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Spectral Matrix Computational Tau Approach for Fractional Differential Equations via Fifth-Kind Chebyshev Polynomials

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ABSTRACT: This paper presents Tau approach for solving fractional differential equations (FDEs) via shifted Chebyshev polynomials of the fifth kind. By leveraging the unique properties of these polynomials, we develop operational matrices that facilitate the approximation of solutions to both linear and nonlinear FDEs. The proposed method employs a tau technique in the matrix form to transform the problem into a solvable algebraic system, ensuring computational efficiency and accuracy. This work presents a rigorous convergence analysis and demonstrates the efficacy of the proposed approach through a series of illustrative examples, showcasing a marked improvement in solution precision relative to conventional methodologies. This research contributes to the growing of work in fractional calculus and offers a robust tool for researchers and practitioners in applied mathematics and engineering.

Key Words: Fractional Differential Equations, Fifth-Kind Chebyshev Polynomials, Operational matrices, Computational Tau Method, Convergence Analysis.

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1. Introduction

Fractional calculus has gained substantial attention in recent years due to its ability to model complex systems with memory and hereditary properties. These systems are commonly encountered in fields such as physics [1,2], engineering [3], finance [4,5], and biology, where classical integer-order derivatives do not adequately capture the underlying dynamics [6,7]. The concept of fractional derivatives and integrals offers a powerful tool for characterizing non-local interactions, anomalous diffusion, and systems exhibiting complex temporal and spatial behavior, making fractional differential equations (FDEs) an essential part of contemporary modeling approaches [8].

Solving FDEs remains a challenging task, as closed-form solutions are not generally available. As a result, the development of efficient numerical methods has become crucial. Among these, spectral methods have gained popularity due to their high accuracy and fast convergence, particularly for problems that are smooth and well-posed [9,10]. These methods rely on expanding the solution in terms of basis functions that are globally defined, allowing for rapid convergence even when using relatively few grid points.

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Chebyshev polynomials (CPs), specifically the Chebyshev polynomials of the first (1kCP) and second (2kCP) kinds, have been extensively used in spectral methods due to their remarkable properties, including near-optimal convergence rates and efficient handling of boundary conditions [11,12]. Furthermore, as a generalization, Jacobi polynomials (JPs) are a class of orthogonal polynomials that depend on two parameters, α and β, and are widely used in approximation theory [13,14,15]. Their generalization to other polynomials, such as CPs, Legendre polynomials (LPs), and more others when substitute with a specific α, β. These polynomials are often used in spectral numerical methods because they provide fast convergence and accurate solutions when discretizing problems on bounded intervals, particularly in the context of solving partial differential equations. In addition, Symmetric polynomials (SPs) are polynomials invariant under any permutation of their four parameters $(\alpha, \beta, \gamma, \delta)$, playing a crucial role in algebraic geometry, numerical analysis, and number theory. Their generalization, such as CPs, LPs, Ultra spherical polynomials (USPs), introduces specific structures used in approximations and interpolation [16,17]. These generalized polynomials are integral in spectral numerical methods, which involve transforming problems into a spectral space for efficient solutions, particularly in solving DEs and FDEs. Recently, there has been a growing interest in the generalization of these classical polynomials to higher-order forms, such as the Chebyshev polynomials of the fifth kind (5kCPs) which are generated from the SPs in a specific values of $\alpha, \beta, \gamma, \delta$. 5kCPs provide improved flexibility and accuracy when applied to problems involving fractional derivatives [18,19]. These generalized CPs have shown promising results in various fields, including the numerical solution of linear and nonlinear FDEs and PDEs [20,21,22,23,24] .

The shifted Chebyshev polynomials of the fifth kind (S5kCPs) have further extended the potential of Chebyshev polynomials by allowing for more efficient representation of functions that exhibit irregular boundary conditions. The shift in the polynomials enables better handling of problems where non-homogeneous or complex boundary conditions are present, such as in many engineering and physical models. This approach has demonstrated superior performance in solving fractional boundary value problems, such as those arising from the Bagley-Torvik equation, a fundamental model used to describe viscoelastic behavior in materials [25].

This study is concerned with proving, finding S5kCPs through out SPs, and studying some of their properties, convergence, total, error bound, and global error analysis. This paper explores the application of S5kCPs with the matrix computational tau scheme in solving fractional boundary value problems, particularly focusing on the linear and non-linear FDEs. By comparing the results obtained using the S5kCPs tau method with those from other numerical methods, we demonstrate its enhanced accuracy and computational efficiency. The findings highlight the superiority of the S5kCPs tau method in solving complex fractional differential equations, offering new insights for researchers and practitioners working in various scientific and engineering disciplines.

2. Fundamental properties of Fifth-kind Chebyshev Polynomials

In this section, we provide an overview of the 5kCPs, which serve as a fundamental tool in the spectral methods employed in this study. We begin by introducing basic concepts from fractional calculus and then proceed to define the 5kCPs. Their orthogonality and recurrence properties, as well as their relationships with the well-established Chebyshev polynomials of the first kind, are explored. These relationships prove essential for the development of the numerical algorithms introduced in subsequent sections.

2.1. Preliminaries

Definition 1. The left-sided Caputo's fractional derivative of order φ, (where $k-1<\varphi<k$) of a Lebesgue integrable function $\phi(\tau)$ is defined as [26]:

$$ {}_{C}D^{\varphi}_{a+}\phi(\tau) = \frac{1}{\Gamma(k-\varphi)} \int_a^\tau \frac{\phi^{(k)}(s)}{(\tau-s)^{\varphi-k+1}}\, ds, \quad \varphi>0, \tau>a, \tag{2.1} $$

where k is the smallest integer greater than φ, $k \in \mathbb{N}$, and $\Gamma(\cdot)$ denotes the Gamma function, and $\phi^{(k)}(\tau)$ represents the k-th derivative of $\phi(\tau)$. The integral is taken over the interval $[a,\tau]$, with a being the lower limit of integration.

Definition 2. For the power function τ^{ν}, the left-sided Caputo's fractional derivative of order φ, and $\varphi>0$ is defined in terms of integer k as follows [27]:

$$ {}_C D^{\varphi}_{a+} \tau^{\nu} = \begin{cases} 0 & \text{if } \nu < \lceil \varphi \rceil , \\ \frac{\Gamma(\nu+1)\tau^{\nu-\varphi}}{\Gamma(\nu+1-\varphi)} & \text{if } \nu \geq \lceil \varphi \rceil , \end{cases} \tag{2.2} $$

where $\lceil . \rceil$ is the ceiling function, $\nu \in \mathbb{N}$.

Definition 3. For a constant C, the left-sided Caputo's fractional derivative of order φ is defined as follows:

$$ {}_C D^{\varphi}_{a+} C = 0, \quad \varphi > 0. \tag{2.3} $$

2.2. Generalized Symmetric Polynomials

The generalized SPs can be derived using the extended Sturm–Liouville theorem for symmetric functions by introducing additional parameters that modify their behavior and properties.

Let $\zeta_i(\tau)$ represent a sequence of symmetric functions that satisfy the following second-order Sturm liouville differential equation:

$$ H_1(\tau)\zeta_i''(\tau) + H_2(\tau)\zeta_i'(\tau) + \left[\lambda_i H_3(\tau) + H_4(\tau) + \frac{1+(-1)^{i+1}}{2} H_5(\tau)\right]\zeta_i(\tau) = 0, \quad i = 0, 1, 2, \ldots, n. \tag{2.4} $$

Where, $H_1(\tau), H_2(\tau), H_3(\tau), H_4(\tau)$, and $H_5(\tau)$ are functions of τ, and $\{\lambda_i\}$ represent a sequence of constants.

As demonstrated by Masjed-Jamei 2006 [16], the functions $H_1(\tau), H_3(\tau), H_4(\tau)$, and $H_5(\tau)$ are even functions, while $H_2(\tau)$ is an odd function. To derive the symmetric class of orthogonal polynomials, the following choices are made for these functions and constants:

$$ H_1(\tau) = \tau^2(\gamma\tau^2 + \delta), \quad H_2(\tau) = \tau(\alpha\tau^2 + \beta), \quad H_3(\tau) = \tau^2, \quad H_4(\tau) = 0, \quad H_5(\tau) = -\beta, $$
$$ \lambda_i = -i(\beta + (i-1)\gamma), $$

where α, β, γ and δ are arbitrary real parameters.

Inserting these choices into the differential equation results in:

$$ \tau^2(\gamma\tau^2 + \beta)\zeta_i''(\tau) + \tau(\alpha\tau^2 + \beta)\zeta_i'(\tau) - (i(\alpha + (i-1)\gamma)\tau^2 + \frac{\beta}{2}(1 + (-1)^{i+1}))\zeta_i(\tau) = 0. \tag{2.5} $$

This equation defines the symmetric class of orthogonal polynomials based on the specified parameter selections.

The solution of Eqn. (2.5) is the generalized polynomial known SPs and defined by:

$$ \mathscr{G}^i_{\alpha,\beta,\gamma,\delta}(\tau) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{\lfloor \frac{i}{2} \rfloor}{k} \left(\prod_{j=0}^{\lfloor \frac{i}{2} \rfloor - (k+1)} \Psi_{i,j,\alpha,\beta,\gamma,\delta} \right) \tau^{i-2k}, \tag{2.6} $$

where,

$$ \Psi_{i,j,\alpha,\beta,\gamma,\delta} = \frac{(2j + (-1)^{i+1} + 2\lfloor \frac{i}{2} \rfloor)\alpha + \gamma}{(2j + (-1)^{i+1} + 2)\beta + \delta}. $$

In addition, Masjed-Jamei [16], introduced the monic (that have leading coefficients equal to 1) symmetric orthogonal polynomials $\overline{\mathscr{G}^i}_{\alpha,\beta,\gamma,\delta}(\tau)$, defined as:

$$ \overline{\mathscr{G}}^i_{\alpha,\beta,\gamma,\delta}(\tau) = \left(\prod_{j=0}^{\lfloor \frac{i}{2} \rfloor - 1} \frac{1}{\Psi_{i,j,\alpha,\beta,\gamma,\delta}} \right) \mathscr{G}^i_{\alpha,\beta,\gamma,\delta}(\tau). \tag{2.7} $$

The monic polynomials $\overline{\mathscr{G}^i}_{\alpha,\beta,\gamma,\delta}(\tau)$ satisfy the following recurrence relation:

$$\overline{\mathscr{G}}^{i+1}_{\alpha,\beta,\gamma,\delta}(\tau) = \tau \cdot \overline{\mathscr{G}}^{i}_{\alpha,\beta,\gamma,\delta}(\tau) + \Lambda^{i}_{\alpha,\beta,\gamma,\delta} \cdot \overline{\mathscr{G}}^{i-1}_{\alpha,\beta,\gamma,\delta}(\tau), \quad i \geq 0,$$

with the following initial values:

$$\overline{\mathscr{G}}^{0}_{\alpha,\beta,\gamma,\delta}(\tau) = 1, \quad \overline{\mathscr{G}}^{1}_{\alpha,\beta,\gamma,\delta}(\tau) = \tau,$$

and

$$\Lambda^{i}_{\alpha,\beta,\gamma,\delta} = \frac{\alpha\beta i^2 + ((\gamma - 2\alpha)\beta - (-1)^i\alpha\delta)i + \frac{1}{2}\delta(\gamma - 2\alpha)\left(1 - (-1)^i\right)}{(2\alpha i + +\gamma - \alpha)\,(2\alpha i + \gamma - 3\alpha)}. \tag{2.8}$$

