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On fixed point theorems in controlled metric type spaces

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ABSTRACT: In this paper, we use control functions to derive fixed point results for mappings under specific conditions within controlled metric type spaces, and we demonstrate the applicability of our results through illustrative examples.

Key Words: b-metric space, extended b-metric space, controlled metric type space, contractive type mappings, fixed points.

Contents

1. Introduction and preliminaries

The theory of fixed points is a foundational component in modern analysis with diverse applications in both theoretical and applied disciplines [1,2]. Fixed point theorems are extensively used to establish the existence and uniqueness of solutions to ordinary and fractional differential equations, often through iterative or approximation techniques (see [3,4,5]). Among these, the Banach contraction principle is particularly notable for its applicability and has been generalized in various ways by relaxing the contraction conditions or by considering more general spaces than classical metric spaces (see [6,7,8,9,10]).

The generalization to b-metric spaces, originally proposed by Bakhtin [11] and later refined by Czerwik [12], provides a broader context for fixed point results. This concept has stimulated a considerable body of work focused on fixed point theorems in such settings (see [13,14,15]). The introduction of strong b-metric spaces by Kirk and Shahzad [16] further enriched this landscape.

Kamran et al. [17] advanced the theory by proposing extended b-metric spaces, governed by control functions defined over the interval $[1, \infty)$. In parallel, Mlaiki et al. [18] developed the framework of controlled metric type spaces by embedding a control function θ into the triangle inequality. Their work demonstrates that controlled metric type spaces and extended b-metric spaces are distinct in general. For a deeper discussion and further developments, the reader is referred to [19,20,21,22,23,24,25,26,27,28,29,30,31,32].

In this paper, we extend fixed point theory by studying contractive-type mappings defined with the aid of control functions. We support our theoretical findings with illustrative examples that highlight their applicability.

The subsequent section presents the basic definitions and notations required for our analysis.

Definition 1.1 **([11,12])** Let Z be a non-empty set and $s \geq 1$. A function $\Delta : Z \times Z \to [0, \infty)$ is called a *b-metric* if, for all $\mu, \nu, \omega \in Z$, the following conditions are met:

1. $\Delta(\mu, \nu) = 0$ if and only if $\mu = \nu$,
2. $\Delta(\mu, \nu) = \Delta(\nu, \mu)$,
3. $\Delta(\mu, \omega) \leq s\,[\Delta(\mu, \nu) + \Delta(\nu, \omega)]$.

The pair (Z, Δ) is then referred to as a *b-metric space*.

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Definition 1.2 **([17])** Let Z be a non-empty set and $\eta : Z \times Z \to [1, \infty)$ a control function. A function $\Delta : Z \times Z \to [0, \infty)$ is called an *extended b-metric* if, for all $\mu, \nu, \omega \in Z$, the following hold:

1. $\Delta(\mu, \nu) = 0$ if and only if $\mu = \nu$,
2. $\Delta(\mu, \nu) = \Delta(\nu, \mu)$,
3. $\Delta(\mu, \nu) \le \eta(\mu, \nu)\, [\Delta(\mu, \omega) + \Delta(\omega, \nu)]$.

The pair (Z, Δ) is then referred to as an extended b-metric space.

Definition 1.3 **([18])** Let Z be a non-empty set and $\eta : Z \times Z \to [1, \infty)$ a control function. A function $\Delta : Z \times Z \to [0, \infty)$ is said to be a *controlled metric type* if, for any $\mu, \nu, \omega \in Z$, the following conditions are satisfied:

1. $\Delta(\mu, \nu) = 0 \iff \mu = \nu$,
2. $\Delta(\mu, \nu) = \Delta(\nu, \mu)$,
3. $\Delta(\mu, \nu) \le \eta(\mu, \omega)\Delta(\mu, \omega) + \eta(\omega, \nu)\Delta(\omega, \nu)$.

The pair (Z, Δ) is then a *controlled metric type space.*

Example 1.1 **([18])** Let $Z = \{1, 2, 3, \ldots\}$ and define $\Delta : Z \times Z \to [0, \infty)$ as:

$$\Delta(\mu, \nu) = \begin{cases} 0, & \text{if } \mu = \nu, \\ \frac{1}{\mu}, & \text{if } \mu \text{ even, } \nu \text{ odd,} \\ \frac{1}{\nu}, & \text{if } \mu \text{ odd, } \nu \text{ even,} \\ 1, & \text{otherwise.} \end{cases}$$

Define $\eta : Z \times Z \to [1, \infty)$ by:

$$\eta(\mu, \nu) = \begin{cases} \mu, & \mu \text{ even, } \nu \text{ odd,} \\ \nu, & \mu \text{ odd, } \nu \text{ even,} \\ 1, & \text{otherwise.} \end{cases}$$

Then Δ is a controlled metric type. However, for $p \ge 2$:

$$\Delta(2p+1, 4p+1) = 1 > \frac{1}{p} = \eta(2p+1, 4p+1)\left[\Delta(2p+1, 2p) + \Delta(2p, 4p+1)\right],$$

showing that Δ is not an extended b-metric with the same control function.

