d)=\xi(\alpha,\mathfrak{c})\ (\mathfrak{b},d)\ +\ (\alpha,\mathfrak{c})\ \xi(\mathfrak{b},d)$$
$$\forall\alpha,\mathfrak{b},\mathfrak{c}\ ,d\in\ R.$$

A mapping ξ is a Jordan Endo-Biderivation on $\mathcal{S}$ if $\forall\ \alpha,\ \mathfrak{c}\in R$, then

$$\xi\left(\alpha^2,\mathfrak{c}^2\right)=\xi(\alpha,\mathfrak{c})(\alpha,\mathfrak{c})+(\alpha,\mathfrak{c})\xi(\alpha,\mathfrak{c})$$

Remark 3.2 *Let $\mathcal{S}=\mathcal{R}\times\mathcal{R}$ and $(\alpha,\mathfrak{b})\in\mathcal{S}$. Then the biadditive mapping $\xi_{(\alpha,\mathfrak{b})}\ :\mathcal{S}\longrightarrow\mathcal{S}$ described by:*

$$\xi_{(\alpha,\mathfrak{b})}\ ((u,y))\ =[(\ u,y),(\alpha,\mathfrak{b})],\forall(u,y)\in\mathcal{S}.$$

is an Endo-Biderivation called the inner Endo-Biderivation on $\mathcal{S}$.

Clearly that every Endo-Biderivation is a Jordan Endo-Biderivation. The converse is in general not correct, the next example clarifies that

Example 3.3 *Let $\mathcal{T}=\mathbb{Q}[u]$ be a polynomial ring over rational numbers with property that $u^2=0$, and I is an ideal of $\mathcal{T}$ produced by u.*

Choo's $\mathcal{R}=\left\{\begin{pmatrix} f & g\\ \alpha & h\end{pmatrix}, f,g,h\in\mathcal{T},\ \text{and}\ \alpha\in I\right\}$. Define $:\mathcal{S}\to\mathcal{S}$, where $\mathcal{S}=\mathcal{R}\times\mathcal{R}$ by

$$\xi\left(\begin{pmatrix} f & g\\ \alpha & h\end{pmatrix},\begin{pmatrix} p & q\\ \imath & k\end{pmatrix}\right)=\left(\begin{pmatrix} 0 & \alpha\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & \imath\\ 0 & 0\end{pmatrix}\right).$$

Then ξ is a Jordan Endo-Biderivation but not Endo-Biderivation.

Example 3.4

1. *Let d and k are derivations of the $\mathcal{R}$ ring and $\mathcal{S}=\mathcal{R}\times\mathcal{R}$, then the bi-additive mapping ξ on $\mathcal{S}$ described by*
$$\xi\left(\alpha,\mathfrak{b}\right)=\ \left(d\left(\alpha\right),k\left(\mathfrak{b}\right)\right),\forall\alpha,\mathfrak{b}\in\mathcal{R}.$$
Then ξ is a Endo-Biderivation on $\mathcal{S}$.

2. *Let $\mathcal{R}=\left\{\begin{pmatrix} \alpha & \mathfrak{b}\\ 0 & c\end{pmatrix}:\alpha,\mathfrak{b},c\in Z\right\}$ be a ring and $\mathcal{S}=\mathcal{R}\times\mathcal{R}$ .Let $\xi:\mathcal{S}\longrightarrow\mathcal{S}$ Defined by*

$$\xi\left(\begin{pmatrix} \alpha & \mathfrak{b}\\ 0 & c\end{pmatrix},\begin{pmatrix} u & y\\ 0 & z\end{pmatrix}\right)=\left(\begin{pmatrix} 0 & \mathfrak{b}\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & y\\ 0 & 0\end{pmatrix}\right)$$ *, in any way* $\begin{pmatrix} \alpha & \mathfrak{b}\\ 0 & c\end{pmatrix},\begin{pmatrix} u & y\\ 0 & z\end{pmatrix}\in\mathcal{R}$.

Then ξ is Endo-Biderivation on $\mathcal{S}$.

Definition 3.5 *$\mathcal{S}=\mathcal{R}\times\mathcal{R}$. A bi-additive mapping $\xi:\mathcal{S}\longrightarrow\mathcal{S}$ is called Quasi Endo-Biderivation on $\mathcal{S}$ if $\xi(\alpha\mathfrak{b},\omega^2)=\xi(\alpha,\ \omega)\ (\mathfrak{b},\ \omega)\ +\ (\alpha,\ \omega)\ \xi(\mathfrak{b},\ \omega)$, for all $\alpha,\mathfrak{b},\ \omega\in\ R$.*

Remark 3.6

1. *Every Endo-Biderivation is a Quasi Endo-Biderivation. In general, the converse is not true.*
2. *Every Quasi Endo-Biderivation is a Jordan Endo-Biderivation. In general, the converse is not true*

For the sake of simplicity, we prove the following Lemma related to the relations that satisfies by the Jordan Endo-Biderivation under suitable conditions.

Lemma 3.7 *Let $\mathcal{S} = \mathcal{R} \times \mathcal{R}$, and $\xi : \mathcal{S} \longrightarrow \mathcal{S}$ is a Jordan Endo-Biderivation, then for all $m,\ t,\ \varrho,\ \mathfrak{b} \in \mathcal{R}$.*

i. $\xi\left(\varrho^2, mt{+}tm\right) = \xi\,(\varrho{,}m)\,(\varrho, t) + (\varrho, m)\,\xi\,(\varrho{,}t) + \xi\,(\ \varrho, t)\,(\ \varrho{,}m) + (\varrho, t)\xi(\varrho{,}m)$

ii. $\xi\left(mt{+}tm,\ \varrho^2\right) = \xi(m, \varrho)(t\ , \varrho) + (m\ , \varrho)\xi(t, \varrho){+}\xi(t, \varrho)\ (m, \varrho) + (t, \varrho)\xi(m, \varrho)$

iii. $\xi\,(\varrho\upsilon + \upsilon\varrho\ , mt{+}tm) = \xi\,(\varrho{,}m)\,(\upsilon{,}t) + (\varrho{,}m)\,\xi\,(\upsilon{,}t) + \xi\,(\upsilon{,}m)\,(\ \varrho{,}t) + (\upsilon, m)\,\xi\ (\varrho{,}t)$
$+\xi(\upsilon{,}t)(\varrho{,}m) + (\upsilon{,}t)\xi(\varrho{,}m) + \xi(\varrho{,}t)(\ \upsilon{,}m) + (\varrho{,}t)\xi(\upsilon{,}m)$