The first five terms take the following forms:

$$\begin{aligned}
\overline{\mathscr{G}}^{2}_{\alpha,\beta,\gamma,\delta}(\tau) &= \tfrac{\beta+\delta}{\alpha+\gamma} + \tau^2,\\
\overline{\mathscr{G}}^{3}_{\alpha,\beta,\gamma,\delta}(\tau) &= \tfrac{(3\beta+\delta)\tau}{3\alpha+\gamma} + \tau^3,\\
\overline{\mathscr{G}}^{4}_{\alpha,\beta,\gamma,\delta}(\tau) &= \tfrac{(\beta+\delta)(3\beta+\delta)}{(3\alpha+\gamma)(5\alpha+\gamma)} + \tfrac{2(3\beta+\delta)\tau^2}{5\alpha+\gamma} + \tau^4,\\
\overline{\mathscr{G}}^{5}_{\alpha,\beta,\gamma,\delta}(\tau) &= \tfrac{(3\beta+\delta)(5\beta+\delta)\tau}{(5\alpha+\gamma)(7\alpha+\gamma)} + \tfrac{2(5\beta+\delta)\tau^3}{7\alpha+\gamma} + \tau^5.
\end{aligned}$$

This formulation uses different symbols to provide a new appearance to the equations and expressions. In the context of this research, the generalized monic function $\overline{\mathscr{G}}^{i}_{\alpha,\beta,\gamma,\delta}(\tau)$ can be employed to generate CPs of various kinds by selecting appropriate values for the parameters α, β, γ, and δ. The function's versatility lies in its ability to represent different families of monic CPs as follows [16,17]: The 1kCPs $\overline{T}_i(\tau)$ can be derived by selecting specific values for the parameters, given by:

$$\overline{T}_i(\tau) = 2^{i-1}\,\overline{\mathscr{G}}^{i}_{-1,1,-1,0}(\tau).$$

In contrast, the 2kCPs $\overline{U}_i(\tau)$ are generated when the parameters are set as:

$$\overline{U}_i(\tau) = 2^{i}\,\overline{\mathscr{G}}^{i}_{-1,1,-3,0}(\tau).$$

By choosing another set of parameter values, specifically $\alpha = -1$, $\beta = 1$, $\gamma = -2$, and $\delta = 0$, the function reduces to the LPs $\overline{P}_i(\tau)$:

$$\overline{P}_i(\tau) = \frac{(2i)!}{(i!)^2 2^i}\,\overline{\mathscr{G}}^{2i}_{-1,1,-2,0}(\tau).$$

The USPs $C_i^a(\tau)$ emerge when the same parameters are assigned, resulting in:

$$C_i^a(\tau) = \frac{2^i (a)_i}{i!}\,\overline{\mathscr{G}}^{2i}_{-1,1,-(2a+1),0}(\tau),$$

where, $(a)_i$ represents the Pochhammer Symbol and, $(a)_i = \dfrac{\Gamma(a+i)}{\Gamma(a)}$. This parametric flexibility allows the function $\overline{\mathscr{G}}^{i}_{\alpha,\beta,\gamma,\delta}(\tau)$ to serve as a unified framework for generating the different kinds of CPs, enabling further exploration and analysis of their properties within a generalized structure.
Also, if we assign specific values to the parameters α, β, γ, and δ, we obtain the fifth kind and sixth kind respectively $\overline{\mathscr{C}}^{i}(\tau)$ and $\overline{\mathcal{Y}}_i(\tau)$ [16,28,29]:
By setting $\alpha = -1$, $\beta = 1$, $\gamma = -3$, and $\delta = 2$, the function $\overline{\mathscr{C}}^{i}(\tau)$ is expressed as:

$$\overline{\mathscr{C}}^{i}(\tau) = \overline{\mathscr{G}}^{i}_{-1,1,-3,2}(\tau).$$

Similarly, by setting $\alpha = -1$, $\beta = 1$, $\gamma = -5$, and $\delta = 2$, the function $\overline{\mathcal{Y}}_i(\tau)$ becomes:

$$\overline{\mathcal{Y}}_i(\tau) = \overline{\mathscr{G}}^{i}_{-1,1,-5,2}(\tau).$$

This demonstrates that the 5kCPs are a special case of the generalized SPs $\overline{\mathscr{G}}^{i}_{\alpha,\beta,\gamma,\delta}(\tau)$ when specific parameter values are substituted.

From this point onward, we focus exclusively on the Chebyshev polynomials of the fifth kind $\overline{\mathscr{C}}^{i}(\tau)$ and their shifted versions.

The orthogonality condition for $\overline{\mathscr{C}}^{i}(\tau)$ is given by:

$$\int_{-1}^{1} \frac{\tau^2}{\sqrt{1-\tau^2}}\, \overline{\mathscr{C}}^{i}(\tau), \overline{\mathscr{C}}^{j}(\tau)\, d\tau = \begin{cases} (-1)^i \left(\prod_{\mathfrak{k}=1}^{i} \Lambda^{\mathfrak{k}}_{-1,1,-3,2}\right) \frac{\pi}{2}, & \text{if } i=j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{2.9}$$

where $\Lambda^{\mathfrak{k}}_{-1,1,-3,2}$ is as defined earlier in (2.8) with the special values of $\alpha, \beta, \gamma, \delta$.

This orthogonality relation can alternatively be expressed as:

$$\int_{-1}^{1} \frac{\tau^2}{\sqrt{1-\tau^2}}\, \overline{\mathscr{C}}^{i}(\tau), \overline{\mathscr{C}}^{j}(\tau)\, d\tau = \begin{cases} \mathfrak{h}_i, & \text{if } i=j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{2.10}$$

where

$$\mathfrak{h}_i = \begin{cases} \frac{\pi}{2^{2i+1}}, & \text{if } i \text{ is even}, \\ \frac{\pi(i+2)}{i2^{2i+1}}, & \text{if } i \text{ is odd}. \end{cases} \tag{2.11}$$

In the following sections, it will be more useful to work with the normalized version of the shifted Chebyshev polynomials of the fifth kind.

2.3. Shifted Orthonormal Chebyshev Polynomials of the Fifth Kind

The 5kCPs are a specific class of orthogonal polynomials. They are defined through a recurrence relation and exhibit properties that make them suitable for numerical applications, particularly in solving fractional differential equations (see also [30,31]). The S5kCPs, denoted by $\mathscr{C}^{i}(\tau)$, can be defined on the interval $[0,1]$ as follows:

$$\mathscr{C}^{i}(\tau) = \frac{1}{\sqrt{\mathfrak{h}_i}}\, \overline{\mathscr{C}}^{i}(2\tau-1),$$

where $\mathfrak{h}_i$ is given in (2.11). From the previous relation, it is evident that $\mathscr{C}^{i}(\tau)$, for $i \geq 0$, are orthonormal over $[0,1]$. Specifically, the orthonormality condition is:

$$\int_{0}^{1} w(\tau)\, \mathscr{C}^{i}(\tau)\, \mathscr{C}^{j}(\tau)\, d\tau = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.12}$$

where the weight function $w(\tau)$ is defined as:

$$w(\tau) = \frac{(2\tau-1)^2}{\sqrt{\tau-\tau^2}}.$$

The S5kCPs, denoted as $\mathscr{C}^{i}(\tau)$, are orthogonal polynomials defined on the closed interval $[0,1]$. They can be generated using the following recurrence relation:

$$\mathscr{C}^{i}(\tau) = (2\tau-1)\left(\mathscr{C}^{i-1}(\tau)\right) + \Lambda_i \mathscr{C}^{i-2}(\tau), \tag{2.13}$$

where $\Lambda_i = \Lambda^{i}_{\alpha,\beta,\gamma,\delta}$, with the special values of $\alpha, \beta, \gamma, \delta$, it is defined in Eqn. (2.8), and the coefficient Λ_i now is given by:

$$\Lambda_i = \frac{-1+(-1)^i + \left(-1+2(-1)^i\right) i - i^2}{4i(1+i)}. \tag{2.14}$$

The first four terms take the following forms:

$$\mathscr{C}^0(\tau) = \sqrt{\tfrac{2}{\pi}},$$
$$\mathscr{C}^1(\tau) = 2\sqrt{\tfrac{2}{3\pi}}(-1+2\tau),$$
$$\mathscr{C}^2(\tau) = 4\sqrt{\tfrac{2}{\pi}}\left(\tfrac{1}{4} - 4\tau + 4\tau^2\right),$$
$$\mathscr{C}^3(\tau) = 8\sqrt{\tfrac{6}{5\pi}}\left(-\tfrac{1}{6} + \tfrac{13\tau}{3} - 12\tau^2 + 8\tau^3\right).$$

Theorem 2.1 *The S5kCPs, $\mathscr{C}^i(\tau)$, can be expressed in terms of the S1kCPs, $T_i^*(\tau)$, through the following relation:*

$$\mathscr{C}^i(\tau) = \sum_{k=0}^{i} g^{i,k} T_k^*(\tau), \tag{2.15}$$

where the coefficients $g^{i,k}$ are defined as:

$$g^{i,k} = 2\sqrt{\frac{2}{\pi}}(-1)^{\frac{i-k}{2}} \cdot \begin{cases} \delta^k, & \text{if both } i \text{ and } k \text{ are even}, \\ \frac{k}{i}, & \text{if both } i \text{ and } k \text{ are odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.16}$$

with δ^k defined as in Eqn. (2.18).

Proof Relation (2.15) with (2.16) is true at $(i = 0)$, so, if we suppose that it is true at $(i-1)$ by mathematical induction it easily proved at (i). For more details and to see another connection with the trigonometric representation for the S5kCPs we refer to [28].

■

Remark 2.1 *The connection formula in Eqn. (2.15) can be decomposed into two distinct connection formulas, as follows in the next corollary. This separation allows for a more detailed and structured representation of the connection between the S5kCPs and the S1kCPs.*

Corollary 2.1 *The S5kCPs, $\mathscr{C}^i(\tau)$, defined in (2.12), can be expressed in terms of the S5kCPs, $T_i^*(\tau)$, by the following connection formulas:*
for even indices:

$$\mathscr{C}^{2i}(\tau) = 2\sqrt{\frac{2}{\pi}} \sum_{r=0}^{i} (-1)^{i+r} \delta_r T_{2r}^*(\tau), \tag{2.17}$$

where δ^r is given by:

$$\delta^r = \begin{cases} \frac{1}{2}, & \text{if } r = 0, \\ 1, & \text{if } r > 0, \end{cases} \tag{2.18}$$

for odd indices:

$$\mathscr{C}^{2i+1}(\tau) = \frac{2\sqrt{2}}{\sqrt{\pi(2i+1)(2i+3)}} \sum_{r=0}^{i} (-1)^{i+r} (2r+1) T_{2r+1}^*(\tau). \tag{2.19}$$

Lemma 2.1 *The S1kCPs, $T_k^*(\tau)$, can be represented in power series form (analytic form), and its inversion formula is given by: (The power series expansion of $T_k^*(\tau)$)*

$$T_k^*(\tau) = k \sum_{r=0}^{k} (-1)^{r+k} \frac{2^{2r}(r+k-1)!}{(2r)!(k-r)!} \tau^r, \tag{2.20}$$

where k denotes the degree of the polynomial.
The inverse relation, expressing τ^k in terms of the polynomials $T^*_j(\tau)$, $(j = 0, 1, 2, ..., k)$, is:

$$\tau^k = 2^{1-2k}(2k)! \sum_{j=0}^{k} \frac{\delta_j}{(k-j)!(k+j)!} T^*_j(\tau), \tag{2.21}$$

where, the coefficients δ^j are as defined in Eqn. (2.18).