Example 1.2 **([18])** Let $Z = \{0, 1, 2\}$, and define $\Delta : Z \times Z \to [0, \infty)$ as:

$$\begin{aligned} \Delta(0,0) &= \Delta(1,1) = \Delta(2,2) = 0, \\ \Delta(0,1) &= \Delta(1,0) = 1, \\ \Delta(0,2) &= \Delta(2,0) = \frac{1}{2}, \\ \Delta(1,2) &= \Delta(2,1) = \frac{2}{5}. \end{aligned}$$

Define a symmetric control function $\eta : Z \times Z \to [1, \infty)$ as:

$$\eta(0,1) = \frac{11}{10}, \quad \eta(1,2) = \frac{5}{4}, \quad \eta(0,2) = 1, \quad \eta(x,x) = 1.$$

Then Δ is a controlled metric type, but:

$$\Delta(0,1) = 1 > \frac{99}{100} = \eta(0,1)\left[\Delta(0,2) + \Delta(2,1)\right],$$

so Δ is not an extended b-metric with $\eta = \theta$.

2. Main Results

Throughout this paper, we work within the framework of a complete controlled metric type space (Z, Δ), associated with a control function $\eta : Z \times Z \to [1, \infty)$. Unless otherwise specified, all results are derived under this setting. Henceforth, the notation FP will be used to represent a fixed point.

Theorem 2.1 *Let (Z, Δ) be a complete controlled metric type space equipped with a control function $\eta : Z \times Z \to [1, \infty)$. Assume there exist constants $\gamma_1, \gamma_2 \in (0, 1]$ and $\gamma_3 \in [0, 1)$ such that the mapping $\tau : Z \to Z$ satisfies*

$$\Delta(\tau\mu, \tau\nu) \leq \gamma_1\eta(\mu, \nu)\Delta(\mu, \nu) + \gamma_2\eta(\mu, \tau\mu)\Delta(\mu, \tau\mu) + \gamma_3\Delta(\nu, \tau\nu), \tag{2.1}$$

for all $\mu, \nu \in Z$.

Suppose there exists $\mu_0 \in Z$, and define the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ by $\mu_n = \tau^n\mu_0$. Let

$$\gamma := \frac{\gamma_1 + \gamma_2}{1 - \gamma_3} < 1,$$

and assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \left(\frac{\eta(\mu_{i+1}, \mu_{i+2})}{\eta(\mu_i, \mu_{i+1})} \cdot \eta(\mu_{i+1}, \mu_m) \right) < \frac{1}{\gamma}. \tag{2.2}$$

Additionally, for every $\mu \in Z$, suppose the limits

$$\lim_{n\to\infty} \eta(\mu_n, \mu) \quad \text{and} \quad \lim_{n\to\infty} \eta(\mu, \mu_n) \tag{2.3}$$

exist and are finite.

If there exists $\mu \in Z$ such that

$$\eta(\mu, \tau\mu) < \frac{1}{\gamma_3},$$

then the mapping τ has a fixed point in Z; that is, there exists $\mu^* \in Z$ such that $\tau(\mu^*) = \mu^*$.

Proof: Let $\mu_0 \in Z$ be the initial point as in the hypothesis. Define the sequence $\{\mu_n\}$ by $\mu_n = \tau^n\mu_0$ for all $n \in \mathbb{N}$. If any two consecutive elements coincide, then τ has a fixed point, and the proof is complete. Otherwise, we show that $\{\mu_n\}$ is a Cauchy sequence.

