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Existence and Uniqueness of Solutions for Loaded Mixed-Type Equation with Fractional Integral Operators *

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ABSTRACT: This paper investigates a boundary value problem for a second-order integro-differential equation of mixed parabolic-hyperbolic type with variable coefficients and fractional loading. The main focus is on establishing the existence and uniqueness of a regular solution under integral gluing conditions imposed on the line of type change. The method of integral equations is employed to study the solvability of the problem. Sufficient conditions for unique solvability are formulated and proven. The results contribute to the theory of mixed-type equations with nonlocal and fractional conditions, which are relevant in various physical models involving memory and hereditary effects.

Key Words: Boundary value problems, integro-differential equations, parabolic-hyperbolic equations, integral gluing condition, existence and uniqueness, loaded equations, Riemann-Liouville fractional derivatives.

Contents

1. Introduction and Formulation of the Problem

Integral and nonlocal conditions, including integral gluing conditions, play a fundamental role in the mathematical modeling of various physical and engineering processes. These conditions help ensure continuity and compatibility of solutions across interfaces and transitions - particularly in the context of mixed-type equations. Applications are vast: in thermoelasticity, such conditions capture heat and stress distributions in deformable bodies; in chemical and environmental systems, they govern diffusive-reaction processes; and in biological models, they regulate spatial-temporal dynamics of interacting populations [1,2,3].

The foundational concept of gluing conditions was introduced by Tricomi in 1957, laying the groundwork for later studies. Building on this, the monograph [6] expanded the framework for differential equations with gluing and integral transmission conditions. Further analytical developments were made in [7], focusing on fractional and nonlocal boundary constraints. Analytical tools for equations involving both integral and fractional operators were explored in [8], supported by the classical framework of fractional calculus presented in [14].

In recent years, significant efforts have been directed toward studying boundary value problems that combine fractional operators with integral or nonlocal gluing conditions. A variety of approaches - including integral equation methods and functional analytic techniques - have been applied to such problems. For example, fractional Caputo-type derivatives were incorporated into mixed-type models in

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[15], while nonlocal and inverse problems were treated in [16,17,18]. The Volterra-type solvability of such systems has been addressed in [19], and solvability results for integral gluing conditions were elaborated in [20,21,22,23].

Broader analytical and numerical investigations of fractional differential equations have gained momentum. The monograph [28] offers an in-depth theoretical foundation for fractional-order equations, while the book by Feehan and Pop [25] presents a comprehensive treatment of control and analysis methods for fractional parabolic and elliptic equations. The recent work [24] studies nonlocal problems involving fractional derivatives with general boundary conditions, contributing novel tools for operator-theoretic analysis. Numerical aspects are also increasingly studied, as seen in [26], which focuses on inverse problems in fractional diffusion, and in the overview by Yang and Yamamoto [27], which outlines inverse problem methodologies for fractional dynamics.

From an application perspective, recent works have shown the relevance of fractional and integral gluing conditions in diverse models - from electrochemical systems [31] and hybrid control frameworks [32] to three-point boundary value problems [30] and diffusion in complex materials [29]. Foundational studies also address elliptic and parabolic-elliptic systems with nonclassical boundary interactions [5,4].

Nevertheless, problems that combine fractional loading, variable coefficients, and integral gluing conditions in parabolic-hyperbolic equations remain insufficiently explored. Addressing this gap is essential for modeling physical processes that exhibit memory, heredity, or anomalous transport - particularly in viscoelastic media, thermal systems, and diffusion in heterogeneous materials [29,33,34,35,36,37,38,39,40,41,42].

Motivated by these considerations, the present work aims to establish the existence and uniqueness of a regular solution to a boundary value problem for a parabolic-hyperbolic integro-differential equation with variable coefficients and integral gluing conditions. By employing the method of integral equations, we contribute to the theoretical framework for such problems, thereby extending the results presented in existing works such as [15], [16], and [29]

Consider a boundary value problem for a loaded second-order integro-differential equation of parabolic-hyperbolic type. Let D be a domain bounded by segments $y = 0$, $x = 1$, $y = 1$ and $x = -1$. We introduce the following notations

$$D_1 = D \cap \{x > 0\}, \;\; D_2 = D \cap \{x < 0\}, \; I = \{(x, y) : x = 0, 0 < y < 1\},$$

$$\gamma_1 = \{(x, y) : -1 < x < 0, \, y = 0\}, \qquad \gamma_2 = \{(x, y) : 0 < y < 1, \, x = 1\},$$

$$D = D_1 \cup D_2 \cup I.$$

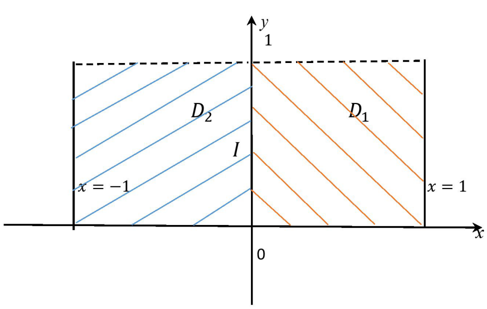

Figure 1: The domain D bounded by $x = -1$, $x = 1$, $y = 0$, and $y = 1$, divided into D_1, D_2, and I.