iv. $\xi\left(mtm,\ \varrho^3\right) = \xi(m, \varrho)(\ t, \varrho)(\varrho, m) + (m, \varrho)\xi(\ t, \varrho)(m, \varrho) + (m, \varrho)(\ t, \varrho)\xi(m, \varrho))$

v. $\xi\left(\varrho^3, mtm\right) {=} \xi\,(\varrho, m)\,(\varrho, t)\,(\ \varrho, m) + (\ \varrho, m)\,\xi\,(\varrho, t)\,(\ \varrho, m) + (\varrho, m)\,(\varrho, t)\,\xi\ \ (\varrho, m)\,.$

vi. $\xi\left(\ \varrho^3, m\mathfrak{b}t{+}t\mathfrak{b}m\right) = \xi\,(\varrho, m)\,(\varrho, \mathfrak{b})\,(\ \varrho, t) + (\varrho, m)\,\xi\,(\varrho, \mathfrak{b})\,(\varrho, t)\ + (\varrho, m)\,(\varrho, \mathfrak{b})\,\xi\,(\varrho, t)$
$+\xi(\varrho, t)\ \ (\varrho, \mathfrak{b})\ \ (\varrho, m) + (\varrho, t)\xi(\varrho, \mathfrak{b})(\ \varrho, m) + (\ \varrho, t)(\ \varrho, \mathfrak{b})\xi(\varrho, m)$

vii. $\xi\left(m\mathfrak{b}t{+}t\mathfrak{b}m,\ \varrho^3\right) =\ \xi\,(m, \varrho)\,(\mathfrak{b}, \varrho)\,(t, \varrho) +\ (m, \varrho)\,\xi\,(\mathfrak{b}, \varrho)\,(t, \varrho)\ \ + (m, \varrho)\,(\mathfrak{b}, \varrho)\,\xi\,(t, \varrho)$
$+\xi\ \ (t, \varrho)\,(\mathfrak{b}, \varrho)\,(m, \varrho) + (t, \varrho)\xi(\mathfrak{b}, \varrho)(m, \varrho) + (t, \varrho)(\mathfrak{b}, \varrho)\xi(m, \varrho)$

Proof:

i. Since ξ is a Jordan Endo-Biderivation, then:

$$\begin{aligned}\xi(\varrho^2, (\Updownarrow + t)^2) &= \xi\,(\varrho, \Updownarrow + t)\,(\ \varrho, \Updownarrow + t) + (\varrho, \Updownarrow + t)\,\xi\ (\varrho, \Updownarrow + t)\\ &= \xi\,(\varrho, m)\,(\varrho, m) + \xi\,(\varrho, m)\,(\varrho{,}t) + \xi\,(\varrho{,}t)\,(\varrho, m)\\ &+ \xi\ (\varrho{,}t)(\varrho{,}t) + (\varrho, m)\xi(\varrho, m) + (\varrho{,}t)\xi(\varrho, m) + (\varrho, m)\xi(\varrho{,}t) + (\varrho{,}\mathfrak{b})\xi(\varrho{,}t)\end{aligned} \tag{3.1}$$

On the other hand

$$\begin{aligned}\xi(\varrho^2, (\Updownarrow + t)^2) &= \xi(\varrho^2,\ \Updownarrow^2 + \Updownarrow t + t\Updownarrow + t^2)\\ &\xi(\varrho^2,\ \Updownarrow^2) + \xi(\varrho^2, \Updownarrow t + t\Updownarrow) + \xi(\varrho^2, t^2)\\ &=\ \xi\ \ (\varrho, \Updownarrow)\,(\varrho, \Updownarrow) + (\varrho, \Updownarrow)\,\xi\,(\varrho, \Updownarrow)\\ &+ \xi\left(\varrho^2, \Updownarrow t + t\Updownarrow\right) + \xi\,(\varrho, t)\,(\ \varrho, t) + (\varrho, t)\ \xi(\varrho, t)\end{aligned}$$

Comparing the last relation with (3.1), we get:

$$\xi\left(\varrho^2, mt{+}tm\right) = \xi(\varrho{,}m)(\varrho, t) + (\varrho, t)\xi(\varrho{,}m) + \xi(\varrho, t)(\varrho{,}m) + (\varrho{,}m)\xi(\varrho, t)$$

ii. Replace ϱ by $\varrho + \upsilon$ in (i)

$$\begin{aligned}\xi\left((\varrho+\upsilon)^2, \Updownarrow t + t\Updownarrow\right) &= \xi\,(\varrho+\upsilon,\ \Updownarrow)\,.\,(\varrho+\upsilon,\ t) + (\varrho+\upsilon,\ t)\,\xi\,(\varrho+\upsilon,\ \Updownarrow)\\ &+ \xi\,(\varrho+\upsilon,\ t)\,(\varrho+\upsilon,\ \Updownarrow)\end{aligned} \tag{3.2}$$

$$\begin{aligned}\xi\,(\varrho,\ \Updownarrow)\,(\varrho,\ t) +\ &\xi\,(\varrho,\ \Updownarrow)\,(\ \upsilon,\ t) +\ \xi\,(\ \upsilon,\ \Updownarrow)\,(\mathfrak{u},\ t)\\ &+ \xi\,(\ \upsilon,\ \Updownarrow)\,.\,(\ \upsilon,\ t) + (\varrho,\ t)\,\xi\,(\varrho,\ \Updownarrow) +\ (\varrho,\ \Updownarrow)\ \xi\,(\ \upsilon,\ \Updownarrow)\\ &+ (\upsilon,\ t)\ \ \xi\,(\varrho,\ \Updownarrow) + (\upsilon,\ t)\,\xi\,(\ \upsilon,\ \Updownarrow) +\ \xi\,(\varrho,\ t)\,(\ \varrho,\ \Updownarrow) + \xi\,(\varrho,\ t)\,(\ \upsilon,\ \Updownarrow)\\ &+\ \xi\,(\upsilon,\ t)\,(x,\ \Updownarrow) +\ \xi\,(\upsilon,\ t)\,.(\sqsupseteq,\ \Updownarrow)\ + (\varrho,\ \Updownarrow)\xi(\ \varrho,\ t)\\ &+ (\varrho,\ \Updownarrow)\ \ \xi(\ \upsilon,\ t)\ \ + (\upsilon,\ \Updownarrow)\ \xi(\varrho, t) +\ (\ \upsilon,\ \Updownarrow)\xi(\ \upsilon,\ t)\end{aligned} \tag{3.3}$$

