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Vertex Energy of Small Integral Graphs

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ABSTRACT: The energy of a graph, introduced in 1970s, is a well studied spectrum based graph invariant that has a notable number of applications in different fields of science. Recently, Arizmendi *et* al. have reworked on this idea by introducing the concept of vertex energy of a graph which reflects upon the way the total energy is distributed among the individual vertices. Further, they have explained a method of computing the energy of a vertex v by means of solving a system of linear equations involving the number of $v-v$ walks of different lengths from v. In the present study, we compute the vertex energies for all non-trivial connected integral graphs of order up to seven using this method.

Key Words: Graph energy, vertex energy, vertex symmetry, integral graphs.

Contents

1. Introduction

The adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$ of a graph $\mathcal{G}$, with its vertex set $V(\mathcal{G}) = \{v_1, ..., v_n\}$, is a square matrix of order n, where the entry a_{ij} is 1 if the vertices v_i and v_j are adjacent and 0 otherwise. It serves as a fundamental tool in spectral graph theory for capturing the structural information of the graph in algebraic form. One of the key algebraic tools derived from the adjacency matrix is the characteristic polynomial, defined as

$$\phi(\mathcal{G}; \lambda) = |\lambda I - \mathcal{A}(\mathcal{G})|,$$

where I is the identity matrix of order n. The roots of the characteristic polynomial are the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $\mathcal{G}$ and play a prominent role in understanding its spectral properties.

The energy of a graph, introduced by Gutman [5], is a spectrum based graph invariant and is defined as the sum of the absolute values of its eigenvalues, i.e,

$$\mathcal{E}(\mathcal{G}) = \sum_{i=1}^{n} |\lambda_i|.$$

The main motivation behind the concept of graph energy is the total π-electron energy in conjugated hydrocarbon molecules, where the eigenvalues represent energy levels in molecular orbital theory. Over the years, graph energy has become a well established invariant with applications in chemistry, physics and network analysis.

In 2018, Arizmendi *et* al. [1] introduced a local version of graph energy, referred to as the energy of a vertex. This concept provides a finer vertex-level distribution of the total graph energy. The energy of a vertex v_i is defined as

$$\mathcal{E}_{\mathcal{G}}(v_i) = |\mathcal{A}|_{ii} \text{ for } i = 1, 2, \cdots, n,$$

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where $|\mathcal{A}| = (\mathcal{A}\mathcal{A}^*)^{\frac{1}{2}}$ and $\mathcal{A}^*$ is the conjugate transpose of the adjacency matrix.

The definition of vertex energy provides an intuitive method to localize spectral contributions and satisfies the additive property

$$\mathcal{E}(\mathcal{G}) = \sum_{i=1}^{n} \mathcal{E}_{\mathcal{G}}(v_i),$$

meaning that total graph energy can be viewed as a sum of the contributions from individual vertices.

In [1], the authors have explored fundamental properties of vertex energy, including general bounds, inequalities and examples. Further, they have computed the vertex energies explicitly for several well known classes of graphs including complete graphs, friendship graphs, cycles and hypercubes. Also, they have explained a method of computing the energy of a vertex v_i by means of solving a system of linear equations involving the number of $v_i - v_i$ walks of different lengths from v_i. Building on this line of research, Ramane *et* al. [9] have computed the energy of the vertices of subdivision graphs of complete graph, complete bipartite graph, cocktail party graph and Petersen graph. Gutman *et* al. [6] have proposed a novel approach for computing vertex energies using eigenvalues and eigenvectors. Following this, Sharathkumar *et* al. [10] have calculated the vertex energy of all non-isomorphic integral trees up to order 30.

Integral graphs are graphs that are characterized as having only integer-valued eigenvalues. The complete graph K_n, cocktail-party graph $CP(n)$ and the complete multipartite graph $K_{n/k,\cdots,n/k}$ are classic examples of integral graphs. For a detailed study on integral graphs, we refer [2,7,13,4,3,8,12].

A fundamental property that reveals the internal structure of a graph is vertex symmetry that highlights the uniformity of a graph from the perspective of its vertices. A graph is said to be vertex-symmetric if, for any two vertices v_i and v_j, there exists an automorphism ϕ such that $\phi(v_i) = v_j$, where an automorphism is a bijective mapping of the vertex set that preserves adjacency relations. As a result, all the vertices in a vertex transitive graph are structurally indistinguishable, sharing identical relational characteristics within the graph. Symmetry plays a crucial role in spectral graph theory, network design and algebraic graph theory, as it ensures uniform structural and algebraic properties in the graph. Several well-known classes of graphs including complete graphs, cycle graphs, Petersen graph, hypercubes and Cayley graphs are vertex symmetric.

The studies on vertex energy illustrate how graph symmetry and structural properties in a graph influence the distribution of energy among the vertices. Also, the corresponding energy decomposition holds theoretical interest and practical usefulness, especially in examining the impact or significance of the nodes/atoms in a network/molecule.

Based on these studies, in this paper, we extend the study of vertex energy by computing it for all non-trivial connected integral graphs of order up to seven [13]. This work aims to further explore the relationship between graph structure and vertex energy within the context of spectral graph theory.