Theorem 2.2 *The explicit analytic form of the S5kCPs, $\mathscr{C}^i(\tau)$, can be expressed as a finite power series as:*

$$\mathscr{C}^i(\tau) = \sum_{p=0}^{i} \alpha_{p,i} \tau^p, \tag{2.22}$$

where, the coefficients $\alpha_{p,i}$ are given by the following formula:

$$\alpha_{p,i} = \begin{cases} \frac{2^{2p+\frac{3}{2}}}{\sqrt{\pi}(2p)!} \sum_{j=\lceil \frac{p+1}{2} \rceil}^{\frac{i}{2}} (-1)^{\frac{i}{2}+j-p} j \delta_j \frac{(2j+p-1)!}{(2j-p)!}, & \text{if } i \text{ is even}, \\ \frac{1}{\sqrt{i(i+2)}} \sum_{j=\lfloor \frac{p}{2} \rfloor}^{\frac{i-1}{2}} (-1)^{\frac{i+1}{2}+j-p} (2j+1)^2 \frac{(2j+p)!}{(2j-p+1)!}, & \text{if } i \text{ is odd}. \end{cases} \tag{2.23}$$

Theorem 2.3 *The inversion formula for the analytic expression (2.22) can be explicitly stated as:*

$$\tau^m = \sum_{n=0}^{m} q_{n,m} \mathscr{C}^n(\tau), \tag{2.24}$$

where, the coefficients $q_{n,m}$ are defined as follows:

$$q_{n,m} = \sqrt{\pi} 2^{-2m-\frac{1}{2}} (2m)! \begin{cases} \frac{2((n+1)^2+m^2+m)}{(m-n)!(n+m+2)!}, & \text{if } n \text{ is even}, \\ \frac{\sqrt{\frac{n+2}{n}}}{(m-n)!(m+n)!} + \frac{\sqrt{\frac{n}{n+2}}}{(m-n-2)!(m+n+2)!}, & \text{if } n \text{ is odd}. \end{cases} \tag{2.25}$$

Proof. The proofs of Theorems 2.2 and 2.3 rely primarily on the application of Lemma 2.1.

■

To enhance the robustness of our findings, we have included additional details to validate our results, confirm established relationships, and address inconsistencies with certain findings reported in [28], furthermore, the next section also, has more details for approximation theory, convergence and error analysis theorems.

3. Approximation, convergence and error analysis

In this part, we perform thorough examination of the convergence and error bound for the shifted orthonormal expansion by utilizing Chebyshev polynomials of the fifth order. Assuming $G(\tau)$ is a square Lebesgue integrable function defined on the interval $[0, 1]$, it can be expressed as a linear combination of the linearly independent S5kCPs vector $Span\{\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \mathscr{C}^2(\tau), \dots\}$, [24,28,30].
Express the function $G(\tau)$ as an infinite sum of S5kCPs:

$$G(\tau) = \sum_{j=0}^{\infty} \mathscr{M}^j \mathscr{C}^j(\tau), \tag{3.1}$$

where, the coefficients $\mathscr{M}^j$ are determined by:

$$\mathscr{M}^j = \int_0^1 G(\tau) \mathscr{C}^j(\tau) w(\tau)\, d\tau, \quad j = 0, 1, 2, \dots. \tag{3.2}$$

In practical applications, we approximate $G(\tau)$ using only the first $(n+1)$ terms of the S5kCPs, so, one writes:

$$G(\tau) \approx G_n(\tau) = \sum_{j=0}^{n} \mathscr{M}^j \mathscr{C}^j(\tau) = \mathscr{M}^T \psi(\tau), \tag{3.3}$$

where, the coefficient vector $\mathscr{M}$ and the shifted Chebyshev vector $\psi(\tau)$ are given by:

$$\mathscr{M}^T = [\mathscr{M}^0, \mathscr{M}^1, \dots, \mathscr{M}^n], \tag{3.4}$$

and

$$\psi(\tau) = [\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \dots, \mathscr{C}^n(\tau)]^T.$$

Lemma 3.1 *Let $\psi(\tau) = [\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \dots, \mathscr{C}^n(\tau)]$ be a vector composed of orthonormal S5kCPs, where $n \in \mathbb{N}$. For $\tau \in [0,1]$, the vector $\psi(\tau)$ can be expressed as:*

$$\psi(\tau) = \mathscr{B}^n T(\tau), \tag{3.5}$$

where, $T(\tau) = [T_0^*(\tau), T_1^*(\tau), T_2^*(\tau), \dots, T_n^*(\tau)]^T$ is a vector of S1kCPs, and $\mathscr{B}^n$ is an $(n+1) \times (n+1)$ lower triangular matrix defined as:

$$\mathscr{B}^n = \begin{pmatrix} \mathscr{B}^{00} & 0 & 0 & \cdots & 0 \\ \mathscr{B}^{10} & \mathscr{B}^{11} & 0 & \cdots & 0 \\ \mathscr{B}^{20} & \mathscr{B}^{21} & \mathscr{B}^{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathscr{B}^{n0} & \mathscr{B}^{n1} & \mathscr{B}^{n2} & \cdots & \mathscr{B}^{nn} \end{pmatrix}.$$

Here, each $\mathscr{B}^{ij}$ represents the coefficients that occupy the entries of the matrix $\mathscr{B}^n$,

$$\mathscr{B}^{ij} = \begin{cases} \mathscr{B}^{2i,2j} = 2(-1)^{i-j} \left(\frac{2}{\pi}\right)^{1/2} \delta^{2j} & \text{and } j \le i, \\ \mathscr{B}^{2i+1,2j+1} = 2(-1)^{i-j} \left(\frac{2}{(2i+1)(2i+3)\pi}\right)^{1/2} \delta^{2j+1}, & \text{and } j \le i. \end{cases} \tag{3.6}$$

Proof: The proof is established by thoroughly analyzing the coefficients in the power series form (2.16), confirming the validity of the lemma. Hence, the proof is complete.

■

Lemma 3.2 *The expression for the left-sided Caputo's fractional derivative of order φ, applied to the S5kCPs is given by:*

$$_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) \approx \sum_{j=0}^{i} \phi_\varphi(i,j) \mathscr{C}^j(\tau), \tag{3.7}$$

where,

$$\phi_\varphi(i,j) = \sum_{s=0}^{i} \frac{\mathscr{B}^{si} \Gamma(s+1) \varepsilon_j^{s,\varphi}}{\Gamma(s-\varphi+1)}, \tag{3.8}$$

and $\varepsilon_j^{s,\varphi}$ is known as:

$$\varepsilon_j^{s,\varphi} = \int_0^1 w(\tau) \tau^{s-\varphi} \mathscr{C}^j(\tau)\, d\tau. \tag{3.9}$$

Proof. From the analytic form (2.22) we can express $_C D_{0+}^{\varphi} \mathscr{C}^i(\tau)$ as :

$$_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) = {_C D_{0+}^{\varphi}} \sum_{s=0}^{i} \alpha^{si} \tau^s = \sum_{s=0}^{i} \alpha^{si} {_C D_{0+}^{\varphi}} \tau^s = \sum_{s=\lceil \varphi \rceil}^{i} \frac{\alpha^{si} \Gamma(s+1)}{\Gamma(s-\varphi+1)} \tau^{s-\varphi},$$

From the inverse relation (2.21), where, $s \geq \lceil \varphi \rceil$

$$\tau^{s-\varphi} \approx \sum_{j=0}^{n} \varepsilon_j^{s,\varphi} \mathscr{C}^j(\tau),$$

and

$$\varepsilon_j^{s,\varphi} = \int_0^1 w(\tau)\tau^{s-\varphi}\mathscr{C}^j(\tau)\,d\tau, \tag{3.10}$$

then,

$${}_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) \approx \sum_{s=\lceil \varphi \rceil}^{i} \sum_{j=0}^{n} \frac{\alpha^{si}\varepsilon_j^{s,\varphi}\Gamma(s+1)}{\Gamma(s-\varphi+1)} \mathscr{C}^j(\tau), \tag{3.11}$$

which verifies the result that is desired. Similar result as (3.11) found in [24,28,30].

■

Corollary 3.1 *The left-sided Caputo's fractional derivative of order φ, for the vector of S5kCPs, denoted as $\psi(\tau) = [\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \dots, \mathscr{C}^n(\tau)]^T$, is expressed in the matrix form as:*

$${}_C D_{0+}^{\varphi} \psi(\tau) \approx \mathcal{D}^{(\varphi)} \psi(\tau). \tag{3.12}$$

Here, $\mathcal{D}^{(\varphi)}$ represents the operational matrix of the left-sided Caputo's fractional derivative of order φ, defined as:

$$\mathcal{D}^{(\varphi)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(\lceil \varphi \rceil, 0) & \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(\lceil \varphi \rceil, 1) & \cdots & \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(\lceil \varphi \rceil, n) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(i, 0) & \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(i, 1) & \cdots & \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(i, n) \end{pmatrix}, \tag{3.13}$$

where, $\phi_\varphi(s, j)$ is defined in (3.8).
Proof: By utilizing Lemma 3.2, the present corollary can be proved as:
Applying Eqs. (3.7) – (3.11), we get:

$$\begin{aligned} {}_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) &\approx \sum_{s=\lceil \varphi \rceil}^{i} \sum_{j=0}^{n} \frac{\alpha^{si}\varepsilon_j^{s,\varphi}\Gamma(s+1)}{\Gamma(s-\varphi+1)} \mathscr{C}^j(\tau), \\ &= \sum_{j=0}^{n} \sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(s, j) \mathscr{C}^j(\tau) = \mathcal{D}^{(\varphi)} \cdot \psi(\tau), \end{aligned} \tag{3.14}$$

where, $\phi_\varphi(s, j)$ is defined in Eq.(3.8), we can express Eq. (3.14) in vector form as:

$${}_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) \approx \left[\sum_{s=\lceil \varphi \rceil}^{i} \phi_\varphi(s, j), \sum_{s=\lceil \alpha \rceil}^{i} \phi_\varphi(s, j), \dots, \sum_{s=\lceil \alpha \rceil}^{i} \phi_\varphi(s, j) \right] \psi(\tau), \quad i = \lceil \alpha \rceil, \dots, n. \tag{3.15}$$

Also, one clearly can write:

$${}_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) = [0, 0, \dots, 0]\psi(\tau), \quad i = 0, 1, \dots, \lceil \varphi \rceil - 1. \tag{3.16}$$

By combining Eqs. (3.15) and (3.16), one obtains the required result.