On the other hand

$$\begin{aligned}\xi\ ((\varrho+\upsilon)^2, \Updownarrow t+t\Updownarrow)) &= \xi(\ \varrho^2+\varrho\upsilon+\upsilon\varrho+\varrho^2\ , \Updownarrow t+t\Updownarrow) \\ &= \xi\ \left(\varrho^2, \Updownarrow t+t\Updownarrow\right)+\xi\left(\varrho\upsilon+\upsilon\ \varrho\ , \Updownarrow t+t\Updownarrow\right)+\xi\left(\upsilon^2, \Updownarrow t+t\Updownarrow\right)\end{aligned}$$

In view of (i), the last relation becomes

$$\begin{aligned}&=\xi\left(\varrho,\ \Updownarrow\right)\left(\varrho,\ t\right)+\left(\varrho,\ t\right)\xi\left(\varrho,\ \Updownarrow\right)+\xi\left(\varrho,\ t\right)\left(\varrho,\ \Updownarrow\right)+\left(\varrho,\ t\right)\xi(\varrho,\ \Updownarrow)+\xi(\varrho\upsilon+\upsilon\varrho, \Updownarrow t \\ &\quad +t\Updownarrow)+\xi(\upsilon,\ \ \Updownarrow)(\upsilon,\ \ t)+(\upsilon,\ \ t)\xi(\upsilon,\ \ \Updownarrow)+\xi(\upsilon,\ t)\ (\upsilon,\ \ \Updownarrow)+(\upsilon,\ \ t)\xi(\upsilon,\ \ \Updownarrow\ )\end{aligned} \tag{3.4}$$

Comparing the relations (3.2) and (3.3), we obtain:

$$\begin{aligned}\xi\left(\varrho\upsilon+\upsilon\varrho\ , \Updownarrow t+t\Updownarrow\right) &= \xi(\ \varrho,\ \ \Updownarrow)(\ \upsilon,\ \ t)+\xi(\upsilon,\ \ \Updownarrow)\ (\varrho,\ t)+(\varrho,\ \ \Updownarrow)\xi(\upsilon,\ \ t)+(\upsilon,\ \ t)\xi(\varrho,\ \ \Updownarrow) \\ &\quad +\xi(\varrho,\ t)\ (\upsilon,\ \ \Updownarrow)+\xi(\upsilon,\ \ \ t)\ (\varrho,\ \Updownarrow)+(\varrho,\ t)\xi(\ \upsilon,\ \ \Updownarrow)+(\upsilon,\ \ \Updownarrow)\ \xi(\varrho,\ t))\end{aligned}$$

iii. Replace (t,ϱ) by $((\Updownarrow t+t\Updownarrow),\varrho^2)$ in (ii)

$$\begin{aligned}\xi\left(\Updownarrow(\Updownarrow t+t\Updownarrow)+(\Updownarrow t+t\Updownarrow)\Updownarrow,\ \varrho^3\right) &= \xi(\Updownarrow,\ \varrho)((\Updownarrow t+t\Updownarrow),\ \varrho^2)+((\Updownarrow t+t\Updownarrow)\ ,\ \varrho^2)\xi(\Updownarrow,\ \varrho) \\ &\quad +\xi((\Updownarrow t+t\Updownarrow),\ \ \varrho^2)\ (\Updownarrow,\ \varrho)+(\Updownarrow,\ \varrho)\xi((\Updownarrow t+t\Updownarrow)\ ,\ \varrho^2) \\ &= \ \ \xi\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)+\xi\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right) \\ &\quad +\left(\Updownarrow,\ \varrho\right).\left(t,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)+\ \left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\ \xi\left(\Updownarrow,\ \varrho\right) \\ &\quad +\ \xi\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)+\left(t,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right) \\ &\quad +\xi\left(t,\varrho\right)\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)+\ \left(\Updownarrow,\varrho\right)\ \xi\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right) \\ &\quad +\ \left(\Updownarrow\ ,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)\left(t\ ,\ \varrho\right)+\left(\Updownarrow\ ,\ \varrho\right)\left(t,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right) \\ &\quad +\ \left(\Updownarrow\ ,\ \varrho\right)\ \xi\ \left(t,\ \varrho\right)\ \left(\Updownarrow,\ \varrho\right)+\ \left(\Updownarrow\ ,\ \varrho\right)\ \left(\Updownarrow\ ,\ \varrho\right)\ \xi(t,\ \varrho)\end{aligned} \tag{3.5}$$

Furthermore

$$\begin{aligned}\xi\left(\Updownarrow(\Updownarrow t+t\Updownarrow)+(\Updownarrow t+t\Updownarrow)\Updownarrow,\ \varrho^3\right) &= \xi(\Updownarrow^2 t+2\Updownarrow t\Updownarrow+t\Updownarrow^2,\ \varrho^3) \\ &= \ \xi\left(\Updownarrow^2 t+t\Updownarrow^2,\ \varrho^3\right)+2\xi(\Updownarrow t\Updownarrow,\ \varrho^3)\end{aligned}$$

Now, Putting $(\Updownarrow^2,\ \ \varrho^2)$ instead of $(\Updownarrow,\ \varrho)$ in (ii), we see:

$$\begin{aligned}\xi\left(\Updownarrow^2 t+t\Updownarrow^2,\ \varrho^3\right) &= \xi\left(\Updownarrow^2,\ \ \varrho^2\right)\left(t,\varrho\right)+\left(t,\varrho\right)\xi\left(\Updownarrow^2,\varrho^2\right) \\ &\quad +\xi\left(t,\varrho\right)\left(\Updownarrow,\varrho^2\right)+\left(\Updownarrow^2,\ \varrho^2\right)\xi\left(t,\ \varrho\right)\end{aligned} \tag{3.6}$$

$$\begin{aligned}&= \xi\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)+\left(\Updownarrow,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)+\left(t,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right) \\ &\quad +\left(t,\varrho\right)\left(\Updownarrow,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)+\xi\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right) \\ &\quad +\left(\Updownarrow,\ \varrho\right)\ \left(\Updownarrow,\ \varrho\right)\ \xi(t,\ \varrho)\end{aligned}$$

Hence

$$\begin{aligned}\xi\left(\Updownarrow(\Updownarrow t+t\Updownarrow)+(\Updownarrow t+t\Updownarrow)\Updownarrow,\ \varrho^3\right) &= 2\xi\left(\Updownarrow t\Updownarrow\ ,\ \varrho^3\right)+\xi\left(\ \Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right) \\ &\quad +\ \left(\Updownarrow,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)+\left(t,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)\left(\Updownarrow,\ \varrho\right) \\ &\quad +\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)+\xi\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)\ \left(\Updownarrow,\ \varrho\right) \\ &\quad +\left(\Updownarrow,\ \varrho\right)\ \left(\Updownarrow,\ \varrho\right)\ \xi(t,\ \varrho)\end{aligned} \tag{3.7}$$