We consider the following linear loaded integro-differential equation:

$$0 = \begin{cases} u_{xx} + a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)u - \sum\limits_{i=1}^{n} d_i(x,y)D_{0y}^{\alpha_i}u(0,y), & \text{in } D_1, \\ u_{xx} - u_{yy} + a_2(x,y)u_x + b_2(x,y)u_y + c_2(x,y)u - \sum\limits_{i=1}^{n} e_i(x,y)D_{0y}^{\beta_i}u(0,y), & \text{in } D_2, \end{cases} \tag{1.1}$$

where the functions $a_i(x,y)$, $b_i(x,y)$, $c_i(x,y)$ $(i = 1,2)$ are sufficiently smooth. Assume that $b_1(x,y) < 0$ in $\overline{D}_1$, and that a_1, b_1, c_1, and their first-order derivatives satisfy the Holder condition in D_1; a_2, $b_2 \in C^2(\overline{D}_2)$, $c_2 \in C^1(\overline{D}_2)$, and

$$d_i(x,y) \in C^1(\overline{D}_1) \cap C^2(D_1), \quad e_i(x,y) \in C^1(\overline{D}_2) \cap C^3(D_2), \quad i = 1,\dots,n.$$

Here, $D_{0y}^{\alpha_i}$ and $D_{0y}^{\beta_i}$ are Riemann-Liouville fractional integral operators of order $\alpha_i < 0$ and $\beta_i < 0$, respectively. If $f(y) \in L(a,b)$ with $a < b < +\infty$, then

$$D_{ay}^{\alpha_i} f(y) = \frac{\operatorname{sgn}(y-a)}{\Gamma(-\alpha_i)} \int_a^y \frac{f(t)}{(y-t)^{1+\alpha_i}}\, dt, \quad y \in (a,b).$$

The function $D_{ay}^{-\alpha_i} f(y)$ is defined almost everywhere on (a,b) and belongs to $L(a,b)$. We also adopt the convention: $D_{ay}^{0} f(y) = f(y), \quad \text{for } \alpha_i = 0$.

We will consider the following problem with special integral gluing conditions for the loaded integro-differential equation (1.1) with variable coefficients.

Problem 1. Find a function $u(x,y)$ such that:
1) $u(x,y) \in C(\overline{D_k}) \cap C^1(D_k \cup I \cup \gamma_k)$ for $k = 1,2$;
2) $u(x,y)$ is a regular solution of equation (1.1) in D_k, $k = 1,2$;
3) The following gluing conditions are satisfied on I:

$$\left.\begin{aligned} \tau_1(y) &= \mu(y)\tau_2(y) + \sigma(y), \\ \nu_1(y) &= \int\limits_0^y \gamma(y,\eta)\nu_2(\eta)d\eta + \delta(y)\nu_2(y) + \xi(y)\tau_2(y) + \theta(y), \end{aligned}\right\} \tag{1.2}$$

where $\tau_1(y) = u(+0,y)$, $\nu_1(y) = u_x(+0,y)$, $\tau_2(y) = u(-0,y)$, $\nu_2(y) = u_x(-0,y)$;

4) The boundary conditions are:

$$u(-1,y) = \varphi_1(y), \quad u_x(1,y) = \varphi_2(y), \quad 0 \le y \le 1, \tag{1.3}$$

$$u(x,0) = \psi_1(x), \quad u_y(x,0) = 0, \quad -1 \le x \le 0, \tag{1.4}$$

$$u(x,0) = \psi_2(x), \quad 0 \le x \le 1. \tag{1.5}$$

Here, $\varphi_1(y), \varphi_2(y), \psi_1(x), \psi_2(x), \mu(y), \sigma(y), \delta(y), \gamma_y(y,\eta), \xi(y), \theta(y)$ are given functions, with $\mu(y) \neq 0$, and

$$\gamma^2(y,\eta) + \delta^2(y) \neq 0.$$

Additionally, we impose the consistency conditions:

$$\varphi_1(0) = \psi_1(-1), \quad \varphi_2(0) = \psi_2'(1), \quad \psi_2(0) = \mu(0)\psi_1(0) + \sigma(0).$$

It is worth noting that the gluing conditions considered here are of a more general nonlocal type involving integral operators. In the case of continuous gluing, the problem reduces to a special case of our formulation, which also yields a new result within the fractional framework. We are now in a position to state our main result.

Theorem 1.1 *If the functions* $\psi_1''(x)$, $\varphi_1''(y)$, $\psi_2''(x)$, $\varphi_2(y)$, $c_1(x,y)$, $\mu'(y)$, $\sigma'(y)$, $\delta'(y)$, $\gamma_y(y,\eta)$, $\xi'(y)$, *and* $\theta'(y)$ *are Hölder continuous;* $\varphi_1'(0)=0$*;* $c_1(x,y)\le 0$*;* $b_1(x,y)=-1$*; and the following compatibility conditions hold:*

$$\begin{cases}\delta(0)\psi_1'(0)+\big(\xi(0)+(a_2(0,0)+\frac{1}{2}b_2(0,0))\delta(0)\big)\,\psi_1(0)+\theta(0)=0, & \text{if } \delta(y)\neq 0,\\ \xi(0)\psi_1(0)+\theta(0)=0, & \text{if } \delta(y)\equiv 0,\end{cases} \tag{1.6}$$

then Problem 1 has a unique solution.

This article is structured as follows. The first section introduces the problem and outlines the foundational concepts necessary for its formulation. Sections 2 and 3 are devoted to a detailed analysis of the problem within the hyperbolic and parabolic subregions of the domain Ω, denoted by Ω_1 and Ω_2, respectively. In each subregion, we derive the core integro-differential relations and examine how these extend to the interface separating the two types-specifically, the line segment AB.

In Section 4, we incorporate nonstandard gluing conditions to formulate a system of integro-differential equations along the interface where the integral discontinuity occurs. These Volterra-type equations [43] are crucial for proving the existence and uniqueness of solutions to the overall problem.