■

Theorem 3.1 ***(see [30]):*** *Let $\mathscr{H}$ be a Hilbert space, and let $\bar{\mathscr{H}}$ be a finite-dimensional subspace of $\mathscr{H}$ (with $\dim \bar{\mathscr{H}} < \infty$). Consider the basis set $\bar{\mathscr{H}} = \{h_1, h_2, \ldots, h_n\}$ within $\bar{\mathscr{H}}$. For any element h in $\mathscr{H}$, let $\bar{h}$ represent the unique best approximation of h from the subspace $\bar{\mathscr{H}}$. The squared norm of the difference between h and $\bar{h}$ is expressed as:*

$$\|h - \bar{h}\|_{L^2}^2 = \frac{Gram(h, h_1, \ldots, h_n)}{Gram(h_1, h_2, \ldots, h_n)},$$

where $Gram(h, h_1, \ldots, h_n)$ denotes the Gram determinant defined by:

$$Gram(h, h_1, \ldots, h_n) = \begin{vmatrix} \langle h, h\rangle & \langle h, h_1\rangle & \cdots & \langle h, h_n\rangle \\ \langle h_1, h\rangle & \langle h_1, h_1\rangle & \cdots & \langle h_1, h_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle h_n, h\rangle & \langle h_n, h_1\rangle & \cdots & \langle h_n, h_n\rangle \end{vmatrix}.$$

Lemma 3.3 *Let $g(\tau) \in L^2[0,1]$ have the optimal approximation according to relation (3.1) is given by:*

$$g(\tau) \approx g_n(\tau) = \sum_{i=0}^{n} \mathscr{M}^i \mathscr{C}^i(\tau) = \mathscr{M}^T \psi(\tau).$$

Then, as n approaches infinity, the limit

$$\lim_{n\to\infty} \|g(\tau) - g_n(\tau)\|_{L^2} = 0.$$

And, assume that the error vector associated with the Caputo's operational matrix is defined as:

$$E^{\varphi} =_C D_{0+}^{\varphi} \psi(\tau) - \mathcal{D}^{(\varphi)} \psi(\tau),$$

where,

$$E^{\varphi} = [e_0^{\varphi}, e_1^{\varphi}, \ldots, e_n^{\varphi}].$$

Therefore,

$$\|e_i^{\varphi}\|_{L^2} \leq \sum_{s=0}^{i} \mathscr{B}^{si} \frac{\Gamma(s+1)}{\Gamma(s-\varphi+1)} \left(\frac{Gram(\tau^{\nu-\varphi}, \mathscr{C}^0(\tau), \ldots, \mathscr{C}^n(\tau))}{Gram(\mathscr{C}^0(\tau), \ldots, \mathscr{C}^n(\tau))} \right).$$

Proof: If we consider $Span\{\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \mathscr{C}^2(\tau), \ldots, \mathscr{C}^n(\tau)\}$ a subspace from Hilbert space $L^2[0,1]$, and th function $\tau^{\nu-\varphi}$ has an approximation from it then it can considered as $\tau^{\nu-\varphi} = \sum_{j=0}^{n} \mathscr{M}_{\varphi}^j \mathscr{C}^j(\tau)$, by utilizing Theorem 3.1 then one can express the following relation:

$$\|\tau^{\nu-\varphi} - \sum_{j=0}^{n} \mathscr{M}_{\varphi}^j \mathscr{C}^j(\tau)\|_{L^2} = \left(\frac{\text{Gram}(\tau^{\nu-\varphi}, \mathscr{C}^0(\tau), \ldots, \mathscr{C}^n(\tau))}{\text{Gram}(\mathscr{C}_0(\tau), \ldots, \mathscr{C}^n(\tau))} \right), \tag{3.17}$$

which indicates that the norm of the ith error term can be expressed as:

$$\|e^{\varphi}\|_{L^2} = \|_C D_{0+}^{\varphi} \mathscr{C}^i(\tau) - \sum_{j=0}^{n} \phi_{\varphi}(i,j) \mathscr{C}^j(\tau)\|_{L^2},$$

$$= \| \sum_{s=\lceil\varphi\rceil}^{i} \frac{\alpha^{si}\Gamma(s+1)}{\Gamma(s-\varphi+1)} \tau^{s-\varphi} - \sum_{s=\lceil\varphi\rceil}^{i} \sum_{j=0}^{n} \frac{\alpha^{si} \varepsilon_j^{s,\varphi} \Gamma(s+1)}{\Gamma(s-\varphi+1)} \mathscr{C}^j(\tau)\|,$$

$$\leq \sum_{s=\lceil\varphi\rceil}^{i} \alpha^{is} \frac{\Gamma(s+1)}{\Gamma(s-\varphi+1)} \|t^{s-\varphi} - \sum_{j=0}^{n} \varepsilon_j^{s,\varphi,} \mathscr{C}^j(\tau)\|_{L^2}.$$

Substituting from Eqn. (3.17), obtain

$$\|e^{\varphi}\|_{L^2} \leq \sum_{s=\lceil\varphi\rceil}^{i} \alpha^{is} \frac{\Gamma(s+1)}{\Gamma(s-\varphi+1)} \left(\frac{\mathrm{Gram}(\tau^{s-\varphi}, \mathscr{C}^0(\tau), \ldots, \mathscr{C}^r(\tau))}{\mathrm{Gram}(\mathscr{C}^0(\tau), \ldots, \mathscr{C}^r(\tau))} \right). \tag{3.18}$$

The final equation conclude the proof.

■

Theorem 3.2 *It is established that S5kCPs are bounded on the interval* $[0,1]$*, and specifically, they satisfy the inequality:*

$$|\mathscr{C}^j(\tau)| < \sqrt{\frac{2}{\pi}}(j+2), \qquad \forall \;\; \tau \in [0,1]. \tag{3.19}$$

Proof: To demonstrate the inequality in Eqn. (3.19) , we consider the following two cases:

Case 1: $j = 2r$: By applying the connection formula in Eqn. (2.17) and the straightforward inequality $|T_j^*(\tau)| \leq 1$, we obtain:

$$\begin{aligned} |\mathscr{C}^j(\tau)| &\leq 2\sqrt{\frac{2}{\pi}} \sum_{l=0}^{r} 1 \\ &= 2\sqrt{\frac{2}{\pi}}(r+1) \\ &= \sqrt{\frac{2}{\pi}}(2r+2) \\ &= \sqrt{\frac{2}{\pi}}(j+2). \end{aligned}$$

Case 2: $j = 2r+1$: Utilizing the connection formula in Eqn. (2.19) along with the inequality $|T_j^*(\tau)| \leq 1$, we find:

$$\begin{aligned} |\mathscr{C}^j(\tau)|| &\leq \frac{2\sqrt{2}}{\sqrt{\pi(2r+1)(2r+3)}} \sum_{l=0}^{r} (2l+1) \\ &= \frac{2\sqrt{2}}{\sqrt{\pi(2r+1)(2r+3)}} (r+1)^2 \\ &< \frac{2\sqrt{2}(r+1)^2}{\sqrt{\pi}(2r+1)} \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi}}(2r+2) \\ &= \frac{\sqrt{2}}{\sqrt{\pi}}(j+2) \\ &< \sqrt{\frac{2}{\pi}}(j+2). \end{aligned}$$

Thus, combining the results from both cases, we conclude that for all $j \geq 0$, the following estimate holds:

$$|\mathscr{C}^j(\tau)| < \sqrt{\frac{2}{\pi}}(j+2), \quad \forall \tau \in [0,1].$$

■

Theorem 3.3 *Let $g(\tau) \in L^2[0,1]$ such that $|g^{(3)}(\tau)| \le L$, and assume it has an expansion of the form given in (3.1). Define the global error as:*

$$E^n(\tau) = \sum_{j=n+1}^{\infty} \mathscr{M}^j \mathscr{C}^j(\tau).$$

Then, the global error is bounded, as follows:

$$|E^n(\tau)| < \frac{3L}{n}.$$

Proof: The complete proof of this theorem found in [31].

■

Theorem 3.4 *Let $g(\tau)$ be a function that satisfies the conditions outlined in Theorem 3.3. Define the approximate solution obtained through the proposed method as:*

$$g_n(\tau) = \sum_{i=0}^{n} \mathscr{M}^i \mathscr{C}^i(\tau).$$

Then, the following inequality holds:

$$\sup_{\tau\in[0,1]} |g(\tau) - g_n(\tau)| \le \frac{3L}{n} + \epsilon^n \|\tilde{\mathscr{M}} - \mathscr{M}\|_2.$$

Proof: Let $g_n(\tau)$, and $\tilde{g}_n(\tau)$, are two different approximate functions according to (3.3) of $g(\tau)$. therefore, we can express the error as follows:

$$|g(\tau) - g_n(\tau)| \le |g(\tau) - \tilde{g}_n(\tau)| + |\tilde{g}_n(\tau) - g_n(\tau)|. \tag{3.20}$$

According to Theorem 3.3, we have

$$|g(\tau) - \tilde{g}_n(\tau)| \le \frac{3L}{n}. \tag{3.21}$$

Additionally, applying the Cauchy–Schwarz inequality, we can write

$$|\tilde{g}_n(\tau) - g_n(\tau)| = \left|\sum_{i=0}^{n} \tilde{\mathscr{M}}^i \mathscr{C}^i(\tau) - \sum_{i=0}^{n} \mathscr{M}^i \mathscr{C}_i(\tau)\right| = \left|\sum_{i=0}^{n} (\tilde{\mathscr{M}}^i - \mathscr{M}^i)\mathscr{C}^i(\tau)\right|,$$

then

$$|\tilde{g}_n(\tau) - g_n(\tau)| \le \sum_{i=0}^{n} |\tilde{\mathscr{M}}^i - \mathscr{M}^i| \cdot \left(\sum_{i=0}^{n} |\mathscr{C}^i(\tau)|^2\right)^{1/2}.$$

From Eqn. (3.19), we have

$$\left(\sum_{i=0}^{n} |\mathscr{C}^i(\tau)|^2\right)^{1/2} \le \epsilon^n = \left(\sum_{i=0}^{n} \frac{2}{\pi}(i+2)^2\right)^{\frac{1}{2}}.$$

Let $\tilde{\mathscr{M}} = [\tilde{\mathscr{M}}^0, \tilde{\mathscr{M}}^1, \dots, \tilde{\mathscr{M}}^n]^T$, and $\mathscr{M} = [\mathscr{M}^0, \mathscr{M}^1, \dots, \mathscr{M}^n]$, then we have:

$$\sum_{i=0}^{n} |\tilde{\mathscr{M}}^i - \mathscr{M}^i| = \left(\sum_{i=0}^{n} |\tilde{\mathscr{M}}^i - \mathscr{M}^i|^2\right)^{\frac{1}{2}} = \|\tilde{\mathscr{M}} - \mathscr{M}\|_2,$$

then,

$$|\tilde{g}_n(\tau) - g_n(\tau)| \le \epsilon^n \|\tilde{\mathscr{M}} - \mathscr{M}\|_2. \tag{3.22}$$

Combining (3.20), (3.21), and (3.22), obtain

$$\sup_{\tau\in[0,1]} |g(\tau) - g_n(\tau)| \le \frac{3L}{n} + \epsilon^n \|\tilde{\mathscr{M}} - \mathscr{M}\|_2.$$

■

Theorem 3.5 *Assume that: ${}_{C}D^{i\varphi}_{0+}g\left(\tau\right)$ is a continuous function through the interval $(0,1]$ where, $i = 0,1,\cdots,n,$ and $\left|{}_{C}D^{i\varphi}_{0+}g\left(\xi\right)\right| \leq \tilde{L}$, where $\tilde{L}$ is a real constant and $0 < \xi \leq \tau$. Consider $\bar{\mathscr{H}_n} = Span\{\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \mathscr{C}^2(\tau), \ldots, \mathscr{C}^n(\tau)\}$ is an n-dimensional subspace. If $g_n\left(\tau\right)$ is the best approximation of the function $g\left(\tau\right)$ out of $\bar{\mathscr{H}_n}$, then the error bound is estimated by:*

$$\parallel g\left(\tau\right) - g_n\left(\tau\right) \parallel_2 \leq \frac{\tilde{L}\zeta^{n,\varphi}}{\Gamma\left(m\alpha+1\right)}, \tag{3.23}$$

where $\zeta^{n,\varphi}$ is defined by:

$$\zeta^{n,\varphi} = \sqrt{\frac{(1+2n\varphi(1+2n\varphi))\sqrt{\pi}\Gamma\left(\frac{1}{2}+2an\right)}{\Gamma(3+2n\varphi)}}, \tag{3.24}$$

which converges for all $\tau \in (0,1]$).