From the contractive condition (2.1), for each $n \in \mathbb{N}$,

$$\Delta(\mu_n, \mu_{n+1}) = \Delta(\tau\mu_{n-1}, \tau\mu_n) \leq (\gamma_1 + \gamma_2)\eta(\mu_{n-1}, \mu_n)\Delta(\mu_{n-1}, \mu_n) + \gamma_3\Delta(\mu_n, \mu_{n+1}).$$

Rewriting, we get:

$$\Delta(\mu_n, \mu_{n+1}) \leq \gamma\eta(\mu_{n-1}, \mu_n)\Delta(\mu_{n-1}, \mu_n),$$

where $\gamma := \frac{\gamma_1+\gamma_2}{1-\gamma_3} < 1$. Iterating this inequality gives:

$$\Delta(\mu_n, \mu_{n+1}) \leq \gamma^n \left(\prod_{j=1}^{n} \eta(\mu_{j-1}, \mu_j) \right) \Delta(\mu_0, \mu_1).$$

To estimate $\Delta(\mu_n, \mu_m)$ for $n < m$, apply the triangle inequality iteratively:

$$\Delta(\mu_n, \mu_m) \leq \sum_{i=n}^{m-1} \left(\prod_{j=n}^{i} \eta(\mu_j, \mu_{j+1}) \right) \Delta(\mu_i, \mu_{i+1}).$$

Substituting from above yields:

$$\Delta(\mu_n, \mu_m) \leq \Delta(\mu_0, \mu_1) \sum_{i=n}^{m-1} \gamma^i \left(\prod_{j=1}^{i} \eta(\mu_j, \mu_{j+1}) \right)^2 .$$

Let $S_p := \sum_{i=0}^{p} \gamma^i \left(\prod_{j=0}^{i} \eta(\mu_j, \mu_{j+1})\right)^2$. Then:

$$\Delta(\mu_n, \mu_m) \leq \Delta(\mu_0, \mu_1)(S_{m-1} - S_{n-1}).$$

By conditions (2.2) and (2.3), the sequence S_p is convergent, so:

$$\lim_{n,m\to\infty} \Delta(\mu_n, \mu_m) = 0.$$

Hence, $\{\mu_n\}$ is a Cauchy sequence. Completeness of (Z, Δ) implies there exists $\mu' \in Z$ such that:

$$\lim_{n\to\infty} \Delta(\mu_n, \mu') = 0.$$

Next, we prove that μ' is a fixed point. By the triangle inequality and the contractive condition:

$$\begin{aligned}
\Delta(\mu', \tau\mu') &\leq \eta(\mu', \mu_{n+1})\Delta(\mu', \mu_{n+1}) + \eta(\mu_{n+1}, \tau\mu')\Delta(\tau\mu_n, \tau\mu') \\
&\leq \eta(\mu', \mu_{n+1})\Delta(\mu', \mu_{n+1}) + \eta(\mu_{n+1}, \tau\mu')\big[\gamma_1\eta(\mu_n, \mu')\Delta(\mu_n, \mu') \\
&\quad + \gamma_2\eta(\mu_n, \tau\mu_n)\Delta(\mu_n, \tau\mu_n) + \gamma_3\Delta(\mu', \tau\mu')\big].
\end{aligned}$$

Taking the limit as $n \to \infty$ and using the assumptions and earlier limits, we obtain:

$$\Delta(\mu', \tau\mu') \leq \gamma_3\eta(\mu', \tau\mu')\Delta(\mu', \tau\mu').$$

Since $\eta(\mu', \tau\mu') < \frac{1}{\gamma_3}$, it follows that $\Delta(\mu', \tau\mu') = 0$, hence $\tau\mu' = \mu'$.

Thus, μ' is a fixed point of τ. Uniqueness may be shown under an additional condition. □

Theorem 2.2 *Let (Z, Δ) be a complete controlled metric type space. Suppose there exist constants $\gamma_1 \in (0, 1]$, $\gamma_2 \in [0, 1]$, and $\gamma_3 \in [0, 1)$, such that the mapping $\tau : Z \to Z$ satisfies*

$$\Delta(\tau\mu, \tau\nu) \leq \gamma_1\eta(\mu, \nu)\Delta(\mu, \nu) + \gamma_2\eta(\mu, \tau\mu)\Delta(\mu, \tau\mu) + \gamma_3\Delta(\nu, \tau\nu), \tag{2.4}$$

for all $\mu, \nu \in Z$. Further, assume that

$$\limsup_{n\to\infty} \eta(\tau^n\mu, \tau^n\nu) < \frac{1}{\gamma_1},$$

for all $\mu, \nu \in Z$. Then, τ admits a unique fixed point in Z.