Comparing the relations (3.5) and (3.6), we get:

$$2\xi\left(\Updownarrow t\Updownarrow\ ,\ \varrho^3\right)=2\ \xi\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)\left(\Updownarrow,\ \varrho\right)+2\left(\Updownarrow,\ \varrho\right)\left(t,\ \varrho\right)\xi\left(\Updownarrow,\ \varrho\right)+2(\Updownarrow\ ,\ \varrho)\ \xi(t,\ \varrho)\ (\Updownarrow,\ \varrho)$$

That is

$$\xi(\Updownarrow t\Updownarrow,\varrho^3)=\xi(\Updownarrow,\varrho)(t,\ \varrho)\ (\Updownarrow,\ \varrho)+(\Updownarrow,\ \varrho)(t,\varrho)\ \xi(\Updownarrow,\varrho)+(\Updownarrow,\varrho)\ \xi(t,\ \varrho)\ (\Updownarrow,\ \varrho)$$

iv. According to (v), we have

$$\xi(\ \varrho^3, \Updownarrow\mathfrak{b}\Updownarrow) = \xi(\varrho,\ \Updownarrow)(\varrho,\ \mathfrak{b})(\varrho,\ \Updownarrow) + (\varrho,\ \Updownarrow)\xi(\varrho,\ \mathfrak{b})\ (\varrho, \Updownarrow) + (\varrho, \Updownarrow)(\varrho,\ \mathfrak{b})\xi(\varrho, \Updownarrow) \tag{3.8}$$

The linearization of ((3.7)) gives:

$$\begin{aligned}\xi(\varrho^3, (\Updownarrow + t)\mathfrak{b}(\Updownarrow + t)) = \xi(\varrho(\Updownarrow + t))(\varrho,\ \mathfrak{b})(\varrho, (\Updownarrow + t))\ + (\varrho, (\Updownarrow + t))\ \xi(\varrho,\ \mathfrak{b})(\ \varrho, (\Updownarrow + t)) \\ + ((\Updownarrow + t)\ ,\ \varrho)(\varrho,\ \mathfrak{b})\ \xi(\varrho, (\Updownarrow + t))\end{aligned}$$

$$\begin{aligned}=&\xi\,(\varrho,\ \Updownarrow)\,(\varrho,\ \mathfrak{b})\,(\varrho, \Updownarrow) + \xi\,(\varrho,\ \Updownarrow)\,(\varrho,\ \mathfrak{b})\,(\ \varrho,\ t) + \xi\,(\varrho,\ t)\,(\ \varrho,\ \mathfrak{b})\,(\varrho,\ \Updownarrow) + \xi\,(\varrho,\ t)\,(\varrho,\ \mathfrak{b})\,(\ \varrho,\ t) \\ &+\ (\varrho,\ \Updownarrow)\,\xi\,(\varrho,\ \mathfrak{b})\,(\varrho, \Updownarrow) + (\varrho,\ \Updownarrow)\,\xi\ \ (\varrho,\ \mathfrak{b})\,(\ \varrho, t) + (\varrho,\ t)\,\xi\,(\varrho,\ \mathfrak{b})\,(\ \varrho, \Updownarrow) + (\varrho,\ t)\,\xi\,(\varrho,\ \mathfrak{b})\,(\varrho, t) \\ &+ (\varrho, \Updownarrow)\,(\varrho,\ \mathfrak{b})\,\xi\,(\ \varrho, \Updownarrow) + (\varrho, \Updownarrow)\,(\varrho,\ \mathfrak{b})\,\xi\,(\varrho, t) + (\varrho, t)\,(\ \varrho,\ \mathfrak{b})\,\xi\,(\varrho, \Updownarrow) + (\varrho, t)\,(\varrho,\ \mathfrak{b})\ \ \xi(\varrho, t)\end{aligned} \tag{3.9}$$

On the other hand

$$\begin{aligned}\xi(\varrho^3, (\Updownarrow + t)\,\mathfrak{b}\,(\Updownarrow + t)) &= \xi(\varrho^3,\ \Updownarrow\mathfrak{b}\Updownarrow + \Updownarrow\mathfrak{b}t\ \ + t\mathfrak{b}\Updownarrow\ +\ t\mathfrak{b}t) \\ &= \xi\,\left(\varrho^3, \Updownarrow\mathfrak{b}\Updownarrow\right) + \xi\,\left(\ \varrho^3, \Updownarrow\mathfrak{b}t + t\mathfrak{b}t\right) + \xi(\varrho^3,\ t\mathfrak{b}t)\end{aligned}$$

$$\begin{aligned}=&\xi\,\left(\ \varrho^3,\ \Updownarrow\mathfrak{b}t + t\mathfrak{b}\Updownarrow\ \right) + \xi\,(\varrho,\ \Updownarrow)\,(\varrho,\ \mathfrak{b})\,(\varrho,\ \Updownarrow) + (\varrho,\ \Updownarrow)\,\xi\,(\ \varrho,\ \mathfrak{b})\,(\varrho, \Updownarrow) + (\varrho, \Updownarrow)\,(\varrho,\ \mathfrak{b})\,\xi\,(\varrho, \Updownarrow) \\ &+ \xi\,(\varrho,\ t)\,(\varrho,\ \mathfrak{b})\,(\ \varrho,\ t) + (\varrho,\ t)\,\xi\,(\ \varrho,\ \mathfrak{b})\,(\varrho, t) + (\varrho, t)\,(\varrho,\ \mathfrak{b})\,\xi\,(\varrho, t)\,.\end{aligned} \tag{3.10}$$

By comparing (3.8) and (3.9), we see:

$$\begin{aligned}\xi\,\left(\ \varrho^3, \Updownarrow\mathfrak{b}t + t\mathfrak{b}\Updownarrow\right) = &\ \xi\,(\varrho,\ \Updownarrow)\,(\ \varrho,\ \mathfrak{b})\,(\ \varrho,\ t) + (\varrho,\ \Updownarrow)\,\xi\,(\ \varrho, \mathfrak{b})\,(\ \varrho, t) \\ &+ (\varrho,\ \Updownarrow)\,(\ \varrho, \mathfrak{b})\,\xi\,(\ \varrho, t) + \xi\,(\varrho, t)\,(\varrho,\ \mathfrak{b})\,(\varrho,\ \Updownarrow) \\ &+ (\varrho, t)\ \xi(\varrho, \mathfrak{b})\ (\varrho,\ \Updownarrow) + (\varrho, t)\ (\ \varrho, \mathfrak{b})\ \xi(\varrho,\ \Updownarrow)\end{aligned}$$

□

Definition 3.8 *Let $\mathcal{S} = \mathcal{R} \times \mathcal{R}$ and $\xi : \mathcal{S} \to \mathcal{S}$ be a Jordan Endo-Biderivation. We define:*

$$\mathcal{X}^y{}_\omega =\ \xi(xy, \omega^2) - \xi(u,\ \ \omega)(\ y,\ \omega)\ \ - (u,\ \ \omega)\ \xi(y,\ \omega)$$

Remark 3.9 *If $\mathcal{X}^y{}_\omega = 0$, then ξ is Quasi Endo-Biderivation on $\mathcal{S}$.*
For the properties of $\mathcal{X}^y{}_\omega$, we present the following result.