2. Preliminaries

Throughout this paper, $\mathcal{G}$ represents a simple, connected and undirected graph of order n, with vertex set $V(\mathcal{G}) = \{v_1, v_2, \cdots, v_n\}$.. The adjacency matrix $\mathcal{A}(\mathcal{G})$ has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, constituting the spectrum of $\mathcal{G}$.

The energy of the graph $\mathcal{G}$ is defined as

$$\mathcal{E}(\mathcal{G}) = \sum_{i=1}^{n} |\lambda_i|.$$

As discussed in [1], the energy of a vertex v_i is given by

$$\mathcal{E}_{\mathcal{G}}(v_i) = |\mathcal{A}(\mathcal{G})|_{ii} \text{ for } i = 1, 2, \cdots, n,$$

where $|\mathcal{A}| = (\mathcal{A}\mathcal{A}^*)^{\frac{1}{2}}$ and $\mathcal{A}^*$ is the conjugate transpose of $\mathcal{A}(\mathcal{G})$.

Further, the energy of a vertex can be computed using Lemma 2.1 and 2.2.

Lemma 2.1 *[1] Let $\mathcal{G}$ be a graph of order n. Then,*

$$\mathcal{E}_{\mathcal{G}}(v_i) = \sum_{j=1}^{n} p_{ij} \left|\lambda_j\right|, \quad i = 1, \cdots, n,$$

where λ_j denotes the j^{th} eigenvalue of $\mathcal{A}(\mathcal{G})$ and

$$\sum_{i=1}^{n} p_{ij} = 1 \quad \text{and} \quad \sum_{j=1}^{n} p_{ij} = 1.$$

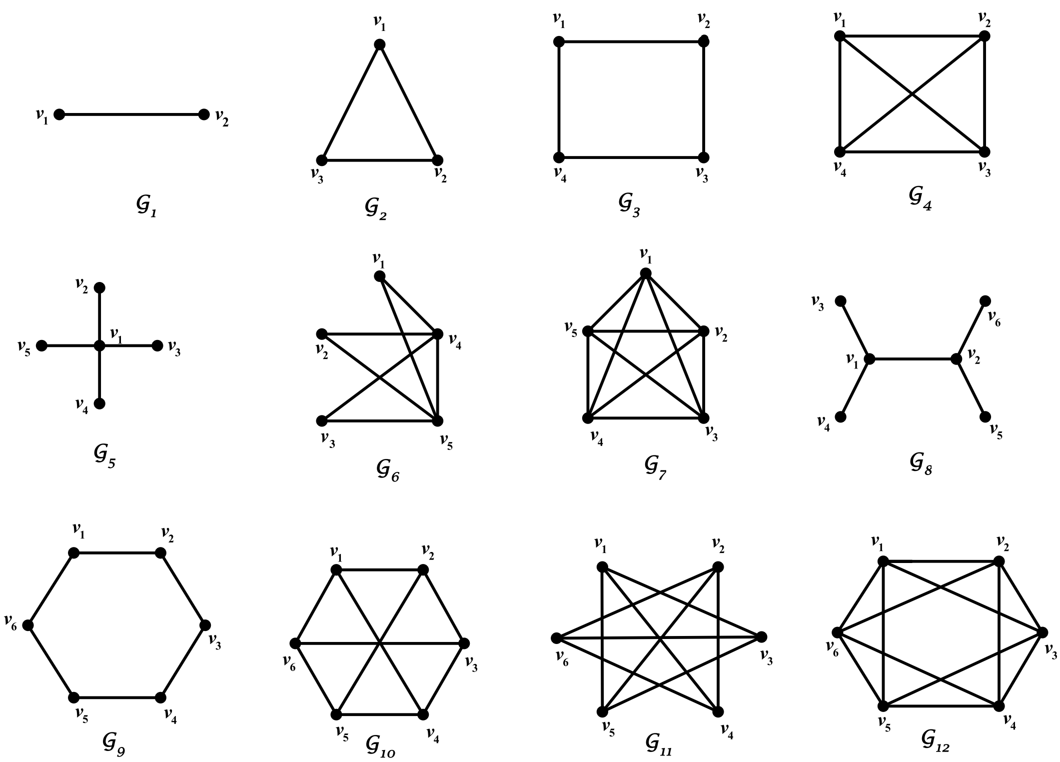
Further, $p_{ij} = u_{ij}^2$, where $U = (u_{ij})$ is the matrix of orthonormal vectors of the eigenvectors of $\mathcal{A}(\mathcal{G})$.

Lemma 2.2 *[1] Let $\mathcal{G}$ be a graph of order n. For $k \in \mathbb{N}$, let $\phi_i\left(A^k\right)$ be the k^{th} moment of A w. r. t. the linear functional ϕ_i. Then,*

$$\phi_i\left(A^k\right) = \sum_{j=1}^{n} p_{ij}\left(\lambda_j^k\right), \qquad i = 1, \cdots, n,$$

with $\phi_i\left(A^k\right)$ being the number of $v_i - v_i$ walks of $\mathcal{G}$ of length k.

3. Figures



4. Vertex energy of integral graphs of order up to seven

As discussed in [11,13], there are only 20 non-trivial connected integral graphs of order up to seven as depicted in Figure 1. In this section, we determine the vertex energies of these integral graphs.