Proof: From Theorem 2.1, there exists $\mu' \in Z$ such that $\tau\mu' = \mu'$. Suppose, for contradiction, there exists another fixed point $\nu' \in Z$, $\nu' \neq \mu'$, with $\tau\nu' = \nu'$. Applying inequality (2.4), we obtain:

$$\begin{aligned}
\Delta(\mu', \nu') &= \Delta(\tau\mu', \tau\nu') \\
&\leq \gamma_1\eta(\mu', \nu')\Delta(\mu', \nu') + \gamma_2\eta(\mu', \mu') \cdot 0 + \gamma_3 \cdot 0 \\
&= \gamma_1\eta(\mu', \nu')\Delta(\mu', \nu').
\end{aligned}$$

Thus,

$$(1 - \gamma_1\eta(\mu', \nu'))\Delta(\mu', \nu') \leq 0.$$

Since $\Delta(\mu', \nu') > 0$, we must have $\gamma_1\eta(\mu', \nu') \geq 1$, contradicting the assumption that $\limsup_{n\to\infty} \eta(\tau^n\mu', \tau^n\nu') < \frac{1}{\gamma_1}$. Therefore, $\mu' = \nu'$, and the fixed point is unique. □

Corollary 2.1 *Let (Z, Δ) be a complete controlled metric type space. Assume that there exist constants $\gamma_1 \in (0, 1]$ and $\gamma_2 \in [0, 1]$ such that the mapping $\tau : Z \to Z$ satisfies the inequality*

$$\Delta(\tau\mu, \tau\nu) \leq \gamma_1\eta(\mu, \nu)\Delta(\mu, \nu) + \gamma_2\eta(\mu, \tau\mu)\Delta(\mu, \tau\mu), \tag{2.5}$$

for all $\mu, \nu \in Z$.

Let $\mu_0 \in Z$ be an initial point, and define the iterative sequence $\mu_n = \tau^n \mu_0$ for all $n \in \mathbb{N}$. Set $\gamma := \gamma_1 + \gamma_2$, and suppose the following condition is satisfied:

$$\sup_{m \geq 1} \lim_{i \to \infty} \left(\frac{\eta(\mu_{i+1}, \mu_{i+2})}{\eta(\mu_i, \mu_{i+1})} \cdot \eta(\mu_{i+1}, \mu_m) \right) < \frac{1}{\gamma}. \tag{2.6}$$

Furthermore, for each $\mu \in Z$, assume:

(i) The limits $\lim_{n\to\infty} \eta(\mu_n, \mu)$ and $\lim_{n\to\infty} \eta(\mu, \mu_n)$ exist and are finite.

(ii) For all $\mu, \nu \in Z$, the inequality $\limsup_{n\to\infty} \eta(\tau^n \mu, \tau^n \nu) < \frac{1}{\gamma_1}$ holds.

Then τ has a unique fixed point in Z.

Proof: This corollary is a direct consequence of Theorem 2.2. By setting $\gamma_3 = 0$, the contractive condition from Theorem 2.2 simplifies to the two-term inequality assumed here. The provided assumptions ensure the conditions of Theorem 2.2 are met, thereby guaranteeing the existence and uniqueness of the fixed point. □

Example 2.1 Let $Z = \{0, 1, 2\}$. Define the mapping $\tau : Z \to Z$ by $\tau(\mu) = \frac{1}{4}\mu$, and define the function $\eta : Z \times Z \to [1, \infty)$ by

$$\eta(\mu, \nu) = 2 + 2\mu + 2\nu.$$

Also, consider a function $\Delta : Z \times Z \to [0, \infty)$ defined by:

$$\Delta(\mu, \nu) = \begin{cases} 0 & \text{if } \mu = \nu, \\ \frac{1}{2} & \text{if } \{\mu, \nu\} = \{0, 1\} \text{ or } \{0, 2\}, \\ \frac{1}{11} & \text{if } \{\mu, \nu\} = \{1, 2\}. \end{cases}$$

Let us redefine η as:

$$\eta(0,0) = \eta(1,1) = \eta(2,2) = \eta(0,2) = 1, \quad \eta(0,1) = \frac{11}{10}, \quad \eta(1,2) = \frac{5}{4}.$$

This ensures that (Z, Δ) is a controlled metric type space.