2. Basic Functional Relations Derived from the Domain D_2

In this section, we consider the following hyperbolic partial differential equation with a fractional source term defined for $x\in\mathbb{R}$ and $y>0$:

$$A_1u\equiv u_{xx}-u_{yy}+a_2(x,y)u_x+b_2(x,y)u_y+c_2(x,y)u=\sum_{i=1}^{n}e_i(x,y)D_{0y}^{\beta_i}u(0,y). \tag{2.1}$$

with the initial conditions

$$u(x,0)=\psi_1(x),\qquad u_y(x,0)=0. \tag{2.2}$$

Theorem 2.1 (Duhamel's Principle for the Hyperbolic Equation) *The solution* $u(x,y)$ *of the initial problem* (2.1)–(2.2) *for the fractional hyperbolic partial differential equation can be represented as*

$$u(x,y)=v(x,y)+\int_0^y w(x,y;\tau)\,d\tau, \tag{2.3}$$

where

1. $v(x,y)$ *is the solution of the homogeneous problem*

$$A_1v(x,y)\equiv v_{xx}-v_{yy}+a_2(x,y)v_x+b_2(x,y)v_y+c_2(x,y)v=0, \tag{2.4}$$

with

$$v(x,0)=\psi_1(x),\quad v_y(x,0)=0; \tag{2.5}$$

2. $w(x,y;\tau)$ *satisfies, for each fixed* $\tau\in[0,y]$,

$$\begin{cases}w_{xx}-w_{yy}+a_2(x,y)w_x+b_2(x,y)w_y+c_2(x,y)w=0, \quad y>\tau,\\ w(x,y;\tau)\big|_{y=\tau}=0,\\ w_y(x,y;\tau)\big|_{y=\tau}=\displaystyle\sum_{i=1}^{n}e_i(x,\tau)\,D_{0\tau}^{\beta_i}u(0,\tau).\end{cases} \tag{2.6}$$

Proof: [Proof of Theorem 2.1] The method of Duhamel involves treating the nonhomogeneous term as a continuous superposition of point sources along the y-axis. To prove the representation (2.3), we verify that the constructed solution $u(x,y)$ satisfies this form. Duhamel's principle allows us to interpret the right-hand side of the equation as a superposition of instantaneous source effects acting at each time

τ. Given the initial problem (2.1)–(2.2), our goal is to demonstrate that the solution can be written in the form (2.3), where $v(x, y)$ solves the associated homogeneous equation (i.e., with zero source term), and $w(x, y; \tau)$ solves the homogeneous equation in the region $y > \tau$, exhibiting a discontinuity in its y-derivative at $y = \tau$, determined by the inhomogeneous term.

Step 1: Construction of the homogeneous solution. Let $v(x, y)$ be the solution to the homogeneous problem, satisfying the same differential operator as the original equation but without the source term, and preserving the original initial and boundary conditions. Since the PDE is linear and, under suitable regularity assumptions, such a solution exists locally for $y > 0$.

Step 2: Construction of the particular solution. We define the auxiliary function $w(x, y; \tau)$, which solves the homogeneous equation for $y > \tau$, with initial data at $y = \tau$ that reflects the contribution of the source term at time τ. Notably, $w(x, y; \tau)$ has a discontinuous derivative at $y = \tau$, modeling the instantaneous effect of the source.

Step 3: Superposition. Due to the linearity of the equation and the continuity of the source term with respect to y, we integrate the effect of each point source over the interval $[0, y]$. According to Duhamel's principle, the solution to the original nonhomogeneous problem (2.1)–(2.2) is given by

$$u(x, y) = v(x, y) + \int_0^y w(x, y; \tau)\, d\tau.$$

This integral accumulates the effect of each instantaneous source at time τ, represented by the derivative jump in w at $y = \tau$.

We now verify that $u(x, y)$ satisfies the initial conditions and the full equation.

Differentiate $u(x, y)$ under the integral:

$$u_{xx}(x, y) = v_{xx}(x, y) + \int_0^y w_{xx}(x, y; \tau)\, d\tau,$$

$$u_{yy}(x, y) = v_{yy}(x, y) + \int_0^y \big(w_{yy}(x, y; \tau) + w_y(x, y; y)\big)\, d\tau.$$

(Similarly for other derivatives, if needed.)

Applying the operator A_1 to $u(x, y)$ and using Leibniz's rule for differentiation under the integral sign, we obtain

$$A_1 u = A_1 v + \int_0^y A_1 w(x, y; \tau)\, d\tau + \left.\frac{\partial w(x, y; \tau)}{\partial y}\right|_{\tau=y}.$$

Since $w(x, y; y) = 0$, the correction term simplifies, and from the initial condition on w_y, we obtain

$$u_{xx} - u_{yy} + a_2(x, y)u_x + \cdots = \int_0^y \sum_{i=1}^n e_i(x, \tau) D_{0\tau}^{\beta_i} u(0, \tau)\delta(y - \tau)\, d\tau = \sum_{i=1}^n e_i(x, y) D_{0y}^{\beta_i} u(0, y),$$

which matches exactly the right-hand side of equation (2.1).

Therefore, $u(x, y)$ satisfies the full equation. From the construction of $v(x, y)$, we also have the initial conditions:

$$u(x, 0) = v(x, 0) + \left.\int_0^y w(x, y; \tau)\, d\tau\right|_{y=0} = \psi_1(x),$$

and

$$u_y(x, 0) = v_y(x, 0) + \frac{\partial}{\partial y} \left.\int_0^y w(x, y; \tau)\, d\tau\right|_{y=0} = 0,$$

because the integral vanishes at $y = 0$.

Hence, the function

$$u(x, y) = v(x, y) + \int_0^y w(x, y; \tau)\, d\tau$$

satisfies both the initial conditions and the nonhomogeneous equation (2.1), completing the proof. □

Consequently, we consider the following solution to the problem, satisfying equation $A_1 v(x,y)=0$, with homogeneous conditions (2.5), which can be written in the form

$$v(x,y)=\frac{1}{2}\left[R(x,y;x+y,0)\psi_1(x+y)+R(x,y;x-y,0)\psi_1(x-y)\right]-$$
$$-\frac{1}{2}\int\limits_{x-y}^{x+y}\left[R_\eta(x,y;\xi,0)+b_2(\xi,0)R(x,y;\xi,0)\right]\psi_1(\xi)\,d\xi, \tag{2.7}$$

where $R(x,y;\xi,\eta)$ is a Riemann function [44].