We begin with some of the previously established results in literature for further reference.

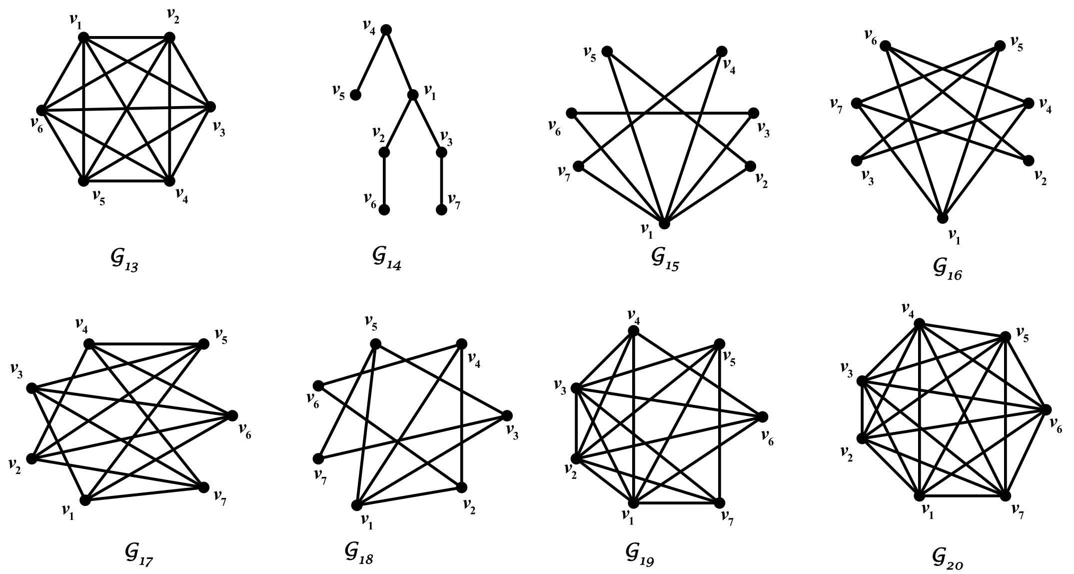


Figure 1: All non-trivial connected integral graphs of order up to seven

Theorem 4.1 *[1] If $\mathcal{G}$ is any vertex-transitive graph with n vertices, then the energy of each vertex v_i, where $i = 1, 2, \cdots, n$ is given by $\mathcal{E}(v_i) = \dfrac{\mathcal{E}(\mathcal{G})}{n}$, $i = 1, 2, \cdots, n$.*

Theorem 4.2 *[1] For the star graph $\mathcal{S}_n$, the energy of each vertex v is given by*

$$\mathcal{E}_{\mathcal{S}_n}(v_i) = \begin{cases} \sqrt{n-1} & \text{if } v_i \text{ is the center of } S_n, \\ \dfrac{1}{\sqrt{n-1}} & \text{otherwise.} \end{cases}$$

Theorem 4.3 *[10] For the graph $G = \mathcal{K}_{1,n} \sim \mathcal{K}_{1,n}$, with u_0 and v_0 being the central vertices of each star and $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ their leaves respectively,*

$$\mathcal{E}_G(v) = \begin{cases} \dfrac{2n+1}{\sqrt{4n+1}} & \text{if } v = u_0, v_0, \\ \dfrac{2}{\sqrt{4n+1}} & \text{otherwise.} \end{cases}$$

Theorem 4.4 *[10] Let $G = \mathcal{SK}_{1,n}$ be the subdivision graph of the star graph $\mathcal{K}_{1,n}$ with v_0 being the central vertex, $v_1, \cdots, v_n$ the leaves and $s_1, \cdots s_n$ the vertices obtained by subdivision, then*

$$\mathcal{E}_G(v) = \begin{cases} \dfrac{n}{\sqrt{n+1}} & \text{if } v = v_0, \\ \dfrac{n^2 - 1 + \sqrt{n+1}}{n\,(n+1)} & \text{if } v = v_i, i = 1, \cdots, n, \\ \dfrac{n - 1 + \sqrt{n+1}}{n} & \text{otherwise.} \end{cases}$$

As a direct consequence of the above theorems, we have the following result.

Proposition 4.1 *For each $v \in \mathcal{G}_n$, for $n \neq 6$, and $n \neq 15, \cdots, 19$ in Fig. 1,*

$$\mathcal{E}_{\mathcal{G}_n}(v) = \begin{cases} 1 & \text{for } n = 1, 3, 10, \\ 1.333333 & \text{for } n = 2, 9, 11, 12, \\ 1.5 & \text{for } n = 4, \\ 1.6 & \text{for } n = 7, \\ 1.666667 & \text{for } n = 13, \\ 1.714826 & \text{for } n = 20, \\ 2 & \text{for } n = 5,\ v = v_1, \\ 0.5 & \text{for } n = 5,\ v = v_2, \cdots, v_5, \\ 1.666667 & \text{for } n = 8,\ v = v_1, v_2, \\ 0.666667 & \text{for } n = 8,\ v = v_3, \cdots, v_6, \\ 1.5 & \text{for } n = 14,\ v = v_1, \\ 1.333333 & \text{for } n = 14,\ v = v_2, v_3, v_4, \\ 0.833333 & \text{otherwise.} \end{cases}$$