(i) Define τ as follows:

$$\tau(0) = 2, \quad \tau(1) = \tau(2) = 1.$$

Let the constants be $\gamma_1 = \gamma_3 = \frac{1}{11}$ and $\gamma_2 = \frac{2}{11}$. Then, the combined contractive constant is:

$$\gamma = \frac{\gamma_1 + \gamma_2}{1 - \gamma_3} = \frac{3/11}{10/11} = \frac{3}{10} < 1.$$

Moreover,

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\eta(\mu_{i+1}, \mu_{i+2})}{\eta(\mu_i, \mu_{i+1})} \cdot \eta(\mu_{i+1}, \mu_m) = 1 < \frac{10}{3} = \frac{1}{\gamma}.$$

(ii) For any $\mu \in Z$, it holds that:

$$\lim_{n\to\infty} \eta(\mu_n, \mu) = \lim_{n\to\infty} \eta(\mu, \mu_n) = 2 + 2\mu,$$

which is finite.

(iii) For any $\mu \in Z$:

$$\eta(\mu, \tau\mu) = 6 < 11 = \frac{1}{\gamma_3}.$$

(iv) For all $\mu, \nu \in Z$, we have:

$$\limsup_{n\to\infty} \eta(\tau^n \mu, \tau^n \nu) = 2 < 11 = \frac{1}{\gamma_1}.$$

(v) The mapping τ satisfies:

$$\Delta(\tau\mu, \tau\nu) \leq \frac{1}{11}\eta(\mu, \nu)\Delta(\mu, \nu) + \frac{2}{11}\eta(\mu, \tau\mu)\Delta(\mu, \tau\mu) + \frac{1}{11}\Delta(\nu, \tau\nu).$$

Therefore, all the conditions of Theorem 2.1 or Theorem 2.2 are satisfied for the given constants. Hence, the mapping τ has a unique fixed point in Z, which is $\mu' = 1$.

Theorem 2.3 *Let (Z, Δ) be a complete controlled metric type space. Suppose $\beta_1, \beta_2 \in [0, 1)$ satisfy*

$$\beta_1 + \beta_2 < 1.$$

Let $\tau : Z \to Z$ be a mapping such that for all $\mu, \nu \in Z$,

$$\Delta(\tau\mu, \tau\nu) \leq \beta_1 \eta(\mu, \nu)\Delta(\mu, \nu) + \beta_2 \eta(\nu, \tau\mu)\Delta(\nu, \tau\mu). \tag{2.7}$$

Fix $\mu_0 \in Z$ and define the iterative sequence $\mu_n = \tau^n \mu_0$ for $n \geq 0$. Suppose the following conditions hold:

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\eta(\mu_{i+1}, \mu_{i+2})}{\eta(\mu_i, \mu_{i+1})} \cdot \eta(\mu_{i+1}, \mu_m) < \frac{1}{\beta_1}, \tag{2.8}$$

and for every $\mu \in Z$,

$$\lim_{n\to\infty} \eta(\mu_n, \mu) \quad \text{and} \quad \lim_{n\to\infty} \eta(\mu, \mu_n) \quad \text{exist and are finite.} \tag{2.9}$$

Additionally, assume

$$\limsup_{n\to\infty} \eta(\tau^n \mu, \tau^n \nu) < \frac{1}{\beta_1 + \beta_2} \quad \text{for all } \mu, \nu \in Z.$$

Then τ admits a unique fixed point in Z.

Proof: Starting from $\mu_0 \in Z$, define the sequence $\mu_{n+1} = \tau\mu_n$. Applying inequality (2.7) with $\mu = \mu_{n-1}$ and $\nu = \mu_n$ yields

$$\Delta(\mu_n, \mu_{n+1}) \leq \beta_1 \eta(\mu_{n-1}, \mu_n)\Delta(\mu_{n-1}, \mu_n) + \beta_2 \eta(\mu_n, \mu_n)\Delta(\mu_n, \mu_n) = \beta_1 \eta(\mu_{n-1}, \mu_n)\Delta(\mu_{n-1}, \mu_n),$$

since $\Delta(\mu_n, \mu_n) = 0$.

By induction,

$$\Delta(\mu_n, \mu_{n+1}) \leq \beta_1^n \left(\prod_{j=1}^{n} \eta(\mu_{j-1}, \mu_j) \right) \Delta(\mu_0, \mu_1).$$

For $m > n$, the controlled triangle inequality implies

$$\Delta(\mu_n, \mu_m) \leq \eta(\mu_n, \mu_{n+1})\Delta(\mu_n, \mu_{n+1}) + \eta(\mu_{n+1}, \mu_m)\Delta(\mu_{n+1}, \mu_m).$$

Iterating this and using the convergence condition (2.8) shows $\{\mu_n\}$ is a Cauchy sequence. By completeness, $\mu_n \to \mu' \in Z$.