Lemma 3.10 *Let $\mathcal{S} = \mathcal{R} \times \mathcal{R}$ and $\xi : \mathcal{S} \to \mathcal{S}$ be a Jordan Endo-Biderivation, then for all u,y,z, $\omega \in \mathcal{R}$.*

1. $\mathcal{X}^y{}_\omega\ \ + \mathcal{Y}^u{}_\omega\ = 0$
2. $\mathcal{X}^{y+z}_\omega = \mathcal{X}^y{}_\omega\ + \mathcal{X}^z{}_\omega$
3. $(\mathcal{X} + \mathcal{Z})^y{}_\omega\ \ = \mathcal{X}^y{}_\omega\ + \mathcal{Z}^y{}_\omega$

Proof:

i.

$$\begin{aligned}\mathcal{X}^y_\omega + \mathcal{Y}^u{}_\omega &= \xi\ \left(uy, \omega^2\right) - \xi\,(u\ ,\omega)\cdot(y\cdot\omega) - (u,\ \omega)\,\xi\,(y,\ \omega) + \xi\,\left(yu, \omega^2\right) \\ &- \xi(y,\ \omega)^{4\cdot\cdot}(u, \omega) - (y, \omega)\xi(u, \omega) \\ &= \xi\,\left(uy + yu, \omega^2\right) - \xi\,(u, \omega)\,(y, \omega) - (u,\ \omega)\xi(y, \omega) - \xi(y, \omega)(u, \omega) - (y,\ \omega)\xi(u, \omega) \\ &= \xi\,(u, \omega)\,(y, \omega) + (u, \omega)\,\xi\,(y, \omega) + \xi\,(y, \omega)\,.\,(u, \omega) + (y, \omega)\,\xi(u, \omega) \\ &- \xi(u, \omega)(y, \omega) - (u, \omega)\xi(y, \omega) - \ \xi(y, \omega)(u, \omega) - (y, \omega)\xi(u, \omega) = 0\end{aligned}$$

ii.

$$\begin{aligned}\mathcal{X}^{y+z}_{\omega} &= \xi(u(y+z),\omega^2)-\xi(u,\omega)((y+z),\omega) - (u,\ \omega)\xi((y+z),\omega)\\ &= \ \xi(\ uy,\ \omega^2)+\xi(uz,\ \ \omega^2)\ -\xi(u,\ \omega)(\ y,\ \omega)-\xi(u,\ \omega)(z,\ \ \ \omega)\\ &\quad - (u,\ \omega)\ \xi(y,\ \omega) - \ \ (u,\ \omega)\xi(z,\ \ \ \omega)\\ &= \mathcal{X}^{y}_{\ \omega}\ +\ \mathcal{X}^{z}{}_{\omega}\end{aligned}$$

iii.

$$\begin{aligned}(\mathcal{X}+\mathcal{Z})^{y}{}_{\omega} &= \xi\left((u+z)\,y,\,\omega^2\right) - \xi\left((u+z)\,,\ \ \omega\right)(\ y,\ \omega) - (u+z),\ \ \omega)\ \xi(y,\ \omega)\\ &= \xi\left(uy,\,\omega^2\right)+\xi\left(zy,\omega^2\right) - \xi\,(z,\ \ \omega)\,(\ y,\ \omega) - (u,\ \omega)\xi(y,\ \omega) - (z,\ \ \omega)\ \xi(y,\ \omega)\\ &= \mathcal{X}^{y}{}_{\omega} +\ \mathcal{Z}^{y}{}_{\omega}.\end{aligned}$$

□

The last result that we needed to prove our main theorem in this work states:

Lemma 3.11 *Let $\mathcal{S}=\mathcal{R}\times\mathcal{R}$ and $\xi:\mathcal{S}\to\mathcal{S}$ be a Jordan Endo-Biderivation, then for any $u,y,z,\ \omega\in\ \mathcal{R}$.*

$$[(u,\omega)\,,\ (\dagger\ ,\ \omega)]\,(z,\ \ \omega)\,\mathcal{X}^{\dagger}{}_{\omega} +\ \mathcal{X}^{y}{}_{\omega}\ \ (z,\ \ \omega)\,[(u,\ \omega),\ (\dagger\ ,\ \omega)] = 0$$

Proof: In view of (iv) of Lemma 3.7, we have:

$$\xi\left(uyu,\ \omega^3\right) = \xi\,(x,\omega)\,(y,\omega)\,.\,(u,\omega) + (u,\omega)\,\xi\,(y,\omega)\,(u,\omega) +(u,\ \omega)(y,\omega)\xi(u,\omega)$$

Replace $(y,\ \omega)$ by $(yzy,\ \omega^3)$ in the above relation to get:

$$\xi\left(uyzyu,\omega^5\right) = \xi\,(u,\ \omega)\left(y\ z\ y,\omega^{\ 3}\right)(u,\ \omega) + (u,\omega\ )\,\xi\left(yzy,\ \omega^{\ 3}\right)(x,\ \ \omega) +(u,\omega)(yzy,\ \omega^3)\xi(u,\omega)$$

Also,

$$\xi\left(yuy,\ \omega^3\right) = \xi\,(y,\ \omega)\,(\ u,\ \omega)\,(y,\ \omega) + (y,\ \omega)\,\xi\,(u,\ \omega)\,(y,\ \omega) +(y,\ \omega)(u,\ \omega)\ \xi(y,\ \omega)$$

Now, putting $(uzu,\ \omega^3)$ instead of $(x,\ \ \omega)$ in above relation, we arrive at:

$$\xi\left(yuzuy,\ \omega^3\right) = \xi\,(y,\ \omega)\left(uzu,\ \omega^3\right)(y,\ \omega) + (y,\ \omega)\xi(uzu,\ \omega^3)(y,\ \omega) + (y,\ \omega)(uzu,\ \omega^3)\xi(y,\ \omega)$$