Therefore, according to Duhamel's principle, if the function $W(x,y,\tau)$ of variables $x\in\mathbb{R},\ y\in\mathbb{R},\ \tau\in\mathbb{R}$ is a smooth solution to the following problem:

$$W_{xx}-W_{yy}+a_2(x,y)W_x(x,y,\tau)+b_2(x,y)W_y(x,y,\tau)+c_2(x,y)W(x,y,\tau)=0,\quad \tau>0, \tag{2.8}$$

$$W(x,y,\tau)\big|_{y=\tau}=0,\qquad W_y(x,y,\tau)\big|_{y=\tau}=\sum_{i=1}^{n}e_i(x,\tau)D_{0\tau}^{\beta_i}u(0,\tau), \tag{2.9}$$

then the Duhamel integral, given by the equality

$$v_1(x,y)=\int\limits_0^y W(x,y-\tau,\tau)\,d\tau,$$

is a solution to the Cauchy problem (2.8)–(2.9) for the loaded integro-differential equation.

Therefore, if we take the substitution $y-\tau=s$, and also use the Cauchy solutions similar to (2.7), we obtain

$$v_1(x,y)=\int\limits_0^y\int\limits_{x-y+\eta}^{x+y-\eta}R(x,y;\xi,0)\sum_{i=1}^{n}e_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta)\,d\xi\,d\eta. \tag{2.10}$$

The obtained expression (2.10) represents the solution of the nonhomogeneous part of equation (2.1). Thus, the solution of the Cauchy problem (2.1)–(2.2) can be written as follows:

$$u(x,y)=\frac{1}{2}\left[R(x,y;x+y,0)\psi_1(x+y)+R(x,y;x-y,0)\psi_1(x-y)\right]-$$
$$-\frac{1}{2}\int\limits_{x-y}^{x+y}\left[R_\eta(x,y;\xi,0)+b_2(\xi,0)R(x,y;\xi,0)\right]\psi_1(\xi)\,d\xi+$$
$$+\frac{1}{2}\int\limits_0^y\int\limits_{x-y+\eta}^{x+y-\eta}R(x,y;\xi,0)E_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta)\,d\xi\,d\eta, \tag{2.11}$$

where $E_i(\xi,\eta)$ depends on $e_i(\xi,\eta)$, and the repeated index i implies summation over $i=1,2,\dots,n$.

$$E_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta)=\sum_{i=1}^{n}e_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta),$$

that is, by the repeating index i we mean the sum from $i=1$ to n.

Direct substitution of expression (2.11) into equation (2.1), taking into account the initial conditions (2.2), as well as equation (2.4) and conditions (2.5), shows that (2.11) is indeed a solution to the problem (2.1)–(2.2).

We apply the condition $u(-0,y)=\tau_2(y)$ to the resulting solution (2.11):

$$2\tau_2(y)=[R(0,y;y,0)\psi_1(y)+R(0,y;-y,0)\psi_1(-y)]-$$

$$-\int_{-y}^{0}[R_\eta(0,y;\xi,0)+b_2(\xi,0)R(0,y;\xi,0)]\,\psi_1(\xi)\,d\xi-$$

$$-\int_{0}^{y}[R_\eta(0,y;\xi,0)+b_2(\xi,0)R(0,y;\xi,0)]\,\psi_1(\xi)\,d\xi+\int_0^y d\eta\int_{-y+\eta}^{y-\eta}R(0,y;\xi,0)E_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta)\,d\xi.$$

As a result, since the function $\psi_1(y)$ is known on $[-1,0]$, but unknown on the interval $[0,1]$, we separate the unknown part and obtain:

$$\psi_1(y)-\int_{y}^{0}[R_\eta(0,y;\xi,0)+b_2(\xi,0)R(0,y;\xi,0)]\,\psi_1(\xi)\,d\xi-$$

$$-\int_{-y}^{0}[R_\eta(0,y;\xi,0)+b_2(\xi,0)R(0,y;\xi,0)]\,\psi_1(\xi)\,d\xi+$$

$$+\int_0^y d\eta\int_{x-y+\eta}^{x+y-\eta}R(0,y;\xi,0)E_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta)\,d\xi+R(0,y;-y,0)\psi_1(-y)=2\tau_2(y).$$

Hence, we obtain the following integral equation [43] for determining the function $\psi_1(y)$ on the unknown interval $(0,1)$:

$$\psi_1(y)-\int_0^y Q(0,y,\xi)\psi_1(\xi)\,d\xi=2\tau_2(y)-R(0,y;-y,0)\psi_1(-y)+$$

$$+\int_{-y}^{0}Q(0,y,\xi)\psi_1(\xi)\,d\xi-\int_0^y D_{0\eta}^{\beta_i}\tau_2(\eta)\,d\eta\int_{x-y+\eta}^{x+y-\eta}R(0,y;\xi,0)E_i(\xi,\eta)\,d\xi,\qquad(2.12)$$

where

$$Q(0,y,\xi)=R_\eta\,(0,y;\xi,0)+b_2(\xi,0)R\,(0,y;\xi,0)\,.$$

Taking into account the properties of the fractional integral operator

$$\int_0^y D_{0\eta}^{\beta_i}\tau_2(\eta)d\eta\int_{x-y+\eta}^{x+y-\eta}E_i(\xi,\eta)R\,(0,y;\xi,0)d\xi=$$

$$=\frac{1}{\Gamma(-\beta_i)}\int_0^y\tau_2(s)ds\int_s^y(\eta-s)^{-1-\beta_i}d\eta\int_{x-y+\eta}^{x+y-\eta}E_i(\xi,\eta)R\,(0,y;\xi,0)d\eta,$$