Theorem 4.5 *For the graph $\mathcal{G}_6$ in Fig. 1, with vertex symmetries $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$,*

$$\mathcal{E}_{\mathcal{G}_6}(v) = \begin{cases} 0.8 & \text{if } v \in \{v_1, v_2, v_3\}, \\ 1.8 & \text{otherwise.} \end{cases}$$

Proof: The characteristic polynomial of $\mathcal{G}_6$ is given by

$$P(\mathcal{G}_6; \lambda) = \lambda^2 (\lambda + 2)(\lambda + 1)(\lambda - 3).$$

As a result, the spectrum of $\mathcal{G}_6$ is $-2, -1, 0^2, 3$, indicating that the distinct eigenvalues of $\mathcal{G}_6$ are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 0$ and $\lambda_4 = 3$.

By analyzing the vertex symmetries of the graph $\mathcal{G}_6$, we observe that there are two distinct vertex symmetries in the form of $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$, so that all the vertices within a symmetry set share the same energy. Since the graph $\mathcal{G}_6$ has four distinct eigenvalues and two types of vertex symmetry, Lemma 2.1 implies that the vertex energies can be determined by solving two 4×4 systems of equations, one corresponding to V_1, while the other pertains to V_2, by equating the first spectral moments with the walk counts, as detailed in the following two cases:

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}$ be the weights of each vertex $v_i \in V_1$ in $\mathcal{G}_6$. Then, we have

$$\begin{aligned} p_{11} + p_{12} + p_{13} + p_{14} &= 1 \\ p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 &= 0 \\ p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 &= 2 \\ p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 &= 2. \end{aligned}$$

On solving the system of equations, we get

$$p_{11} = 0.2, p_{12} = 0, p_{13} = 0.666667, p_{14} = 0.133333.$$

The energy associated with each vertex $v_i \in V_1$ is then given by

$$\mathcal{E}_{\mathcal{G}_6}(v_i) = \sum_{j=1}^{4} p_{1j} |\lambda_j| = 0.8.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}$ be the weights of any vertex $v_i \in V_2$ in $\mathcal{G}_6$. Then, we have

$$\begin{aligned} p_{21} + p_{22} + p_{23} + p_{24} &= 1 \\ p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 &= 0 \\ p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 &= 4 \\ p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 &= 6. \end{aligned}$$

Solving the system of equations, we get

$$p_{21} = 0.2, p_{22} = 0.5, p_{23} = 0, p_{24} = 0.3.$$

Hence, the energy corresponding to each vertex $v_i \in V_2$ is given by

$$\mathcal{E}_{\mathcal{G}6}(v_i) = \sum_{j=1}^{4} p_{2j} |\lambda_j| = 1.8.$$

□

Theorem 4.6 *For the graph $\mathcal{G}_{15}$ in Fig. 1, with vertex symmetries $\{v_1\}$ and $\{v_2, \cdots, v_7\}$,*

$$\mathcal{E}_{\mathcal{G}15}(v) = \begin{cases} 2.4 & \text{if } v = v_1, \\ 1.266667 & \text{otherwise.} \end{cases}$$

Proof: The characteristic polynomial of $\mathcal{G}_{15}$ is given by

$$P(\mathcal{G}_{15}; \lambda) = (\lambda + 2)(\lambda + 1)^3 (\lambda - 1)^2 (\lambda - 3).$$

As a result, the spectrum of $\mathcal{G}_{15}$ is $-2, -1^3, 1^2, 3$, indicating that the distinct eigenvalues of $\mathcal{G}_{15}$ are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1$ and $\lambda_4 = 3$.

By examining the vertex symmetries of the graph $\mathcal{G}_{15}$, we find that that there are two sets with distinct vertex symmetries, namely $V_1 = \{v_1\}$ and $V_2 = \{v_2, \cdots, v_7\}$, so that all the vertices within a set share the same energy. As the graph $\mathcal{G}_{15}$ has four distinct eigenvalues and two types of vertex symmetry, Lemma 2.1 implies that the energy of the vertices of $\mathcal{G}_{15}$ can be obtained by solving two 4×4 systems of equations, one corresponding to the vertex set V_1, while the other pertains to V_2. These are obtained by equating the spectral moments with the corresponding walk counts, as detailed in the following two cases.