Set $\upsilon = uyzyu + yuzuy$, then

$$\begin{aligned}\xi\left(\upsilon,\ \omega^5\right) &= \xi(\ uyzyu + yuzuy,\ \omega^5)\\ &= \ \xi\left(uyzyu,\ \omega^5\right) + \xi(\ yuzuy,\ \omega^5)\\ &= \ \xi\,(u,\ \omega)\left(yzy,\ \omega^{\ 3}\right)(u,\ \omega\ ) + (u,\ \omega)\,\xi\left(yzy,\ \omega^3\right)(u,\ \omega)\\ &\quad + (u,\ \omega\ )\left(yzy,\ \omega^3\right)\xi\,(u,\ \omega) +\xi\,(y,\ \omega)\left(uzu,\ \omega^3\right)(y,\ \omega\ )\\ &\quad +(y,\ \omega\ )\xi(uzu,\ \omega^3)(y,\ \omega) + (y,\ \omega)(uzu,\ \omega^3)\xi(y,\omega)\end{aligned}$$

$$\begin{aligned}&= (u,\omega)\,(y,\omega)\,(z,\ \ \omega)\,(y,\omega)\,(u,\omega) + (u,\omega)\,\xi\,(y,\omega)\,(z,\omega)\,(y,\ \omega)\,(u,\omega) + (u,\omega)\,(y,\omega)\ \ \xi\,(z,\omega)\,(y,\omega)\,(u,\omega)\\ &\quad + (u,\omega)\,(y,\omega)\,(z,\omega)\,\xi\,(y,\omega)\,(u,\omega) + (u,\omega)\cdot(y,\omega)\,(z,\ \ \omega)\,(y,\omega)\,\xi\,(u,\omega) +\xi\,(y,\omega)\,(u,\ \omega)\,(z,\ \ \omega)\,(u,\omega)\,(y,\omega)\\ &\quad + (y,\omega)\,\xi\,(u,\ \omega)\,(\ z,\ \ \omega)\,(u,\ \omega)\,(y,\ \omega) + (y,\ \omega)\,(u,\ \omega)\,\xi\,(z,\ \omega)\,(u,\ \omega)\,(y,\ \omega)\\ &\quad + (y,\omega)\cdot(u,\omega)\,(z,\ \omega)\,\xi\,(u,\ \omega)\,(y,\omega) + (y,\ \omega)(uzu,\omega^3)(u,\omega)(z,\ \omega)(u,\omega)\xi(y,\omega)\end{aligned} \tag{3.11}$$

On the other hand, by replacing $(\alpha,\ \ \omega)$ by $(uy,\ \ \omega^2)$ and $(b,\ \ \omega)$ by $(yu,\ \ \omega^2)$ in (iiv) of Lemma 3.7, we arrive at:

$$\xi\,(uy)\,z\,(yu) + (yu)z(uy),\ \omega^5) =\xi(\ \upsilon,\ \omega^5)$$

$$=\xi\left(uy,\ \omega^2\right)(z,\ \ \omega)\left(yu,\ \omega^2\right)+\left(uy,\ \omega^2\right)\xi\left(z,\ \ \omega\right)\left(yu,\ \omega^2\right)+\left(uy,\ \omega^2\right)(z,\ \ \omega)\,\xi\left(yu,\ \omega^2\right)$$
$$+\,\xi\left(yu,\ \omega^2\right)(z,\ \ \omega)\left(uy,\ \omega^2\right)+\left(yu,\ \omega^2\right)\xi\left(z,\ \ \omega\right)\left(uy,\ \omega^2\right)+(yu,\ \omega^2)(z,\ \omega)\xi(uy,\ \omega^2) \tag{3.12}$$

By comparing relations (3.10) and (3.11) suggests that:

$$(y,\ \omega)\,(u,\omega)\,(z,\ \ \omega)\,\mathcal{X}^y_\omega+(u,\ \ \omega)\,(y,\ \omega)\,(z,\ \ \omega)\,\mathcal{Y}^u_\omega+\mathcal{X}^y_\omega\,(z,\ \ \omega)\,(y,\ \omega)\,(u,\ \ \omega)$$
$$+\,\mathcal{Y}^u_\omega\ \ (z,\ \ \omega)\,(u,\ \omega)\,(y,\omega)=0$$

Since $\mathcal{X}^y_\omega$ = - $\mathcal{Y}^u_\omega$, then

$$\left[(u,\ \omega)\,,\ (y,\ \omega)\right](z,\ \ \omega)\,\mathcal{X}^y_\omega{+}\mathcal{X}^y_\omega\ \ (z,\ \ \omega)\ \left[(u,\ \omega),\ (y,\ \omega)\right]\ =0.$$

□

The conditions that make the concepts of Endo-Biderivation and Jordan Endo-Biderivation equivalent are given in the main result below.

Theorem 3.12 *Assume $\mathcal{R}$ be a P.R, and $\xi:\mathcal{S}\to\mathcal{S}$ be a Jordan Endo-Biderivation as $\mathcal{S}=\mathcal{R}\times\mathcal{R}$ is a ring of characteristic ≠2, then ξ is Endo-Biderivation on $\mathcal{S}$.*

Proof: If $\mathcal{R}$ is a commutative, so is $\mathcal{S}$. Now, from (iii) of Lemma 3.7, we have:

$$\xi\,(uy+yu,\ \omega\upsilon{+}\upsilon\omega)=\xi\,(u{,}\omega)\,(y{,}\upsilon)+(u,\ \omega)\,\xi\,(y,\ \upsilon)\,{+}\xi\,(y{,}\omega)\,(u{,}\upsilon)+(y,\omega)\,\xi\,(u{,}\upsilon)\,{+}\xi\,(y{,}\upsilon)\,(u{,}\omega)$$
$$+\,(y{,}\upsilon)\ \ \xi\,(u{,}\omega)\,{+}\xi(u{,}\upsilon)(y{,}\omega)+(u{,}\upsilon)\xi(y{,}\omega)$$

That is

$$2\xi\,(uy,\omega\upsilon{+}\upsilon\omega)=2\ \xi\,(u,\ \omega)\,(y{,}\upsilon)\,{+}2\,(u{,}\omega)\,\xi\,(y{,}\upsilon)\,{+}2\xi(y{,}\omega)(u{,}\upsilon)+2(y{,}\omega)\ \xi(u{,}\upsilon)$$

The 2-torsinity free of $\mathcal{S}$ leads us to:

$$\xi\,(uy,\omega\upsilon{+}\upsilon\omega)=\xi(u{,}\omega)(y{,}\upsilon)+(u{,}\omega)\xi(y,\ \upsilon){+}\xi(y{,}\omega)(u,\upsilon)+(y{,}\omega)\ \xi(u,\ \upsilon)$$