substitution in our equation (2.12), then, after several substitutions from the relation, we obtain the following equation for $0\le y\le 1$:

$$\psi_1(y)-\int_0^y Q(0,y,\eta)\psi_1(\eta)\,d\eta=2\tau_2(y)-\int_0^y H(y,\eta)\tau_2(\eta)\,d\eta-g_1(y),\qquad(2.13)$$

where

$$H(y,\eta)=\frac{1}{\Gamma(-\beta_i)}\int\limits_{\eta}^{y}(s-\eta)^{-1-\beta_i}\,ds\int\limits_{-y+s}^{y-s}E_i(\xi,s)R(0,y;\xi,0)\,d\xi,$$

$$g_1(y)=R(0,y;-y,0)\psi_1(-y)-\int\limits_{-y}^{0}Q(0,y,\eta)\psi_1(\eta)\,d\eta.$$

Assuming that the function on the right-hand side of equation (2.13) is known for the time being, and denoting the resolvent of the kernel of equation (2.13) by $Z_1(y,s)$, we can express the solution of the equation in the following form:

$$\psi_1(y)=2\tau_2(y)+\int_0^y X_1(y,\eta)\tau_2(\eta)\,d\eta+g_1^*(y),\qquad 0\le y\le 1,$$

where

$$X_1(y,\eta)=2Z_1(y,\eta)+H(y,\eta)+\int\limits_{\eta}^{y}Z_1(y,\xi)H(y,\xi)\,d\xi,$$

$$\overline{g}_1(y)=g_1(y)+\int\limits_{0}^{y}Z_1(y,\eta)g_1(\eta)\,d\eta.$$

Thus, the function $\psi_1(y)$, as it appears in expression (2.11), can be extended over the entire segment $[-1,1]$ in the following form:

$$\tilde{\psi}_1(y)=\begin{cases}2\tau_2(y)+\int\limits_{0}^{y}Z_1(y,\eta)\tau_2(\eta)\,d\eta+\overline{g}_1(y), & 0\le y\le 1,\\ \psi_1(y), & -1\le y\le 0.\end{cases}\tag{2.14}$$

Proceeding in this manner, the function $\tilde{\psi}_1(y)$ can be extended over the entire real axis, $(-\infty,+\infty)$, as shown in [29]. Accordingly, the solution of equation (2.1), subject to the initial conditions

$$u(x,0)=\tilde{\psi}_1(x),\qquad u_y(x,0)=0,\qquad -1\le x\le 0,$$

is defined as follows:

$$\begin{aligned}u(x,y)=\tfrac{1}{2}\Big[R(x,y;x+y,0)\tilde{\psi}_1(x+y)+R(x,y;x-y,0)\tilde{\psi}_1(x-y)\Big]-\\ \tfrac{1}{2}\int\limits_{x-y}^{x+y}\left[R_\eta(x,y;\xi,0)+b_2(\xi,0)R(x,y;\xi,0)\right]\tilde{\psi}_1(\xi)\,d\xi+\\ +\tfrac{1}{2}\int\limits_{0}^{y}\int\limits_{x-y+\eta}^{x+y-\eta}R(x,y;\xi,0)E_i(\xi,\eta)D_{0\eta}^{\beta_i}\tau_2(\eta)\,d\xi\,d\eta.\end{aligned}\tag{2.15}$$

Now, applying the condition $u_x(-0,y)=\nu_2(y)$ to the solution (2.15), we obtain:

$$\begin{aligned}\nu_2(y)=\tau_2'(y)+\tau_2(y)\left[R_x(0,y;y,0)-R_\eta(0,y;y,0)-b_2(y,0)\right]+\\ +\int\limits_{0}^{y}T(y,s)\,\tau_2(s)\,ds+g_{11}(y),\qquad 0<y<1.\end{aligned}\tag{2.16}$$

where

$$T(y,s) = \tfrac{1}{2}\left[R_x(0,y;y,0)X_1(y,s) + X'_{1x}(y,s) - X_1(y,s)\right] +$$

$$+\int\limits_s^y \frac{(\eta-s)^{-1-\beta_i}}{2\Gamma(-\beta_i)}\left[R(0,y;y-\eta,0)E_i(y-\eta,\eta) - R(0,y;-y+\eta,0)E_i(-y+\eta,\eta)\right]d\eta +$$

$$+\frac{1}{2\Gamma(-\beta_i)}\int\limits_s^y (\eta-s)^{-1-\beta_i}\,d\eta \int\limits_{-y+\eta}^{y-\eta} R_x(0,y;\xi,0)E_i(\xi,\eta)\,d\xi - \left(2Q_x(0,y,s) + \int\limits_s^y Q_x(0,y,\xi)X_1(\xi,s)\,d\xi\right),$$

$$g_{11}(y) = \tfrac{1}{2}\left[R_x(0,y;y,0)\,\overline{g}_1(y) + \overline{g}'_1(y) + R_x(0,y;y,0)\psi_1(y) + R(0,y;-y,0)\psi'_1(y)\right.$$
$$\left.-\big(R_\eta(0,y;y,0)+b_2(y,0)\big)\overline{g}_1(y) + \big(R_\eta(0,y;y,0)+b_2(y,0)R(0,y;-y,0)\big)\psi_1(y)\right]$$
$$-\int\limits_{-y}^{0} Q_x(0,y,\xi)\psi_1(\xi)\,d\xi - \int\limits_0^y Q_x(0,y,\xi)\overline{g}_1(\xi)\,d\xi.$$

Thus, we have derived the first principal integro-differential relation (2.16), corresponding to the hyperbolic region D_2 over the interval I.