To show μ' is fixed by τ, note that

$$\Delta(\mu', \tau\mu') \leq \eta(\mu', \mu_{n+1})\Delta(\mu', \mu_{n+1}) + \eta(\mu_{n+1}, \tau\mu')\Delta(\mu_{n+1}, \tau\mu').$$

Using (2.7) and continuity conditions, the right-hand side tends to zero as $n \to \infty$, hence $\tau\mu' = \mu'$.

Uniqueness follows by applying (2.7) to two fixed points μ', ν' and using the assumption on the limit superior of η, which forces

$$\Delta(\mu', \nu') = 0 \implies \mu' = \nu'.$$

□

Corollary 2.2 *Let (Z, Δ) be a complete controlled metric type space. Suppose there exists a constant $\beta \in (0,1)$ such that the mapping $\tau : Z \to Z$ satisfies*

$$\Delta(\tau\mu, \tau\nu) \leq \beta\, \eta(\mu,\nu)\, \Delta(\mu,\nu)$$

for all $\mu, \nu \in Z$.

Fix an element $\mu_0 \in Z$ and define the sequence $\{\mu_n\}$ by $\mu_n = \tau^n \mu_0$ for all $n \in \mathbb{N}$. Assume further that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\eta(\mu_{i+1}, \mu_{i+2})}{\eta(\mu_i, \mu_{i+1})} \cdot \eta(\mu_{i+1}, \mu_m) < \frac{1}{\beta}.$$

Moreover, suppose that for every $\mu \in Z$:

i) The limits $\lim_{n\to\infty} \eta(\mu_n, \mu)$ and $\lim_{n\to\infty} \eta(\mu, \mu_n)$ exist and are finite.

ii) The value $\limsup_{n\to\infty} \eta(\tau^n\mu, \tau^n\nu)$ exists and satisfies

$$\limsup_{n\to\infty} \eta(\tau^n\mu, \tau^n\nu) < \frac{1}{\beta}.$$

Then τ admits a unique fixed point in Z.

Proof: The conclusion is an immediate consequence of Theorem 2.3 upon setting $\beta_1 = \beta$ and $\beta_2 = 0$. □

Example 2.2 Consider the set $Z = \{0, 1, 2, \ldots\}$ and define the mapping $\tau : Z \to Z$ by

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu = 1, \\ \mu - 1, & \text{if } \mu > 1. \end{cases}$$

Let the control function $\eta : Z \times Z \to [1, \infty)$ be given by

$$\eta(\mu, \nu) = 1 + |\mu - \nu|,$$

and the metric-type function $\Delta : Z \times Z \to [0, \infty)$ be defined as

$$\Delta(\mu, \nu) = \begin{cases} 0, & \text{if } \mu = \nu, \\ \frac{1}{\mu+\nu}, & \text{if } \mu \neq \nu. \end{cases}$$

Then the following statements hold:

i) (Z, Δ) forms a complete controlled metric type space.

ii) Setting $\beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{1}{3}$, we have $\beta = \beta_1 + \beta_2 = \frac{5}{6} < 1$. For $\mu_0 = 0$ and the sequence $\mu_n = \tau^n \mu_0$, condition (2.8) is fulfilled.

iii) For each $\mu \in Z$, the limits $\lim_{n\to\infty} \eta(\mu_n, \mu)$ and $\lim_{n\to\infty} \eta(\mu, \mu_n)$ exist and are finite, equaling $1 + |\mu|$.

iv) For all $\mu, \nu \in Z$,

$$\limsup_{n\to\infty} \eta(\tau^n\mu, \tau^n\nu) = 1 < \frac{6}{5} = \frac{1}{\beta_1 + \beta_2}.$$

v) For every $\mu, \nu \in Z$, the inequality

$$\Delta(\tau\mu, \tau\nu) \leq \frac{1}{2}\eta(\mu,\nu)\Delta(\mu,\nu) + \frac{1}{3}\eta(\nu,\tau\mu)\Delta(\nu,\tau\mu)$$

holds.

Thus, all the conditions of Theorem 2.3 are satisfied, and the mapping τ admits a unique fixed point in Z, namely $\mu' = 1$.

3. Conclusion

Extending the Banach contraction principle to more general spaces is fundamental for advancing fixed point theory. This has been realized by introducing diverse contraction conditions in generalized spaces like controlled metric type spaces and extended b-metric spaces. The integration of these new contraction types within controlled metric type spaces allows for a richer and more adaptable fixed point theory. Our study utilized control functions to establish fixed point theorems for contractive mappings and provided examples illustrating the applicability and strength of these results.

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