Replacing $(u\ ,\ \omega)$ and $(y,\ \upsilon)$ by $(u{,}\upsilon)$) and $(y,\ \omega)$ respectively, in the last relation gives:

$$2\xi\,(uy,\upsilon\omega)=\xi(u,\upsilon)\,(y,\omega)+(u,\upsilon)\,\xi\,(y\ ,\ \ \omega)\,{+}\xi\,(y{,}\omega)\,(u,\ \upsilon)+(y{,}\omega)\xi(u,\upsilon)$$
$$=2\xi\,(u,\upsilon)\,(y,\omega)+2\xi(y,\omega)$$

Hence, by 2-torsinity free of $\mathcal{S}$, we conclude:

$$\xi\,(uy,\upsilon\omega)=\xi(u,\upsilon)\ \ (y,\omega)+(u,\upsilon)\xi(y,\omega)$$

Suppose $\mathcal{R}$ is noncommutative ring, then by Lemma 3.11, we have:

$$\left[(u,\ \omega)\,,(y,\omega)\right](z,\omega)\,\mathcal{X}^y_\omega+\mathcal{X}^y_\omega\,(z,\omega)\left[(u,\omega)\,,(y,\omega)\right]=0,for\ \ u,y,\omega{\in}\mathcal{R}.$$

According to Lemma 2.1 suggests that:

$$\left[(u,\ \omega)\,,(y,\omega)\right](z,\omega)\,\mathcal{X}^y_\omega{=}\ 0,\ \ for\ \ u,y,\omega{\in}\mathcal{R}.$$

By prime property together with a non-commutatively of $\mathcal{S}$, we conclude that: $\mathcal{X}^y_\omega$ =0, for $y,\ \omega\in\mathcal{R}$
That is

$$\xi\left(uy,\omega^2\right){=}\xi\,(u,\omega)\cdot(y,\omega)+(u,\omega)\xi(y,\omega) \tag{3.13}$$

Putting $\omega+\upsilon$ instead of ω, we get:

$$\xi(uy{,}(\omega+\upsilon)^2)=\xi\,(u,\omega)\,(y,\omega)+(u,\omega)\xi(y,\omega){+}\xi(u,\omega)(y,\upsilon){+}\ (u,\ \omega)\,\xi\,(y,\upsilon)\,{+}\xi(u,\ \upsilon)(y,\omega)$$
$$+\,(u,\ \upsilon)\xi(y,\ \omega){+}\xi(u,\upsilon)(y,\upsilon)+(u,\ \upsilon)\xi(y,\upsilon)$$
$$=\xi\left(uy{,}\omega^2\right)+\ \xi(uy{,}\upsilon^2){+}\xi(uu{,}\omega\upsilon{+}\upsilon\omega)$$

Hence

$$\xi\left(uy,\omega v+v\omega\right)=\xi\left(u,\omega\right)\left(y,v\right)+\left(u,\omega\right)\xi\left(y,v\right)+\xi\left(u,v\right).\left(y,\ \omega\right)+\left(u,v\right)\xi(y,\omega) \tag{3.14}$$

Also

$$\begin{aligned}\xi\left(uy+yu,\omega v+v\omega\right)=\xi\left(u,\omega\right)\left(y,\ v\right)+\left(u,\omega\right)\xi\left(y,v\right)+\xi\left(y,\ \omega\right)\left(u,v\right)+\left(y,\ \omega\right)\ \xi\left(u,v\right)\\+\xi\left(y,v\right)\left(u,\ \omega\right)+\left(y,\ v\right)\xi\left(u,\ \omega\right)+\xi\left(u,v\right)\left(y,\omega\right)+\left(u,v\right)\xi(y,\omega)\end{aligned} \tag{3.15}$$

Comparing relation (3.13) and (3.14)

$$\xi\left(uy,\omega v+v\omega\right)=\xi\left(u,\omega\ \right)\cdot\left(y,v\right)+\left(u,\omega\right)\xi\left(y,v\right)+\xi\left(u,v\right)\left(y,\omega\right)+\left(u,v\right)\xi(y,\omega)$$

Putting $(u,\ \omega)$ and $(y,\ v)$ instead of $(y,\ \omega)$ and $(u,\ v)$ respectively, in the last relation gives:

$$2\xi\left(uy,\ \omega v\right)=2\xi(u,\omega)\ \left(y,v\right)+\left(u,\omega\right)\xi(y,\ v)$$

The 2-torsionity free leads us to:

$$\xi\left(uy,\omega v\right)=\xi\left(u,\omega\right)\left(y,v\right)+\left(u,\omega\right)\xi\left(y,v\right).$$

□

Definition 3.13 *Let $\mathcal{S}=\mathcal{R}\times\mathcal{R}$. A Bi-additive mapping $\xi:\mathcal{S}\longrightarrow\mathcal{S}$ is referred to as Jordan triple Endo-*
Biderivation on $\mathcal{S}$ if:

$$\xi\left(\alpha\mathfrak{b}\alpha,\ \varrho^{3}\right)=\xi\left(\alpha,\ \varrho\right)\left(\mathfrak{b},\ \varrho\right)\left(\alpha,\varrho\right)+\ \left(\alpha,\ \varrho\right)\xi\left(\mathfrak{b},\ \varrho\right)\left(\alpha,\varrho\right)+\left(\alpha,\varrho\right)(\mathfrak{b},\varrho)\xi(\alpha,\varrho)$$

$$\forall\alpha,\ \mathfrak{b},\varrho\in\mathcal{R}.$$

Example 3.14 *In Example 3.4 "Item 1", where*

$$\xi\left(\alpha\mathfrak{b}\alpha,\varrho^{3}\right)=\ \left(d\left(\alpha\mathfrak{b}\alpha\right),k\left(\varrho^{3}\right)\right),\forall\alpha,\ \mathfrak{b},\varrho\in\mathcal{R}.$$

then ξ is Jordan triple Endo.-Biderivation.

Also in Example 3.4 "Item 2", then ξ is Jordan triple Endo-Biderivation.

Proposition 3.15 *Every Jordan Endo-Biderivations on prime ring $\mathcal{R}\times\mathcal{R}$ of characteristic $\neq 2$, is Jordan triple Endo- Biderivation on $\mathcal{R}\times\mathcal{R}$.*

Proof: Assume $\xi:\mathcal{S}\longrightarrow\mathcal{S}$ be a Jordan Endo-Biderivation as $\mathcal{S}=\mathcal{R}\times\mathcal{R}$.