Proof: suppose that: the expansion of the continuous function $g\left(\tau\right)$ through the interval $(0,1]$, in terms of the generalized Taylor's series [32,30], is given as:

$$g\left(\tau\right) = \sum_{i=0}^{n-1} \frac{\tau^{i\varphi}}{\Gamma\left(i\varphi+1\right)}\ {}_{C}D^{\varphi}_{0+}g\left(0\right) + \frac{\tau^{n\varphi}}{\Gamma\left(n\varphi+1\right)}\ {}_{C}D^{\varphi}_{0+}g\left(\xi\right), \tag{3.25}$$

for $\xi \in [0,\tau]$. From the previous definition of the Taylor generalized series we can write:

$$\left|g\left(\tau\right) - \sum_{i=0}^{n-1} \frac{\tau^{i\varphi}}{\Gamma\left(i\varphi+1\right)}\ {}_{C}D^{\varphi}_{0+}g\left(0\right)\right| \leq \frac{\tau^{n\varphi}}{\Gamma\left(n\varphi+1\right)} \left|\ {}_{C}D^{\varphi}_{0+}g\left(\xi\right)\right| = \frac{\tilde{L}\tau^{n\varphi}}{\Gamma\left(n\varphi+1\right)}.$$

Since, $g_n\left(\tau\right)$ is the best approximation of $g\left(t\right)$ in $\bar{\mathscr{H}_n}$, hence, the squared norm (or distance) between the original function $g(\tau)$ and its best approximation $g_n(\tau)$ is less than or equal to the squared norm between $g(\tau)$ and any other function in the subspace $\bar{\mathscr{H}_n}$, then,

$$\parallel g\left(\tau\right) - g_n\left(\tau\right) \parallel_2 \leq \parallel g\left(\tau\right) - \sum_{i=0}^{n-1} \frac{\tau^{i\varphi}}{\Gamma\left(i\varphi+1\right)}\ {}_{C}D^{\varphi}_{0+}g\left(0\right) \parallel_2,$$

or

$$\int_0^1 \left|g\left(\tau\right) - g_n\left(\tau\right)\right|^2 \omega(\tau)d\tau \leq \frac{\tilde{L}^2}{\Gamma\left(n\varphi+1\right)^2} \int_0^1 \tau^{2n\varphi}\omega(\tau)d\tau,$$

in other form

$$\parallel g\left(\tau\right) - g_n\left(\tau\right) \parallel_2 \leq \frac{\tilde{L}}{\Gamma\left(n\varphi+1\right)} \sqrt{\frac{(1+2n\varphi(1+2n\varphi))\sqrt{\pi}\Gamma\left(\frac{1}{2}+2an\right)}{\Gamma(3+2n\varphi)}},$$

then the error bound is:

$$\parallel g\left(\tau\right) - g_n\left(\tau\right) \parallel_2 \leq \frac{\tilde{L}\zeta^{n,\varphi}}{\Gamma\left(n\varphi+1\right)},$$

which proves the theorem.

■

4. Operational matrices of derivative

In this section, we will present the operational matrices for the derivatives of the S5kCPs. Assuming $G(\tau)$ is a square Lebesgue integrable function defined through the interval $[0,1]$, and $\psi(\tau), T(\tau)$ are defined as previous.

Proposition 4.1 *The first order derivative of the vector $\psi(\tau)$ with respect to τ can be represented as:*

$$\frac{d\psi(\tau)}{d\tau} = \mathcal{D}^{(1)}\psi(\tau), \tag{4.1}$$

where, $\mathcal{D}^{(1)}$ is the operational matrix of derivatives, and represented as $(n+1)\times(n+1)$ matrix, defined as:

$$\mathcal{D}^{(1)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \sum_{s=1}^{i}\phi_1(1,0) & \sum_{s=1}^{i}\phi_1(1,1) & \cdots & \sum_{s=1}^{i}\phi_1(1,n) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=1}^{i}\phi_1(i,0) & \sum_{s=1}^{i}\phi_1(i,1) & \cdots & \sum_{s=1}^{i}\phi_1(i,n) \end{pmatrix}, \tag{4.2}$$

where, $\phi_1(s,j) = \alpha^{si}\; s\; q_{j,s-1}$, in addition, α^{si} and $q_{j,s-1}$ are previously defined in (2.23), (2.25).

Proof. The expression for the first order S5kCPs relation (2.22) is given by:

$$D^1\mathscr{C}^i(\tau) = D^1\sum_{s=0}^{i}\alpha^{si}\tau^s = \sum_{s=0}^{i}\alpha^{si}s\tau^{s-1},$$

from the inverse relation (2.21), where, $s \geq 1$,

$$\tau^{s-1} = \sum_{j=0}^{s-1} q_{j,s-1}\mathscr{C}^j(\tau),$$

and $q_{j,s-1}$ defined early in (2.25), then,

$$D^1\mathscr{C}^i(\tau) = \sum_{s=1}^{i}\sum_{j=0}^{s-1}\alpha^{si}\; s\; q_{j,s-1}\mathscr{C}^j(\tau), \tag{4.3}$$

or, in the other form

$$D^1\mathscr{C}^i(\tau) = \sum_{j=0}^{s-1}\sum_{s=1}^{i}\phi_1(s,j)\mathscr{C}^j(\tau) = \mathcal{D}^{(1)}\cdot\psi(\tau),$$

where, $\phi_1(s,j) = \alpha^{si}\; s\; q_{j,s-1}$, we can express the previous equation in vector form as:

$$D^1\mathscr{C}^i(\tau) = \left[\sum_{s=1}^{i}\phi_1(s,j), \sum_{s=1}^{i}\phi_1(s,j), \ldots, \sum_{s=1}^{i}\phi_1(s,j)\right]\psi(\tau), \quad i = 1,2,\ldots,n. \tag{4.4}$$

Also, one clearly can write:

$$D^1\mathscr{C}^i(\tau) = [0,0,\ldots,0]\psi(\tau), \quad at, \quad i = 0. \tag{4.5}$$

By combining Eqs. (4.4) and (4.5), one obtains the required result.

∎

Proposition 4.2 *For any positive integer n, the n-th derivative of the S5kCPs vector $\psi(\tau)$ can be represented using the operational matrix $\mathcal{D}^{(1)}$ (4.2) as:*

$$\frac{d^n\psi(\tau)}{d\tau^n} = \mathcal{D}^{(n)}\psi(\tau), \tag{4.6}$$

where, $\mathcal{D}^{(n)} = (\mathcal{D}^{(1)})^n$, for $n = 1, 2, 3, \ldots$. The matrix $\mathcal{D}^{(n)}$ represents the operational matrix corresponding to the n-th order derivative.

Proposition 4.3 *The first order derivative of $T(\tau) = [T_0^*(\tau), T_1^*(\tau), \ldots, T_n^*(\tau)]^T$, with respect to τ can then be represented as:*

$$\frac{dT(\tau)}{d\tau} = \mathscr{D}^{(1)}T(\tau), \tag{4.7}$$

where, $\mathscr{D}^{(1)}$ is the ordinary operational matrix of derivative, which represented as an $(n+1)\times(n+1)$ matrix, defined as:

$$\mathscr{D}^{(1)} = (d^{ij}) = \begin{cases} \frac{4i}{\epsilon^j}, & \text{for } j = 0, 1, \ldots, \quad i = j + k, \\ k = 1, 3, 5, \ldots, n, & \text{if } n \text{ is odd}, \\ k = 1, 3, 5, \ldots, n-1, & \text{if } n \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}$$

For example, when n is even, $\mathscr{D}^{(1)}$ is given by:

$$\mathscr{D}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & \cdots & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n-1 & 0 & 2n-2 & 0 & 2n-2 & \cdots & 0 & 0 \\ 0 & 2n & 0 & 2n & 0 & \cdots & 2n & 0 \end{pmatrix}.$$

Proposition 4.4 *For any positive integer n, the n-th derivative of the S1kCPs vector $T(\tau)$ can be represented using the operational matrix $\mathscr{D}^{(1)}$ as:*

$$\frac{d^nT(\tau)}{d\tau^n} = \mathscr{D}^{(n)}T(\tau), \tag{4.8}$$

where, $\mathscr{D}^{(n)} = (\mathscr{D}^{(1)})^n$, for $n = 1, 2, 3, \ldots$. The matrix $\mathscr{D}^{(n)}$ represents the operational matrix corresponding to the n-th derivative.

Proposition 4.5 *The left-sided Caputo's fractional derivative of order φ for the vector of S1kCPs, denoted as $T(\tau) = [T_0^*(\tau), T_1^*(\tau), \ldots, T_n^*(\tau)]^T$, is expressed as:*

$$_C D_{0+}^{\varphi} T(\tau) \approx \mathscr{D}^{(\varphi)}T(\tau), \tag{4.9}$$

and the matrix $\mathscr{D}^{(\varphi)}$ is the $(n+1)\times(n+1)$ operational matrix of the fractional derivative in the following form:

$$\mathscr{D}^{(\varphi)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{r=\lceil\varphi\rceil}^{n} \mathcal{S}_{\lceil\varphi\rceil,0,i} & \sum_{r=\lceil\varphi\rceil}^{n} \mathcal{S}_{\lceil\varphi\rceil,1,i} & \cdots & \sum_{r=\lceil\varphi\rceil}^{n} \mathcal{S}_{\lceil\varphi\rceil,j,i} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r=\lceil\varphi\rceil}^{n} \mathcal{S}_{r,0,i} & \sum_{r=\lceil\varphi\rceil}^{n} \mathcal{S}_{m,1,i} & \cdots & \sum_{r=\lceil\varphi\rceil}^{n} \mathcal{S}_{n,j,i} \end{pmatrix}, \tag{4.10}$$

also, the coefficients $\mathcal{S}_{r,j,i}$ is given by:

$$\mathcal{S}_{r,j,i} = \frac{\sqrt{\pi}}{h_j}\sum_{l=0}^{j}\frac{(-1)^{i+r+j+l}i(i+r-1)!j(j+l-1)!2^{2r+2l}\Gamma(r+1)\Gamma(r+l-\varphi+\frac{1}{2})}{(i-r)!(j-l)!(2r)!(2l)!\Gamma(r-\varphi+1)\Gamma(r+l-\varphi+1)}. \tag{4.11}$$

proof. By using Eqns.(2.2) and (2.20) we get

$$\begin{aligned} {}_C D^{\varphi}_{0+} T^*_i(\tau) &= i\textstyle\sum_{r=0}^{i}(-1)^{r+i}\frac{2^{2r}(r+i-1)!}{(2r)!(i-r)!}\, {}_C D^{\varphi}_{0+}\tau^r \\ &= i\textstyle\sum_{r=0}^{i}(-1)^{r+i}\frac{2^{2r}(r+i-1)!}{(2r)!(i-r)!}\cdot\frac{\Gamma[r+\varphi]}{\Gamma[r+\varphi-1]}\tau^{r-\varphi}, \qquad r=\lceil\varphi\rceil,\dots,n. \end{aligned} \tag{4.12}$$