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}$ be the weights of the vertex v_1 in $\mathcal{G}_{15}$. Then, we have

$$\begin{aligned} p_{11} + p_{12} + p_{13} + p_{14} &= 1 \\ p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 &= 0 \\ p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 &= 6 \\ p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 &= 6. \end{aligned}$$

Solving the system of equations, we get

$$p_{11} = 0.6, p_{12} = 0, p_{13} = 0, p_{14} = 0.4.$$

Thus, the energy of the vertex v_1 is given by

$$\mathcal{E}_{\mathcal{G}15}(v_1) = \sum_{j=1}^{4} p_{1j} |\lambda_j| = 2.4.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}$ be the weights of any vertex $v_i \in V_2$ in $\mathcal{G}_{15}$. Then, we have

$$\begin{aligned} p_{21} + p_{22} + p_{23} + p_{24} &= 1 \\ p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 &= 0 \\ p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 &= 2 \\ p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 &= 2. \end{aligned}$$

On solving the system of equations, we get

$$p_{21} = 0.066667, p_{22} = 0.5, p_{23} = 0.333333, p_{24} = 0.1.$$

Therefore, the energy corresponding to each vertex $v_i \in V_2$ becomes

$$\mathcal{E}_{\mathcal{G}_{15}}(v_i) = \sum_{j=1}^{4} p_{2j} |\lambda_j| = 1.266667.$$

□

Theorem 4.7 *For the graph $\mathcal{G}_{16}$ in Fig. 1, with vertex symmetries $\{v_1\}$, $\{v_2, v_3\}$ and $\{v_4, v_5, v_6, v_7\}$,*

$$\mathcal{E}_{\mathcal{G}_{16}}(v) = \begin{cases} 1.6 & \text{if } v = v_1, \\ 1.066667 & \text{if } v \in \{v_2, v_3\}, \\ 1.566667 & \text{otherwise.} \end{cases}$$

Proof: The characteristic polynomial of $\mathcal{G}_{16}$ is given by

$$P(\mathcal{G}_{16}; \lambda) = \lambda (\lambda + 2)^2 (\lambda + 1) (\lambda - 1)^2 (\lambda - 3).$$

As a result, the spectrum of $\mathcal{G}_{16}$ is $-2^2, -1, 0, 1^2, 3$, indicating that the distinct eigenvalues of $\mathcal{G}_{16}$ are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 0, \lambda_4 = 1$ and $\lambda_5 = 3$.

By examining the vertex symmetries of the graph $\mathcal{G}_{16}$, we find that there are three distinct vertex symmetry sets namely, $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$ and $V_3 = \{v_4, v_5, v_6, v_7\}$, so that all the vertices within a set share the same energy. The graph $\mathcal{G}_{16}$ has five distinct eigenvalues and three types of vertex symmetry, so that by Lemma 2.1, the vertex energies can be determined by solving three 5×5 systems of equations, one each corresponding to the vertex sets V_1, V_2 and V_3. These are obtained by equating the spectral moments with the corresponding walk counts, as detailed in the following three cases:

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ be the weights of the vertex v_1 in $\mathcal{G}_{16}$. Then, we have

$$\begin{aligned} p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1 \\ p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0 \\ p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= 4 \\ p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 4 \\ p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= 28. \end{aligned}$$

By solving the system of equations, we obtain

$$p_{11} = 0.4, p_{12} = 0, p_{13} = 0.333333, p_{14} = 0, p_{15} = 0.266667.$$

Hence, the energy of the vertex v_1 becomes

$$\mathcal{E}_{\mathcal{G}_{16}}(v_i) = \sum_{j=1}^{5} p_{1j} |\lambda_j| = 1.6.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}, p_{25}$ be the weights of any vertex $v_i \in V_2$ in $\mathcal{G}_{16}$. Then, we have

$$\begin{aligned} p_{21} + p_{22} + p_{23} + p_{24} + p_{25} &= 1 \\ p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 + p_{25}\lambda_5 &= 0 \\ p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 + p_{25}\lambda_5^2 &= 2 \\ p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 + p_{25}\lambda_5^3 &= 0 \\ p_{21}\lambda_1^4 + p_{22}\lambda_2^4 + p_{23}\lambda_3^4 + p_{24}\lambda_4^4 + p_{25}\lambda_5^4 &= 10. \end{aligned}$$

On solving the system of equations, we get

$$p_{21} = 0.266667, p_{22} = 0, p_{23} = 0.333333, p_{24} = 0.333333, p_{25} = 0.066667.$$

Thus, the vertex energy that corresponds to each vertex $v_i \in V_2$ is given by

$$\mathcal{E}_{\mathcal{G}}(v_i) = \sum_{j=1}^{5} p_{2j} |\lambda_j| = 1.066667.$$

Case 3: Let $p_{31}, p_{32}, p_{33}, p_{34}, p_{35}$ be the weights for any vertex $v_i \in V_3$ in $\mathcal{G}_{16}$. Then, we have

$$\begin{aligned} p_{31} + p_{32} + p_{33} + p_{34} + p_{35} &= 1 \\ p_{31}\lambda_1 + p_{32}\lambda_2 + p_{33}\lambda_3 + p_{34}\lambda_4 + p_{35}\lambda_5 &= 0 \\ p_{31}\lambda_1^2 + p_{32}\lambda_2^2 + p_{33}\lambda_3^2 + p_{34}\lambda_4^2 + p_{35}\lambda_5^2 &= 3 \\ p_{31}\lambda_1^3 + p_{32}\lambda_2^3 + p_{33}\lambda_3^3 + p_{34}\lambda_4^3 + p_{35}\lambda_5^3 &= 2 \\ p_{31}\lambda_1^4 + p_{32}\lambda_2^4 + p_{33}\lambda_3^4 + p_{34}\lambda_4^4 + p_{35}\lambda_5^4 &= 17. \end{aligned}$$