3. Basic Functional Relations Derived from the Domain D_1

At this stage, we begin the analysis of the problem within the parabolic part of the domain

$$D_1 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\}.$$

We consider the following fractional parabolic equation:

$$u_{xx} + a_1(x,y)\,u_x + b_1(x,y)\,u_y + c_1(x,y)\,u = \sum_{i=1}^{n} d_i(x,y)\,D_{0y}^{\alpha_i}u(0,y), \quad (x,y)\in D_1, \tag{3.1}$$

subject to the initial and boundary conditions:

$$u(0,y) = \tau_1(y), \qquad u_x(1,y) = \varphi_2(y), \qquad u(x,0) = \psi_2(x), \tag{3.2}$$

where $\alpha_i < 0$, and $\tau_1(y)$ is assumed to be an unknown function to be determined. Additionally, we take $b_1(x,y) = -1$ throughout the domain.

Due to the linearity of equation (3.1), and in view of the superposition principle, we split the solution into two parts:

$$u(x,y) = u^{(h)}(x,y) + u^{(\tau)}(x,y), \tag{3.3}$$

where $u^{(h)}(x,y)$ denotes the solution to the homogeneous problem (i.e., with vanishing right-hand side) subject to the prescribed boundary and initial conditions, and $u^{(\tau)}(x,y)$ denotes the particular solution of the nonhomogeneous problem, where the source term involves the unknown function $\tau_1(y)$ and satisfies homogeneous boundary conditions.

Consequently, we will have

$$u_x(0,y) = u_x^{(h)}(0,y) + u_x^{(\tau)}(0,y), \tag{3.4}$$

which makes it possible to apply the condition $u_x(0,y) = \nu_1(y)$ to the obtained solution of the problem (3.1)–(3.2).

Thus, we proceed to identify the representation of the nonhomogeneous solution. Assume that the Green's function $U(x,\xi;t)$ represents the fundamental solution of the homogeneous forward parabolic operator [44]. Then, the particular solution $u^{(\tau)}$ can be expressed, by Duhamel's principle, as

$$u^{(\tau)}(x,y) = \int_0^y \int_0^1 U(x,\xi;y-s)\,f(\xi,s)\,d\xi\,ds, \tag{3.5}$$

where the nonhomogeneous source term is given by

$$f(x,y)=\sum_{i=1}^{n}\frac{d_i(x,y)}{\Gamma(-\alpha_i)}\int_0^y (y-s)^{-\alpha_i-1}\,\tau_1(s)\,ds.$$

Differentiating with respect to x at $x=0$ gives

$$\begin{aligned}u_x^{(\tau)}(0,y)&=\int_0^y\int_0^1\frac{\partial U}{\partial x}(0,\xi;y-s)\,f(\xi,s)\,d\xi\,ds\\&=\sum_{i=1}^{n}\int_0^y\int_0^1\frac{\partial U}{\partial x}(0,\xi;y-s)\cdot\frac{d_i(\xi,s)}{\Gamma(-\alpha_i)}\int_0^s(s-\eta)^{-\alpha_i-1}\,\tau_1(\eta)\,d\eta\,d\xi\,ds.\end{aligned}$$

Changing the order of integration, we explicitly identify the kernel:

$$u_x^{(\tau)}(0,y)=\sum_{i=1}^{n}\int_0^y\tau_1(\eta)\left[\int_\eta^y\int_0^1\frac{\partial U}{\partial x}(0,\xi;y-s)\cdot\frac{d_i(\xi,s)}{\Gamma(-\alpha_i)}(s-\eta)^{-\alpha_i-1}\,d\xi\,ds\right]d\eta. \tag{3.5}$$

Solutions of the homogeneous problem (i.e., with vanishing right-hand side), subject to the prescribed boundary and initial conditions (3.2), at $b_1(x,y)=-1$, can be written in the form

$$u^{(h)}(x,y)=\int_0^1\psi_2(\xi)\,G(x,y;\xi,0)\,d\xi+\int_0^y\tau_1(\eta)\,G_\xi(x,y;0,\eta)\,d\eta.$$

$$u^{(h)}(x,y)=\int_0^1\psi_2(\xi)\,G(x,y;\xi,0)\,d\xi+\int_0^y\tau_1(\eta)\,G_\xi(x,y;0,\eta)\,d\eta+$$

$$\int_0^y\varphi_2(\eta)\,G(x,y;1,\eta)\,d\eta-\int_0^y\int_0^1 G(x,y;\xi,\eta)\,V(\xi,\eta)\,d\xi\,d\eta. \tag{3.6}$$

Here, $G(x,y;\xi,\eta)$ is the Green's function of the mixed problem for the heat equation [35], and the function $V(x,y)$ is the solution to the following integral equation:

$$V(x,y)-\int_0^y\int_0^1\Xi(x,y;\xi,\eta)\,V(\xi,\eta)\,d\xi\,d\eta=$$

$$p_1(x,y)+\int_0^y L_1(x,y;\eta)\,\tau_1'(\eta)\,d\eta-\int_0^y\Xi(x,y;1,\eta)\,\varphi_2(\eta)\,d\eta. \tag{3.7}$$

Here,

$$\Xi(x,y;\xi,\eta)=a_1(x,y)\,G_x(x,y;\xi,\eta)+c_1(x,y)\,G(x,y;\xi,\eta),$$

$$L_1(x,y;\eta)=a_1(x,y)\,N(x,y;0,\eta)-\int_\eta^y c_1(x,y)\,G_\xi(x,y;0,t)\,dt,$$

where $p_1(x,y)$ is a continuously differentiable function depending on the given data, and

$$N(x,y;0,\eta)=\frac{1}{4\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-4n)^2}{4(y-\eta)}\right]-\exp\left[-\frac{(x-2-4n)^2}{4(y-\eta)}\right]\right\}.$$