Next, we approximate $\tau^{r-\varphi}$ using a shifted Chebyshev polynomial series consisting of $(n+1)$ terms, we have

$$\tau^{r-\alpha} \approx \sum_{j=0}^{n} A_{r,j}T^*_j(\tau), \tag{4.13}$$

where,

$$\begin{aligned} A_{r,j} &= \frac{1}{h_j}\int_0^1 \tau^{r-\varphi}T^*_j(\tau)\,d\tau = \frac{1}{h_j}\sum_{l=0}^{j}(-1)^{l+j}\frac{2^{2l}(l+j-1)!}{(2l)!(j-l)!}\int_0^1\frac{1}{\sqrt{\tau-\tau^2}}\tau^{r-\varphi+l}\,d\tau, \\ &= \frac{\sqrt{\pi}}{h_j}\sum_{l=0}^{j}(-1)^{l+j}\frac{2^{2l}(l+j-1)!\,\Gamma(r-\varphi+l+\frac{1}{2})}{(2l)!\,(j-l)!\,\Gamma(r-\varphi+l+1)}, \end{aligned} \tag{4.14}$$

and $h_j = \dfrac{\epsilon_j\pi}{2}$, such that $\epsilon_j = 1$ for $k \geq 1$ and $\epsilon_0 = 2$. Applying Eqs. (4.12)–(4.14), then, we get:

$$\begin{aligned} {}_C D^{\varphi}_{0+} T^*_i(\tau) &\approx \sum_{r=\lceil\varphi\rceil}^{i}\sum_{j=0}^{n}(-1)^{r+i}\frac{2^{2r}i(r+i-1)!}{(2r)!(i-r)!}\frac{\Gamma(r+\varphi)}{\Gamma(r+\varphi-1)}A_{r,j}\ \ T^*_j(\tau) \\ &= \sum_{j=0}^{n}(\sum_{r=\lceil\varphi\rceil}^{i}\mathcal{S}_{r,j,i})T^*_j(\tau) = M\cdot T^*_j(\tau), \quad i=\lceil\alpha\rceil,\dots,n, \end{aligned} \tag{4.15}$$

where, $\mathcal{S}_{r,j,i}$ is defined in Eq.(4), we can express Eq. (4.15) in vector form as:

$${}_C D^{\varphi}_{0+} T^*_i(\tau) \approx \left[\sum_{r=\lceil\alpha\rceil}^{i}\mathcal{S}_{r,0,i}, \sum_{r=\lceil\varphi\rceil}^{i}\mathcal{S}_{r,1,i},\dots,\sum_{r=\lceil\varphi\rceil}^{i}\mathcal{S}_{r,j,i}\right]\Phi(\tau), \quad i=\lceil\varphi\rceil,\dots,n. \tag{4.16}$$

Also, we can write:

$${}_C D^{\varphi}_{0+} T^*_k(\tau) = [0,0,\dots,0]\Phi(\tau), \quad i=0,1,\dots,\lceil\varphi\rceil-1. \tag{4.17}$$

By combining Eqs. (4.16) and (4.17), obtain the required result.

■

Numerous prior works have demonstrated the construction of operational matrices for Caputo's fractional derivatives. See [33] for an illustration.

Proposition 4.6 *The ordinary and fractional derivatives operate the well-defined function $G(\tau)$, which, approximated in terms of the S5kCPs as: $G(\tau) = \mathscr{M}^T\psi(\tau)$, (3.3), according to the operational matrices of derivatives given in (4.6) and (3.12) as:*

$$D^n G(\tau) = \mathscr{M}^T\cdot\mathcal{D}^{(n)}\cdot\psi(\tau), \tag{4.18}$$

$${}_C D^{\varphi}_{0+} G(\tau) = \mathscr{M}^T\cdot\mathcal{D}^{(\varphi)}\cdot\psi(\tau). \tag{4.19}$$

Proof. Since, the approximation of well-defined function in (3.3)

$$G(\tau) = \mathscr{M}^T \psi(\tau),$$

$$D^n G(\tau) = \mathscr{M}^T D^n \psi(\tau) = \mathscr{M}^T \cdot \mathcal{D}^{(n)} \cdot \psi(\tau).$$

Similarly,

$${}_C D^{\varphi}_{0+} G(\tau) = \mathscr{M}^T_C D^{\varphi}_{0+} \psi(\tau) = \mathscr{M}^T \cdot \mathcal{D}^{(\varphi)} \cdot \psi(\tau).$$

∎

Proposition 4.7 *The ordinary and fractional derivatives operate the well-defined function* $G(\tau) = \mathscr{M}^T \psi(\tau)$, (3.3) *according to the operational matrices* (4.8) *and* (4.9) *and using the connection relation* (3.5) *given as:*

$$D^n G(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(n)} \cdot (\mathscr{B}^n)^{-1} \psi(\tau). \tag{4.20}$$

$${}_C D^{\varphi}_{0+} G(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(\varphi)} \cdot (\mathscr{B}^n)^{-1} \psi(\tau). \tag{4.21}$$

Proof. Since, the approximation of the well-defined function in (3.3)

$$G(\tau) = \mathscr{M}^T \psi(\tau),$$

$$D^n G(\tau) = \mathscr{M}^T D^n \psi(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot D^n T(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(n)} T(\tau),$$

or

$$D^n G(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(n)} \cdot (\mathscr{B}^n)^{-1} \psi(\tau).$$

Similarly,

$${}_C D^{\varphi}_{0+} T(\tau) \approx \mathscr{D}^{(\varphi)} T(\tau),$$

$${}_C D^{\varphi}_{0+} G(\tau) = \mathscr{M}^T \cdot {}_C D^{\varphi}_{0+} \psi(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot {}_C D^{\varphi}_{0+} T(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(\varphi)} \cdot T(\tau).$$

or

$${}_C D^{\varphi}_{0+} G(\tau) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(\varphi)} \cdot (\mathscr{B}^n)^{-1} \psi(\tau).$$

∎

As demonstrated in Propositions 4.6 and 4.7, equivalent relationships can be established using distinct matrices. These relationships not only support the following remark but also provide a framework for developing two distinct schemes to approximate the function $G(\tau)$ and its derivatives of both integer and fractional orders.

Remark 4.1 Using the connection relation (3.5), according to the operational matrices of derivatives given in (4.1), (3.12), (4.8), and (4.9) the following matrices are equivalent.

$$\begin{aligned} \mathcal{D}^{(\varphi)} &= \mathscr{B}^n \cdot \mathscr{D}^{(\varphi)} \cdot (\mathscr{B}^n)^{-1}, \\ \mathcal{D}^{(n)} &= \mathscr{B}^n \cdot \mathscr{D}^{(n)} \cdot (\mathscr{B}^n)^{-1}. \end{aligned} \tag{4.22}$$

These formulations provide a robust framework for analyzing and solving FDEs involving the S5kCPs in the matrix computational discretization technique.

5. Linear and Nonlinear FDEs

In this section, we demonstrate the significance of the operational matrices (4.18), (4.19), (4.20), and (4.21) for ordinary and fractional derivatives by applying them to solve some types of FDEs using matrix tau approach.

5.1. Linear FDEs

For the equation to be linear, each term involving $G(\tau)$ and its derivatives should only appear to the first power, and products of $G(\tau)$ or its derivatives must not exist. If we assume that the equation can be expressed without products of dependent variables, a linear representation could resemble the following [34]:

$$\sum_{h=0}^{n_1} P_h(\tau)G^{(\varphi_h)}(\tau) = f(\tau), \tag{5.1}$$

and the suggested initial conditions are:

$$G^{(\varphi_h)}(\tau_h) = \xi_h, \qquad h = 0, \ldots, (n_1 - 1), \tag{5.2}$$

where, $f(\tau)$ is well-defined function, and, $G^{(\varphi_h)}(\tau) = {}_CD_{0+}^{\varphi_h}G(\tau)$, to solve the linear Eqn. (5.1), we approximate $G(\tau)$ using S5kCPs as:

$$G(\tau) \approx \sum_{j=0}^{n} \mathscr{M}^j\mathscr{C}^j(\tau), = \mathscr{M}^T\psi(\tau), \tag{5.3}$$

where $\mathscr{M}^j$ represents the coefficients vector, $\mathscr{M} = [\mathscr{M}^0, \mathscr{M}^1, \ldots, \mathscr{M}^n]^T$, is the vector of unknown coefficients, and $\psi(\tau) = [\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \ldots, \mathscr{C}^n(\tau)]^T$, is the vector of S5kCPs, therefore, $n > n_1$.
Using the operational matrices given in propositions 4.6 and 4.7 for fractional derivatives, Using tau's numerical approach, the residual can be constructed as follows:

$$R_n(\tau) = \sum_{h=0}^{n_1} P_h(\tau)\, {}_CD_{0+}^{\varphi_h}\mathscr{M}\psi(\tau) - f(\tau), \tag{5.4}$$

or,

$$R_n(\tau) = \sum_{h=0}^{n_1} P^h \cdot \mathscr{M}^T \cdot \mathcal{D}^{(\varphi_h)}\psi(\tau) - F^T \cdot \psi(\tau), \tag{5.5}$$

$$R_n(\tau) = \sum_{h=0}^{n_1} P^h \cdot \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(\varphi_h)} \cdot (\mathscr{B}^n)^{-1}\psi(\tau) - F^T \cdot \psi(\tau), \tag{5.6}$$

such that,

$$P^h = \begin{pmatrix} P_h(\tau) & 0 & \cdots & 0 \\ 0 & P_h(\tau) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_h(\tau) \end{pmatrix}, \qquad F = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}, \qquad f_j = \int_0^1 f(\tau)w(\tau)\mathscr{C}^j d\tau.$$

By applying the tau method by enforcing the orthogonality of the residual:

$$\langle R_n(\tau), \mathscr{C}^j(\tau)\rangle = \int_0^1 R_n(\tau)\mathscr{C}^j(\tau)w(\tau)\, d\tau = 0, \quad j = 0, 1, \ldots, (n - n_1 - 1). \tag{5.7}$$

In addition, $(n_1 - 1)$ equations are produced directly from the initial conditions (5.2), and using the formation of the operational matrices as shown below:

$$G^{(\varphi_h)}(\tau_h) = \mathscr{M}^T \cdot \mathcal{D}^{(\varphi_h)} \cdot \psi(\tau_h) = [\xi_h], \qquad h = 0,\ 1,\ 2,\ \ldots, (n_1 - 1), \tag{5.8}$$

or, using the second scheme as:

$$G^{(\varphi_h)}(\tau_h) = \mathscr{M}^T \cdot \mathscr{B}^n \cdot \mathscr{D}^{(\varphi_h)} \cdot (\mathscr{B}^n)^{-1}\psi(\tau_h) = [\xi_h], \qquad h = 0,\ 1,\ 2,\ \ldots, (n_1 - 1). \tag{5.9}$$

Solving the system (5.7) together with (5.8) or (5.9), yields the unknown coefficients $\mathscr{M}^T$, and the approximate solution is then obtained from Eq. (5.3).