Solving the system of equations, we get

$$p_{31} = 0.266667, p_{32} = 0.25, p_{33} = 0, p_{34} = 0.333333, p_{35} = 0.15.$$

Thus, the energy associated with each vertex $v_i \in V_3$ is given by

$$\mathcal{E}_{\mathcal{G}}(v_i) = \sum_{j=1}^{5} p_{3j} |\lambda_j| = 1.566667.$$

□

Theorem 4.8 *For the graph $\mathcal{G}_{17}$ in Fig. 1, with vertex symmetries $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7\}$,*

$$\mathcal{E}_{\mathcal{G}_{17}}(v) = \begin{cases} 1.642857 & \text{if } v \in \{v_1, v_2, v_3, v_4\}, \\ 1.142857 & \text{otherwise.} \end{cases}$$

Proof: The characteristic polynomial of $\mathcal{G}_{17}$ is given by

$$P(\mathcal{G}_{17}; \lambda) = \lambda^2 (\lambda + 3) (\lambda + 1)^2 (\lambda - 1) (\lambda - 4).$$

As a result, the spectrum of $\mathcal{G}_{17}$ is $-3, -1^2, 0^2, 1, 4,$, indicating that the distinct eigenvalues of $\mathcal{G}_{17}$ are $\lambda_1 = -3, \lambda_2 = -1, \lambda_3 = 0, \lambda_4 = 1$ and $\lambda_5 = 4$.

By examining the vertex symmetries of the graph $\mathcal{G}_{17}$, we find that there are two sets with distinct vertex symmetries, namely $V_1 = \{v_1, v_2, v_3, v_4\}$ and $V_2 = \{v_5, v_6, v_7\}$, so that all the vertices within a symmetry set share the same energy. Since the graph $\mathcal{G}_{17}$ has five distinct eigenvalues and two types of vertex symmetry, Lemma 2.1 implies that the energy of the vertices of $\mathcal{G}_{17}$ can be obtained by solving two 5×5 systems of equations, one corresponding to V_1 and the other pertains to V_2. These are obtained by equating the spectral moments with the corresponding walk counts, as detailed in the following two cases.

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ be the weights of any vertex $v_i \in V_1$ in $\mathcal{G}_{17}$. Then, we have

$$\begin{aligned} p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1 \\ p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0 \\ p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= 4 \\ p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 6 \\ p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= 46. \end{aligned}$$

By solving the system of equations, we obtain

$$p_{11} = 0.107143, p_{12} = 0.5, p_{13} = 0, p_{14} = 0.25, p_{15} = 0.142857.$$

Thus, the energy of each vertex $v_i \in V_1$ becomes

$$\mathcal{E}_{\mathcal{G}_{17}}(v_i) = \sum_{j=1}^{5} p_{1j}\,|\lambda_j| = 1.642857.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}, p_{25}$ be the weights of any vertex $v_i \in V_2$. Then, we have

$$\begin{aligned}
p_{21} + p_{22} + p_{23} + p_{24} + p_{25} &= 1\\
p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 + p_{25}\lambda_5 &= 0\\
p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 + p_{25}\lambda_5^2 &= 4\\
p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 + p_{25}\lambda_5^3 &= 4\\
p_{21}\lambda_1^4 + p_{22}\lambda_2^4 + p_{23}\lambda_3^4 + p_{24}\lambda_4^4 + p_{25}\lambda_5^4 &= 52.
\end{aligned}$$

On solving the system of equations, we get

$$p_{21} = 0.190476, p_{22} = 0, p_{23} = 0.666667, p_{24} = 0, p_{25} = 0.142857.$$

Thereby, the energy of each vertex $v_i \in V_2$ will be

$$\mathcal{E}_{\mathcal{G}_{17}}(v_i) = \sum_{j=1}^{5} p_{2j}\,|\lambda_j| = 1.142857.$$

□

Theorem 4.9 *For the graph $\mathcal{G}_{18}$ in Fig. 1, with vertex symmetries $\{v_1\}$, $\{v_2, v_3, v_4, v_5\}$ and $\{v_6, v_7\}$,*

$$\mathcal{E}_{\mathcal{G}_{18}}(v) = \begin{cases} 1.6 & \text{if } v = v_1,\\ 1.566667 & \text{if } v \in \{v_2, v_3, v_4, v_5\},\\ 1.066667 & \text{if } v \in \{v_6, v_7\}. \end{cases}$$

Proof: The characteristic polynomial of $\mathcal{G}_{18}$ is given by

$$P(\mathcal{G}_{18}; \lambda) = \lambda(\lambda+2)(\lambda+1)^3(\lambda-2)(\lambda-3).$$

As a result, the the spectrum of $\mathcal{G}_{18}$ is $0, -2, -1^3, 2, 3$, indicating that the distinct eigenvalues of $\mathcal{G}_{18}$ are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 0, \lambda_4 = 2$ and $\lambda_5 = 3$.