Equation (3.7) is a Volterra-type integral equation in two variables. Taking into account the properties of the functions $a_1(x,y)$ and $c_1(x,y)$, along with Theorem 1.1, we can readily verify that the solution has the following form:

$$V(x,y)=\tilde{p}_1(x,y)+\int_0^y\bar{L}_1(x,y;t)\,\tau_1'(t)\,dt. \tag{3.8}$$

where

$$\tilde{p}_1(x,y)=p_1(x,y)-\int_0^y\left(\Xi_1(x,y;\eta)\,\varphi_2(\eta)-\int_0^1\widetilde{\Xi}(x,y;\xi,\eta)\,p_1(\xi,\eta)\,d\xi\right)d\eta,$$

$$\bar{L}_1(x,y;t) = L_1(x,y;t) + \int_0^y \int_0^1 \widetilde{\Xi}(x,y;\xi,\eta)\, L_1(\xi,\eta;t)\, d\xi\, d\eta,$$

$$\Xi_1(x,y;t) = \Xi(x,y;t) + \int_0^y \int_0^1 \widetilde{\Xi}(x,y;\xi,\eta)\, \Xi(\xi,\eta;t)\, d\xi\, d\eta,$$

where $\widetilde{\Xi}$ is the resolvent of the kernel $\Xi(x,y;\xi,\eta)$. Substituting this result into expression (3.6), we obtain the following representation:

$$u^{(h)}(x,y) = \int_0^1 \psi_2(\xi)\, G(x,y;\xi,0)\, d\xi + \int_0^y \varphi_2(\eta)\, G(x,y;1,\eta)\, d\eta - \int_0^y \int_0^1 G(x,y;\xi,\eta)\, \tilde{p}_1(x,y)\, d\xi\, d\eta$$
$$+ \int_0^y \tau_1(\eta)\, G_\xi(x,y;0,\eta)\, d\eta - \int_0^y \tau_1'(t)\, dt \int_t^y \int_0^1 \bar{L}_1(\xi,\eta;t)\, G(x,y;\xi,\eta)\, d\xi\, d\eta. \tag{3.9}$$

Now, applying the condition $u_x(0,y) = \nu_1(y)$ to the resulting solution of the mixed problem (3.1)–(3.2), with reference to (3.4), (3.6), and (3.9), we obtain the following integro-differential relation with respect to the functions $\tau_1(y)$ and $\nu_1(y)$:

$$\nu_1(y) = \int_0^y K(y,t)\, \tau_1(t)\, dt - \int_0^y \Phi_{1x}(y,t)\, \tau_1'(t)\, dt + \Phi_{0x}(y). \tag{3.10}$$

where

$$K(y,t) = G_{\xi x}(0,y;0,t) + \sum_{i=1}^n \int_t^y \int_0^1 U_x(0,\xi;y-\eta)\, \frac{d_i(\xi,\eta)}{\Gamma(-\alpha_i)}\, (\eta - t)^{-\alpha_i - 1}\, d\xi\, d\eta,$$

$$\Phi_{1x}(y,t) = \int_t^y \int_0^1 \overline{L}_1(\xi,\eta;t)\, G_x(0,y;\xi,\eta)\, d\xi\, d\eta,$$

and

$$\Phi_{0x}(y) = \int_0^y \left[\varphi_2(\eta)\, G_x(0,y;1,\eta) - \int_0^1 G_x(0,y;\xi,\eta)\, \widetilde{p}_1(\xi,\eta)\, d\xi \right] d\eta$$
$$+ \int_0^1 \psi_2(\xi)\, G_x(0,y;\xi,0)\, d\xi.$$

Thus, relation (3.10) constitutes our second principal integro-differential relation, derived from the domain D_1 over the interval I.

4. Connecting System of Equations on the Line of Intersection of the Integral Discontinuity

In this section, we provide a proof of Theorem 1.1 by employing the fundamental relations derived in Sections 2 and 3. We begin by formulating a system of integro-differential equations for the functions $\tau_1(y)$, $\tau_2(y)$, $\nu_1(y)$, and $\nu_2(y)$, based on expressions (2.16), (3.10), and the gluing conditions (1.2) defined within the parabolic and hyperbolic subdomains D.

$$0 = \begin{cases} \nu_1(y) - \int\limits_0^y K(y,t)\, \tau_1(t)\, dt + \int\limits_0^y \Phi_{1x}(y,t)\, \tau_1'(t)\, dt = \Phi_{0x}(y), \\ \nu_2(y) - \tau'_2(y) - \Pi(y)\tau_2(y) - \int\limits_0^y T(y,t)\tau_2(t)dt = g_{11}(y), \\ \tau_1(y) - \mu(y)\tau_2(y) = \sigma(y), \\ \nu_1(y) - \int\limits_0^y \gamma(y,t)\nu_2(t)dt - \delta(y)\nu_2(y) - \xi(y)\tau_2(y) = \theta(y). \end{cases} \tag{4.1}$$

where

$$\Pi(y) = R_x(0, y; y, 0) - R_\eta(0, y; y, 0) - b_2(y, 0). \tag{4.2}$$

Eliminating the intermediate functions $\tau_1(y)$, $\nu_1(y)$, and $\nu_2(y)$ by combining the first and second equations of the system with the third discontinuous gluing condition, we reduce system (4.1) to a Volterra integral equation of the third kind with respect to $\tau_2'(y)$:

$$\delta(y)\,\tau_2'(y) + \int_0^y \left[\delta(y)\,\Pi(y) + \xi(y) + \mu(t)\,\Phi_{1x}(y,t) - \gamma(y,t) - \int_t^y \alpha(y,s)\,ds\right]\tau_2'(t)\,dt =$$

$$= -F(y) - \delta(y)\,\Pi(y)\,\psi_1(0) - \xi(y)\,\psi_1(0). \tag{4.3}$$

where

$$\Phi_{1x}(y,t) = \int_t^y \int_0^1 \overline{L_1}(\xi,\eta;t)\,G_x(0,y;\xi,\eta)\,d\xi\,d\eta,$$

$$\alpha(y,t) = \mu(t)\,K(y,t) + \mu'(t)\,\Phi_{1x}(y,t) - \gamma(y,t)\,\Pi(t) - \delta(y)\,T(y,t),$$

$$F(y) = \Phi_{0x}(y) + \int_0^y \left(\sigma(t) K(y,t) + g_{11}(t)\,\gamma(y,t)\right) dt +$$

$$+\,\sigma'(y) \int_0^y \Phi_{1x}(y,t)\,dt + \delta(y)\,g_{11}(y) + \theta(y).$$