5.2. Nonlinear FDEs

For the nonlinear case, the equation retains the nonlinear interactions between $G(\tau)$ and its fractional derivatives [35]:

$$\sum_{k=0}^{n_1}\sum_{h=0}^{n_2} Q_{k,h}(\tau)G^k(\tau)G^{(\nu_h)} + \sum_{k=1}^{n_3}\sum_{h=0}^{n_4} P_{k,h}(\tau)G^{(h)}(\tau)g^{(\varphi_h)} = f(\tau). \tag{5.10}$$

and the connected conditions are:

$$G(\tau_h) = \xi_h, \qquad h = 0,\ 1,\ 2,\ \ldots, (n_l - 1). \tag{5.11}$$

As in the linear case, we approximate $G(\tau)$, $G^{(h)}(\tau)$, $G^{(\nu_h)}(\tau)$, and $G^{(\varphi_h)}(\tau)$, using S5kCPs via two proposed schemes, according to tau approach, then, the residual for nonlinear case (5.10), (5.11) is given by:

$$R_n(\tau) = \sum_{k=0}^{n_1}\sum_{h=0}^{n_2} Q_{k,h}(\tau)(\mathscr{M}^T\psi(\tau))^k \mathscr{M}^T\ {}_C D_{0+}^{(\nu_h)}\psi(\tau) + \sum_{k=1}^{n_3}\sum_{h=0}^{n_4} P_{k,h}(\tau)\mathscr{M}^T D^{(h)}\psi(\tau)\mathscr{M}_C^T D_{0+}^{(\varphi_h)}\psi(\tau) - f(\tau).$$

Using the relations found in propositions 4.6, and 4.7 we get:

$$R_n(\tau) = \sum_{k=0}^{n_1}\sum_{h=0}^{n_2} Q^{k,h}\cdot(\mathscr{M}^T\psi(\tau))^k\cdot\mathscr{M}^T\cdot\mathcal{D}^{(\nu_h)}\psi(\tau) + \sum_{k=1}^{n_3}\sum_{h=0}^{n_4} P^{k,h}\cdot\mathscr{M}^T\cdot\mathcal{D}^{(h)}\psi(\tau)\cdot\mathscr{M}^T\cdot\mathcal{D}^{(\varphi_h)}\psi(\tau) - F\cdot\psi(\tau). \tag{5.12}$$

and,

$$\begin{gathered} R_n(\tau) = \textstyle\sum_{k=0}^{n_1}\sum_{h=0}^{n_2} Q^{k,h}\cdot(\mathscr{M}^T\psi(\tau))^k\cdot\mathscr{M}^T\cdot\mathscr{B}^n\cdot\mathscr{D}^{(\nu_h)}\cdot(\mathscr{B}^n)^{-1}\psi(\tau) \\ +\textstyle\sum_{k=1}^{n_3}\sum_{h=0}^{n_4} P^{k,h}\cdot\mathscr{M}^T\cdot\mathscr{B}^n\cdot\mathscr{D}^{(h)}(\mathscr{B}^n)^{-1}\cdot\psi(\tau)\cdot\mathscr{M}^T\cdot\mathscr{B}^n\cdot\mathscr{D}(\varphi_h)\cdot(\mathscr{B}^n)^{-1}\psi(\tau) - F\cdot\psi(\tau). \end{gathered} \tag{5.13}$$

Therefore,

$$Q^{k,h} = \begin{pmatrix} Q_{k,h}(\tau) & 0 & \cdots & 0 \\ 0 & Q_{k,h}(\tau) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & Q_{k,h}(\tau) \end{pmatrix}, \quad P^{k,h} = \begin{pmatrix} P_{k,h}(\tau) & 0 & \cdots & 0 \\ 0 & P_{k,h}(\tau) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_{k,h}(\tau) \end{pmatrix},$$

$$F = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}, \qquad f_j = \int_0^1 f(\tau)w(\tau)\mathscr{C}^j\,d\tau.$$

Since this is a nonlinear system, we apply iterative techniques, to minimize the residual and solve for the unknown coefficient vector $\mathscr{M}$. Applying the tau method leads us to enforce the orthogonality condition in the same manner:

$$\langle R_n(\tau), \mathscr{C}^j(\tau)\rangle = \int_0^1 R_n(\tau)\mathscr{C}^j(\tau)w(\tau)\,d\tau = 0, \quad j = 0, 1, \ldots, (n - n_l - 1). \tag{5.14}$$

Eqn. (5.14) produces $(n-n_l-1)$ non-linear algebric equations, such that n_l is the greatest for $l = 0,\ 1,\ 2, 3$ and 4, and it is noted that $n > n_l$. In a similar case we can construct the system using (5.14) and the conditions (5.11) as:

$$G(\tau_h) = \mathscr{M}^T\cdot\psi(\tau_h) = [\xi_h], \qquad h = 0,\ 1,\ 2,\ \ldots, (n_l - 1).$$

This results in a system of nonlinear equations that can be solved iteratively, to find the coefficient vector $\mathscr{M}$.

6. Test Examples

In this section, we introduce a test cases to showcase the precision and effectiveness of the proposed method utilizing S5kCPs. The method is applied to a fractional differential equation, with the numerical results compared to the exact solution. The evaluation of the method's performance is conducted through comprehensive error analysis and an assessment of computational efficiency.

Example 1. Consider the initial value problem of Bagley Torvik equation [36,37,38,39].
The general form of the Bagley-Torvik equation is:

$$D^2 G(\tau) + D^{(\varphi)} G(\tau) + G(\tau) = f(\tau), \quad 0 \leq \tau \leq 1, \tag{6.1}$$

subject to the initial conditions:

$$G(0) = 0, \quad G'(0) = 0. \tag{6.2}$$

The Bagley–Torvik equation is solved using the tau method for three different cases with the results compared to other numerical methods to assess the proposed approach's reliability. The method is applied to the problem for $\varphi = \frac{3}{2}$, within the interval $[0, 1]$.

Case 1: When $f(\tau) = 2 + 4\sqrt{\frac{\tau}{\pi}} + \tau^2$.
By applying the method described in Section 5.1 with $n = 3$, we approximate the function $G(\tau)$ using S5kCPs, represented by the following series expansion:

$$G(\tau) \approx \sum_{j=0}^{n} \mathscr{M}^j \mathscr{C}^j(\tau), = \mathscr{M}^T \psi(\tau),$$

where the vector $\psi(\tau) = [\mathscr{C}^0(\tau), \mathscr{C}^1(\tau), \ldots, \mathscr{C}^n(\tau)]^T$ is the vector of S5kCPs, therefore, $n > n_1$, and the coefficient vector $\mathscr{M}^j$ is given by:

$$\mathscr{M} = [\mathscr{M}^0, \mathscr{M}^1, \ldots, \mathscr{M}^n]^T.$$

From Eqns. (5.5-5.7) the residual of the fractional differential equation is formulated as:

$$R_n(\tau) = \sum_{h=0}^{n_1} P^h [\mathscr{M} \cdot \mathscr{B}^n \cdot \mathscr{D}^2 \cdot (\mathscr{B}^n)^{-1} \cdot \psi(\tau) + \mathscr{M} \cdot \mathscr{B}^n \cdot \mathscr{D}^{\frac{3}{2}} \cdot (\mathscr{B}^n)^{-1} \cdot \psi(\tau) + \mathscr{M} \cdot \psi(\tau) - F^T \cdot \psi(\tau)].$$

Here, we have

$$\mathscr{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 6 & 0 & 12 & 0 \end{pmatrix},$$

$$\mathscr{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 \end{pmatrix},$$

$$\mathscr{D}^{\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{64}{\pi^{3/2}} & \frac{128}{3\pi^{3/2}} & -\frac{128}{15\pi^{3/2}} & \frac{128}{35\pi^{3/2}} \\ -\frac{128}{3\pi^{3/2}} & \frac{768}{5\pi^{3/2}} & \frac{768}{7\pi^{3/2}} & -\frac{256}{9\pi^{3/2}} \end{pmatrix},$$

$$\mathscr{B}^3 = \begin{pmatrix} \sqrt{\frac{2}{\pi}} & 0 & 0 & 0 \\ 0 & 2\sqrt{\frac{2}{3\pi}} & 0 & 0 \\ -\sqrt{\frac{2}{\pi}} & 0 & 2\sqrt{\frac{2}{\pi}} & 0 \\ 0 & -2\sqrt{\frac{2}{15\pi}} & 0 & 2\sqrt{\frac{6}{5\pi}} \end{pmatrix},$$

and

$$\psi(\tau) = \begin{pmatrix} \sqrt{\frac{2}{\pi}} \\ 2\sqrt{\frac{2}{3\pi}}\,(-1+2x) \\ 2\sqrt{\frac{2}{\pi}}\left(-\frac{3}{2}+2(-1+2x)^2\right) \\ 2\sqrt{\frac{2}{15\pi}}\left(1-2x+3(-3(-1+2x)+4(-1+2x)^3)\right) \end{pmatrix}.$$

So that:

$$P_0(\tau) = 1, \quad P_1(\tau) = 1, \quad P_2(\tau) = 1.$$

Table (1) provides a comparison of the exact and numerical solutions, along with the absolute errors for different methods at $n = 5$. The results demonstrate that the S5KCPs achieves superior accuracy, with the absolute errors remaining exceptionally small across all values of τ, where the exact solution is $G(\tau) = \tau^2$. In contrast, the other methods show relatively larger errors, particularly at intermediate values of τ, highlighting the high precision and reliability of the S5KCPs for solving the given problem. Figure (1) presents a clear comparison between the exact solution and the numerical solution obtained through the proposed method. As shown in the plot, the numerical solution closely matches the exact solution, indicating the high accuracy and effectiveness of the method in approximating the solution of the fractional differential equation. Figure(2) illustrates the comparison between the absolute error and the values of τ. As observed, the absolute error decreases as τ increases, demonstrating the convergence of the numerical method with respect to the chosen parameter. This behavior emphasizes the efficiency of the S5kCPs in reducing the error.

Table 1: Comparison of exact and numerical solutions along with absolute errors for different methods, for example 1, case 1.

τ	Exact Solution	Numerical Solution	S5KCPs, $n = 5$	[37]	[38]	[39]
0.0	0.0	3.87×10^{-16}	3.87×10^{-16}	-	-	-
0.1	0.01	0.0100	3.97×10^{-16}	8.74×10^{-9}	9.64×10^{-11}	1.40×10^{-13}
0.2	0.04	0.0400	3.89×10^{-16}	8.17×10^{-9}	3.86×10^{-10}	5.60×10^{-13}
0.3	0.09	0.0900	3.88×10^{-16}	8.17×10^{-9}	8.67×10^{-10}	1.26×10^{-12}
0.4	0.16	0.1600	3.88×10^{-16}	8.34×10^{-9}	1.54×10^{-9}	2.24×10^{-12}
0.5	0.25	0.2500	3.88×10^{-16}	8.59×10^{-9}	2.41×10^{-9}	3.50×10^{-12}
0.6	0.36	0.3600	4.44×10^{-16}	8.90×10^{-9}	3.47×10^{-9}	5.04×10^{-12}
0.7	0.49	0.4900	3.33×10^{-16}	9.28×10^{-9}	4.72×10^{-9}	6.87×10^{-12}
0.8	0.64	0.6400	3.33×10^{-16}	9.72×10^{-9}	6.17×10^{-9}	8.97×10^{-12}
0.9	0.81	0.8100	3.33×10^{-16}	1.02×10^{-8}	7.81×10^{-9}	1.14×10^{-11}
1.0	1.00	1.0000	4.44×10^{-16}	1.09×10^{-8}	9.64×10^{-9}	1.40×10^{-11}

Case 2: Here, $f(\tau)$ is selected such that the exact solution $G(\tau) = \sin(\alpha\tau)$ [40,41,42]. Table (2) shows that increasing n improves accuracy, with $n = 12$ yielding the smallest errors when $\alpha = 1$. While errors increase with larger τ, higher n consistently ensures better precision, emphasizing the advantages of higher-order methods. Table (3) highlights the superior accuracy of the S5kCPs method, with the the mean absolute error (MAE) approaching machine precision as n increases. The another methods exhibit larger errors, particularly for higher values of φ. Figure (3) shows the reduction in absolute error as n increases, demonstrating that higher values of n lead to more accurate numerical solutions.