By examining the vertex symmetries of the graph $\mathcal{G}_{18}$, we find that there are three sets with distinct vertex symmetries, namely $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3, v_4, v_5\}$ and $V_3 = \{v_6, v_7\}$, so that all the vertices within a set share the same energy. The graph $\mathcal{G}_{18}$ has five distinct eigenvalues and three types of vertex symmetry, so that by Lemma 2.1, the vertex energies can be determined by solving three 5×5 systems of equations, one corresponding to the vertex V_1, another to V_2 and the third one for V_3. These are obtained by equating the spectral moments with the corresponding walk counts, as detailed in the following three cases:

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ be the weights of the vertex v_1 in $\mathcal{G}_{18}$. Then, we have

$$\begin{aligned}
p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1\\
p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0\\
p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= 4\\
p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 4\\
p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= 28.
\end{aligned}$$

By solving the system of equations, we get

$$p_{11} = 0.4, p_{12} = 0, p_{13} = 0.333333, p_{14} = 0, p_{15} = 0.266667.$$

Thereby, the energy of the vertex v_1 becomes

$$\mathcal{E}_{\mathcal{G}_{18}}(v_1) = \sum_{j=1}^{5} p_{1j} |\lambda_j| = 1.6.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}, p_{25}$ be the weights of any vertex $v_i \in V_2$ in $\mathcal{G}_{18}$. Then, we have

$$\begin{aligned} p_{21} + p_{22} + p_{23} + p_{24} + p_{25} &= 1 \\ p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 + p_{25}\lambda_5 &= 0 \\ p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 + p_{25}\lambda_5^2 &= 3 \\ p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 + p_{25}\lambda_5^3 &= 4 \\ p_{21}\lambda_1^4 + p_{22}\lambda_2^4 + p_{23}\lambda_3^4 + p_{24}\lambda_4^4 + p_{25}\lambda_5^4 &= 17. \end{aligned}$$

Solving the system of equations, we get

$$p_{21} = 0.1, p_{22} = 0.583333, p_{23} = 0, p_{24} = 0.166667, p_{25} = 0.15.$$

Hence, the energy of each vertex $v_i \in V_2$ becomes

$$\mathcal{E}_{\mathcal{G}_{18}}(v_i) = \sum_{j=1}^{5} p_{2j} |\lambda_j| = 1.566667.$$

Case 3: Let $p_{31}, p_{32}, p_{33}, p_{34}, p_{35}$ be the weights for any vertex $v_i \in V_3$ in $\mathcal{G}_{18}$. Then, we have

$$\begin{aligned} p_{31} + p_{32} + p_{33} + p_{34} + p_{35} &= 1 \\ p_{31}\lambda_1 + p_{32}\lambda_2 + p_{33}\lambda_3 + p_{34}\lambda_4 + p_{35}\lambda_5 &= 0 \\ p_{31}\lambda_1^2 + p_{32}\lambda_2^2 + p_{33}\lambda_3^2 + p_{34}\lambda_4^2 + p_{35}\lambda_5^2 &= 2 \\ p_{31}\lambda_1^3 + p_{32}\lambda_2^3 + p_{33}\lambda_3^3 + p_{34}\lambda_4^3 + p_{35}\lambda_5^3 &= 2 \\ p_{31}\lambda_1^4 + p_{32}\lambda_2^4 + p_{33}\lambda_3^4 + p_{34}\lambda_4^4 + p_{35}\lambda_5^4 &= 10. \end{aligned}$$

Solving the system of equations, we get

$$p_{31} = 0.1, p_{32} = 0.333333, p_{33} = 0.333333, p_{34} = 0.166667, p_{35} = 0.066667.$$

Therefore, the energy of each vertex $v_i \in V_3$ is given by

$$\mathcal{E}_{\mathcal{G}_{18}}(v_i) = \sum_{j=1}^{5} p_{3j} |\lambda_j| = 1.066667.$$

□

Theorem 4.10 *For the graph $\mathcal{G}_{19}$ in Fig. 1, with vertex symmetries $\{v_1, v_2, v_3\}$ and $\{v_4, \cdots, v_7\}$,*

$$\mathcal{E}_{\mathcal{G}_{19}}(v) = \begin{cases} 1.904762 & \text{if } v \in \{v_1, v_2, v_3\}, \\ 1.571429 & \text{if otherwise.} \end{cases}$$

Proof: The characteristic polynomial of $\mathcal{G}_{19}$ is given by

$$P\left(\mathcal{G}_{19};\lambda\right) = (\lambda+2)\,(\lambda+1)^4\,(\lambda-1)\,(\lambda-5)\,.$$

As a result, the spectrum of $\mathcal{G}_{19}$ is $-2, -1^4, 1, 5$, indicating that the distinct eigenvalues of $\mathcal{G}_{19}$ are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1$ and $\lambda_4 = 5$.

By examining the vertex symmetries of the graph $\mathcal{G}_{19}$, we find that that there are two distinct vertex symmetry sets, namely $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, \cdots, v_7\}$, so that all the vertices within a set share the same energy. As the graph $\mathcal{G}_{19}$ has four distinct eigenvalues and two types of vertex symmetry, Lemma 2.1 implies that the energy of the vertices of $\mathcal{G}_{19}$ can be obtained by solving two 4×4 systems of equations, one corresponding to V_1 and another to V_2. These are obtained by equating the spectral moments with the corresponding walk counts, as detailed in the following two cases.