If $\delta(y) \neq 0$, then (4.3) becomes:

$$\tau_2'(y) + \frac{1}{\delta(y)} \int_0^y \left[\delta(y)\,\Pi(y) + \xi(y) + \mu(t)\,\Phi_{1x}(y,t) - \gamma(y,t) - \int_t^y \alpha(y,s)\,ds\right]\tau_2'(t)\,dt =$$

$$= -\frac{F(y)}{\delta(y)} - \Pi(y)\,\psi_1(0) - \frac{\xi(y)}{\delta(y)}\,\psi_1(0), \tag{4.4}$$

which represents a Volterra integral equation of the second kind, featuring a weak singularity in the case $0 < \delta(y) < 1$. Denoting the kernel and the right-hand side by $K_1(y,t)$ and $F_1(y)$, respectively, we have

$$\tau_2'(y) + \frac{1}{\delta(y)} \int_0^y K_1(y,t)\,\tau_2'(t)\,dt = F_1(y), \qquad 0 < \delta(y) < 1. \tag{4.5}$$

We next analyze the asymptotic behavior of the right-hand side as $y \to 1$ and $y \to 0$:

$$F_1(y) = -\frac{F(y)}{\delta(y)} - \Pi(y)\,\psi_1(0) - \frac{\xi(y)}{\delta(y)}\,\psi_1(0) = R_1(y) + R_2(y) + R_3(y), \tag{4.6}$$

where

$$|R_1(y)| = \left|\frac{1}{\delta(y)}\int_0^y \sigma(t)\,K(y,t)\,dt\right| + \left|\frac{1}{\delta(y)}\int_0^y \sigma'(y)\,\Phi_{1x}(y,t)\,dt\right| + \left|\frac{1}{\delta(y)}\Phi_{0x}(y)\right| +$$

$$+\frac{1}{\delta(y)}\left|\int_0^y \gamma(y,\eta)\,g_{11}(\eta)\,d\eta\right| + |g_{11}(y)| + \left|\frac{1}{\delta(y)}\theta(y)\right| \le \frac{c_1}{y^\varepsilon},\ \ \varepsilon < 1,$$

$$|R_2(y)| = |\Phi(y)\,\psi_1(0)| \le c_2, \quad \text{and similarly, } |R_3(y)| \le c_3 = \text{const}.$$

Hence, the right-hand side is uniformly bounded, i.e., $|F_1(y)| \le \text{const}$, for some constant $C > 0$. Analogously, for the kernel, we obtain

$$|K_1(y,t)| \le |\delta(y)|\,|\Pi(y)| + |\xi(y)| + |\mu(t)|\,|\Phi(y,t)| + |\gamma(y,t)| + \left|\int_0^y A(y,s)\,ds\right| \le c_4,$$

therefore, $K_1(y,t)$ is bounded and continuous on the square $[0,1]\times[0,1]$, where $c_i = \text{const}$, $i = 1,2,3,\ldots$. According to classical results in the theory of integral equations (see, e.g., [34]), integral equations of the second kind with continuous kernels and weakly singular right-hand sides (with singularity order less than one) admit a unique solution.

If $\delta(y) = 0$, equation (4.3) reduces to a Volterra equation of the first kind:

$$\int_0^y K_2(y,t)\,\tau_2'(t)\,dt = F_2(y), \tag{4.7}$$

where

$$K_2(y,t) = \xi(y) - \gamma(y,t) + \mu(t)\,\Phi_{1x}(y,t) - \int_t^y \alpha(y,s)\,ds, \quad F_2(y) = -F(y) - \psi_1(0)\,\xi(y).$$

From the existence and uniqueness results for the integral equation obtained above [43], together with the assumptions of Theorem 1.6, it follows that the derivative $\tau_2'(y)$ is uniquely determined. Consequently, the auxiliary functions $\tau_1'(y)$, $\nu_1'(y)$, and $\nu_2'(y)$ can be uniquely reconstructed from the original system (4.1). This establishes the unique solvability of the problem with an integral discontinuity for the loaded equation (2.1) with variable coefficients. Therefore, Theorem 1.1 is proven.

5. Conclusion

In this study, we investigated a boundary value problem for a mixed-type integro-differential equation involving fractional loading and nonlocal (integral) gluing conditions. Utilizing the framework of integral equations, we established the existence and uniqueness of a regular solution under suitable smoothness and compatibility assumptions. The results obtained contribute to the broader theory of mixed-type equations with fractional and integral characteristics.

The techniques developed here pave the way for extending the analysis to more complex scenarios, including nonlinear or degenerate cases. In particular, the integral formulation provides a robust toolset for addressing intricate boundary behaviors and discontinuities inherent in such problems.

Furthermore, we demonstrated that the boundary value problem for a second-order loaded integro-differential equation with an integral gluing condition is well-posed and uniquely solvable under natural assumptions on the kernel and source terms. The methodology presented confirms the effectiveness of integral equation techniques in tackling challenging problems in parabolic-hyperbolic systems, and lays a solid foundation for future analytical and numerical developments in this field.

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