Replace mt+tm for t in Lemma 3.7 (ii) we get

$$\begin{aligned}\xi\left(m\left(mt+tm\right)+\left(mt+tm\right)m,\varrho^{3}\right)=\xi\left(m,\varrho\right)\left(mt+tm,\varrho^{2}\right)+\left(m,\varrho\right)\xi\left(mt+tm,\varrho^{2}\right)\\+\xi\left(mt+tm,\varrho^{2}\right)\left(m,\varrho\right)+\ \left(mt+tm,\varrho^{2}\right)\xi(m,\varrho)\end{aligned}$$

$$\begin{aligned}=\ \xi\left(m,\ \varrho\right)\left(m,\varrho\right)\left(t,\varrho\right)+\xi\left(m,\varrho\right)\left(t,,\varrho\right)\left(m,\varrho\right)+\left(m,\varrho\right)\xi\left(m,\varrho\right)\left(t,\varrho\right)+\left(m,\varrho\right)\left(m,\varrho\right)\xi\left(t,\varrho\right)\\+(m,\varrho)\xi(t,\varrho)(m,\varrho)+(m,\varrho)(t,\ \varrho)\xi(m,\varrho)\\+\xi\left(m,\varrho\right)\left(t,\varrho\right)\left(m,\varrho\right)+\left(m,\varrho\right)\xi\left(t,\varrho\right)\left(m,\varrho\right)\\+\xi\left(t,\varrho\right)\left(m,\varrho\right)\left(m,\varrho\right)+(t,\varrho)\xi(m,\varrho)\left(m,\varrho\right)\\+\left(m,\varrho\right)\left(t,\varrho\right)\xi\left(m,\ \varrho\right)+\left(t,\varrho\right)\left(m,\varrho\right)\xi\left(m,\ \varrho\right).\end{aligned} \tag{3.16}$$

On the other hand

$$\begin{aligned}\xi\left(m\left(mt+tm\right)+\left(mt+tm\right)m,\ \varrho^{3}\right)&=\xi(mmt+mtm+mtm+tmm,\varrho^{3})\\&=\xi\left(mmt,\varrho^{3}\right)+\xi\left(tmm,\varrho^{3}\right)+\xi(mtm+mtm,\varrho^{3})\end{aligned}$$

$$= \xi(\Updownarrow, \varrho)(m, \varrho)(t, \varrho) + (\Updownarrow, \varrho)\xi(m, \varrho)(t, \varrho) + (\Updownarrow, \varrho)(m, \varrho)\xi(t, \varrho)$$
$$+ \xi(t, \varrho)(m, \varrho)(\varrho, \Updownarrow) + (t, \varrho)\xi(m, \varrho)(\Updownarrow, \varrho) + (t, \varrho)(m, \varrho)\xi(\Updownarrow, \varrho) + \xi(mtm + mtm, \varrho^3)$$

Comparing (3.15) and the last relation becomes

$$\begin{aligned}\xi\left(2mtm, \varrho^3\right) &= \xi\,(m, \varrho)\,(t, \varrho)\,(m, \varrho) + \ (m, \varrho)\,\xi\,(t, \varrho)\,(m, \varrho) + (m, \varrho)\,(t, \varrho)\,\xi\,(m, \varrho)\\ &+ \xi(m, \varrho)(t, \varrho)(m, \varrho) + (m, \varrho)\,(t, \varrho)\,\xi\,(m, \varrho) + (m, \varrho)\xi(t, \varrho)(m, \varrho)\\ \xi\left(2mtm, \varrho^3\right) &= 2(\xi\,(m, \varrho)\,(t, \varrho)\,(m, \varrho) + \ (m, \varrho)\,\xi\,(t, \varrho)\,(m, \varrho) + (m, \varrho)\,(t, \varrho)\,\xi\,(m, \varrho))\end{aligned}$$

By 2-torsion free we get

$$\xi\left(mtm, \varrho^3\right) = \xi\,(m, \varrho)\,(t, \varrho)\,(m, \varrho) + \ (m, \varrho)\,\xi\,(t, \varrho)\,(m, \varrho) + (m, \varrho)\,(t, \varrho)\,\xi\,(m, \varrho)$$

Thus it's Jordan triple Endo-Biderivation on $\mathcal{R} \times \mathcal{R}$. □

Corollary 3.16 *Every Endo- Biderivation on semiprime ring $\mathcal{R} \times \mathcal{R}$ of characteristic $\neq$2, is Jordan triple Endo-Biderivation on $\mathcal{R} \times \mathcal{R}$.*

Proof: Let ξ be Endo-Bidrivation on $\mathcal{R} \times \mathcal{R}$ by Definition 3.1, we have ξ is Jordan Endo-Biderivation on $\mathcal{R} \times \mathcal{R}$ and hence by Proposition 3.15, then ξ is Jordan triple Endo-Biderivation on $\mathcal{R} \times \mathcal{R}$. □

Lemma 3.17 *Let $\mathcal{R} \times \mathcal{R}$ be 2-torsion free semiprime ring and ξ be endo-Biderivation on $\mathcal{R} \times \mathcal{R}$, if ξ^2=0 then ξ = 0.*

Proof: By ξ^2=0

$$\xi(\xi(mt, \varrho^2) = 0$$

$$\xi(\xi\,(m,\ \varrho)\,(t\ ,\ \varrho) + (m\ ,\ \varrho)\,\xi\,(t,\ \varrho)\,) = 0$$

$$\xi(\xi\,(m,\ \varrho)\,(t,\varrho)\,) + \xi(\,(m\ ,\ \varrho)\,\xi\,(t,\ \varrho)) = 0$$

$$\xi(\xi\,(m, \varrho))\,(t, \varrho) + \xi(m, \varrho)\xi(t, \varrho) + \xi\,(m, \varrho)\,\xi\,(t, \varrho) + (m, \varrho)\xi(\xi\,(t, \varrho)) = 0$$

$$\xi^2\,(m, \varrho)\,(t, \varrho) + \xi(m, \varrho)\xi(t, \varrho) + \xi\,(m, \varrho)\,\xi\,(t, \varrho) + (m, \varrho)\xi^2\,(t, \varrho) = 0$$

By assumption we have

$$2\ \xi(m, \varrho)\xi(t, \varrho) = 0$$

By 2-torsion free

$$\xi(m, \varrho)\xi(t, \varrho) = 0$$

Right multiply by $\xi(m, \varrho)$ we have

$$\xi(m, \varrho)\xi(t, \varrho)\xi(m, \varrho) = 0$$

By semiprimness

$$\xi(m, \varrho) = 0$$

Hence ξ=0 □

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