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}$ be the weights of any vertex $v_i \in V_1$ in $\mathcal{G}_{19}$. Then, we have

$$\begin{aligned}
p_{11} + p_{12} + p_{13} + p_{14} &= 1\\
p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 &= 0\\
p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 &= 6\\
p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 &= 22.
\end{aligned}$$

On solving the system of equations, we get

$$p_{11} = 0.142857, p_{12} = 0.666667, p_{13} = 0, p_{14} = 0.190476.$$

Thus, the energy of each vertex $v_i \in V_1$ will be

$$\mathcal{E}_{\mathcal{G}_{19}}(v_i) = \sum_{j=1}^{4} p_{1j}\,|\lambda_j| = 1.904762.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}$ be the weights of any vertex $v_i \in V_2$ in $\mathcal{G}_{19}$. Then, we have

$$\begin{aligned}
p_{21} + p_{22} + p_{23} + p_{24} &= 1\\
p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 &= 0\\
p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 &= 4\\
p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 &= 12.
\end{aligned}$$

Solving the system of equations, we get

$$p_{21} = 0.142857, p_{22} = 0.5, p_{23} = 0.25, p_{24} = 0.107143.$$

Therefore, the energy of each vertex $v_i \in V_2$ will be

$$\mathcal{E}_{\mathcal{G}_{19}}(v_i) = \sum_{j=1}^{4} p_{2j}\,|\lambda_j| = 1.571429.$$

□

5. Conclusion

Calculating the vertex energy of an arbitrary graph poses significant challenges, as the process involves solving the characteristic polynomial, determining the eigenvalues of the adjacency matrix and then solving systems of equations based on the number of distinct eigenvalues and symmetries in it. The complexity increases specially when graphs spectra are non-integral. In this context, our focus has been

Table 1: Vertex energies of all non-trivial connected integral graphs of order up to seven

Sl. No.	n	*Name of the graph*	*Number of vertex symmetries*	*Vertex Energies*	
1	2	$\mathcal{G}_1$	1	1	
2	3	$\mathcal{G}_2$	1	1.333333	
3	3	$\mathcal{G}_3$	1	1	
4	4	$\mathcal{G}_4$	1	1.5	
5	5	$\mathcal{G}_5$	2	$\{v_1\}$	2
				$\{v_2, \cdots, v_5\}$	0.5
6	5	$\mathcal{G}_6$	2	$\{v_1, v_2, v_3\}$	0.8
				$\{v_4, v_5\}$	1.8
7	5	$\mathcal{G}_7$	1	1.6	
8	6	$\mathcal{G}_8$	2	$\{v_1, v_2\}$	1.666667
				$\{v_3, \cdots, v_6\}$	0.666667
9	6	$\mathcal{G}_9$	1	1.333333	
10	6	$\mathcal{G}_{10}$	1	1	
11	6	$\mathcal{G}_{11}$	1	1.333333	
12	6	$\mathcal{G}_{12}$	1	1.333333	
13	6	$\mathcal{G}_{13}$	1	1.666667	
14	7	$\mathcal{G}_{14}$	3	$\{v_1\}$	1.5
				$\{v_2, v_3, v_4\}$	1.333333
				$\{v_5, v_6, v_7\}$	0.833333
15	7	$\mathcal{G}_{15}$	2	$\{v_1\}$	2.4
				$\{v_2, \cdots, v_7\}$	1.266667
16	7	$\mathcal{G}_{16}$	3	$\{v_1\}$	1.6
				$\{v_2, v_3\}$	1.066667
				$\{v_4, \cdots, v_7\}$	1.566667
17	7	$\mathcal{G}_{17}$	2	$\{v_1, \cdots, v_4\}$	1.642857
				$\{v_5, v_6, v_7\}$	1.142857
18	7	$\mathcal{G}_{18}$	3	$\{v_1\}$	1.6
				$\{v_2, \cdots, v_5\}$	1.566667
				$\{v_6, v_7\}$	1.066667
19	7	$\mathcal{G}_{19}$	2	$\{v_1, v_2, v_3\}$	1.904762
				$\{v_4, \cdots, v_7\}$	1.571429
20	7	$\mathcal{G}_{20}$	1	1.714286	

restricted to integral graphs, where the eigenvalues of the adjacency matrix are integers. This restriction has not only simplified the computational challenges but also has provided a structured framework for analyzing the distribution of energy across individual vertices. In particular, we have determined the vertex energies of all non-trivial connected integral graphs of order up to 7 as enlisted in Table 1.

This work opens several promising directions for future research. In particular, the computation of vertex energy can be extended to specific classes of graphs such as k-partite graphs, trees, Cayley graphs, powers of graphs and distance-regular graphs, each offering unique structural characteristics that may yield new insights into vertex energy distribution. Also, future research could explore the potential relationships between vertex energy and intrinsic properties of vertices. By identifying and analyzing these correlations, we can better understand how energy is distributed within a graph and assess the relative importance of individual vertices. Such investigations could also contribute to the discovery of new graph-theoretic invariants in spectral graph theory, or enhance existing ones by incorporating vertex energy into their analysis.

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