Case 3: In this scenario, the function $f(\tau) = 5\tau + \frac{8\tau^{\frac{3}{2}}}{\sqrt{\pi}} + \tau^3$ is selected, resulting in the exact solution $G(\tau) = \tau^3 - \tau$, with the initial conditions $G(0) = 0$ and $G'(0) = -1$. Table (4) presents a comparison of error values for four distinct numerical techniques with $n = 5$ and varying values of τ. The results indicate that the S5KCPs technique consistently produces the smallest errors at each point, significantly

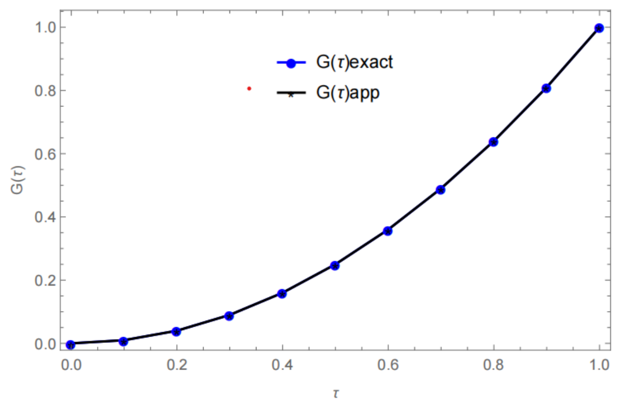

Figure 1: Comparison between the exact and the numerical solution for example 1, case 1.

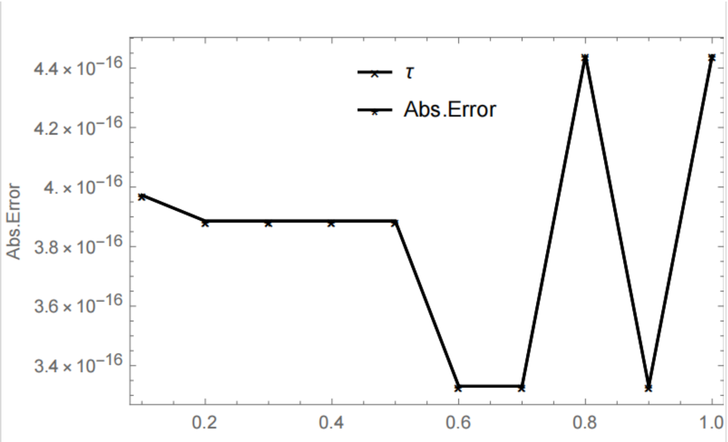

Figure 2: Comparison between Absolute error and τ values for example 1, case 1.

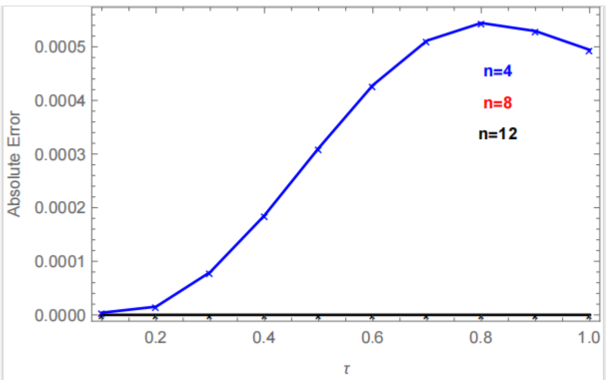

Figure 3: Absolute analysis with different n values for example 1, case 2

Table 2: Absolute errors for $n = 4$, $n = 8$, and $n = 12$ at various τ values, along with the exact solution, for example 1, case 2.

τ	Exact Solution	$n = 4$	$n = 8$	$n = 12$
0.0	0.000000	1.006×10^{-16}	1.084×10^{-17}	3.642×10^{-17}
0.1	0.0998334	4.065×10^{-6}	2.766×10^{-11}	1.640×10^{-11}
0.2	0.198669	1.532×10^{-5}	1.392×10^{-11}	1.287×10^{-11}
0.3	0.29552	7.998×10^{-5}	6.299×10^{-11}	1.418×10^{-11}
0.4	0.389418	1.855×10^{-4}	1.137×10^{-10}	8.602×10^{-11}
0.5	0.479426	3.110×10^{-4}	6.064×10^{-10}	2.419×10^{-10}
0.6	0.564642	4.284×10^{-4}	1.062×10^{-9}	1.194×10^{-9}
0.7	0.644218	5.115×10^{-4}	1.166×10^{-9}	2.377×10^{-9}
0.8	0.717356	5.443×10^{-4}	1.045×10^{-9}	3.339×10^{-9}
0.9	0.783327	5.293×10^{-4}	1.021×10^{-9}	4.166×10^{-9}
1.0	0.841471	4.943×10^{-4}	1.031×10^{-9}	4.965×10^{-9}

Table 3: Comparison of MAE between the methods for example 1, case 2.

n	$\alpha = 1$			$\alpha = 4\pi$		
	S5KCPs	FTM [41]	CSM [42]	S5KCPs	FTM [41]	CSM [42]
4	5.4×10^{-4}	2.7×10^{-4}	3.4×10^{-4}	2.6×10^{-2}	2.5×10^{-2}	3.9×10^{0}
8	1.1×10^{-9}	3.5×10^{-7}	4.3×10^{-7}	4.2×10^{-9}	3.5×10^{-4}	4.7×10^{-1}
16	7.9×10^{-16}	4.2×10^{-10}	1.8×10^{-8}	2.3×10^{-16}	4.2×10^{-9}	3.5×10^{-5}

outperforming the other approaches. The error values for S5KCPs are smaller than those for the other methods, demonstrating its superior accuracy and effectiveness. While the errors generally increase as τ grows, S5KCPs remains the most precise even at higher values of τ, making it a highly reliable and efficient solution. Figure (4) illustrates the comparison between the exact solution and the numerical solution obtained using the proposed method, in addition, figure (5) shows the differences between the numerical and the exact solution. This indicates the high accuracy and effectiveness of the numerical method in approximating the exact solution.

Table 4: Comparison of Error Values for Different Methods with $n = 5$ for example 1, case 3.

	$n = 5$			
τ	[37]	[38]	[39]	S5KCPs
0.1	1.04×10^{-7}	1.85×10^{-8}	2.9432×10^{-13}	8.32×10^{-17}
0.2	1.41×10^{-7}	3.71×10^{-8}	2.5719×10^{-12}	2.78×10^{-17}
0.3	1.76×10^{-7}	5.59×10^{-8}	8.8158×10^{-12}	0.0
0.4	2.19×10^{-7}	7.51×10^{-8}	2.1010×10^{-11}	0.0
0.5	2.75×10^{-7}	9.48×10^{-8}	4.1136×10^{-11}	5.55×10^{-17}
0.6	3.49×10^{-7}	1.15×10^{-7}	7.1180×10^{-11}	1.11×10^{-16}
0.7	4.47×10^{-7}	1.36×10^{-7}	1.1312×10^{-10}	5.55×10^{-17}
0.8	5.77×10^{-7}	1.58×10^{-7}	1.6895×10^{-10}	1.67×10^{-16}
0.9	7.49×10^{-7}	1.81×10^{-7}	2.4064×10^{-10}	1.38×10^{-16}

Example 2. Consider the non-linear fractional differential equation [43]:

$$D^{\varphi}G(\tau) - 6G^2(\tau) = f(\tau), \tag{6.3}$$

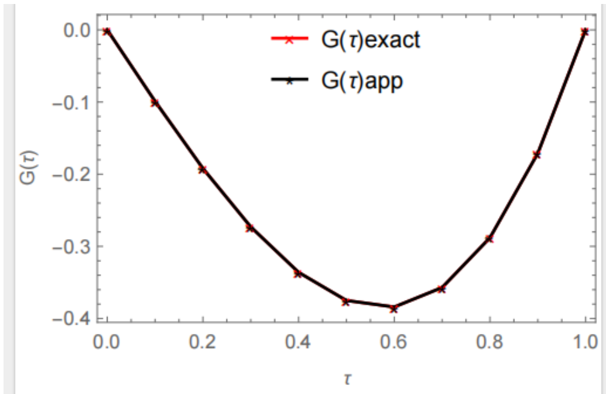

Figure 4: The exact solution values represent the analytical solution for the given τ for example 1, case 3.

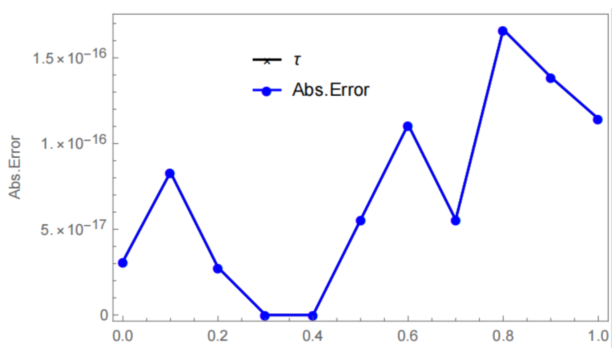

Figure 5: The differences between the numerical and exact solution for example 1, case 3.

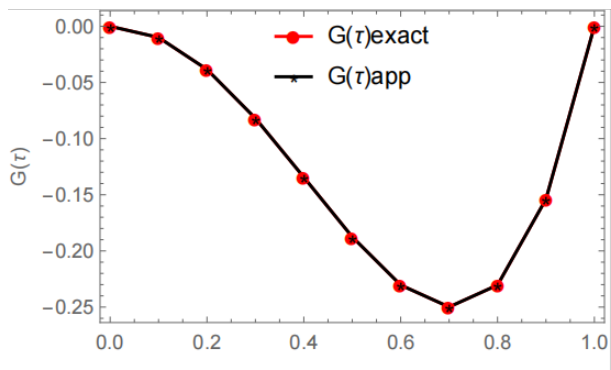

Figure 6: Comparison of the exact and approximate solutions for example 2.

with the initial conditions:

$$G(0) = 0, \quad G'(0) = 1. \tag{6.4}$$

This example investigates a fractional differential equation with specific initial conditions and a non-homogeneous term to produce the exact solution

$$G(\tau) = \tau^{2\varphi} - \tau^2.$$

Table (5) presents a comparison of absolute errors obtained using the S5kCPs and another numerical method, highlighting the accuracy of S5KCPs for $\varphi = 2$. Figure (6) illustrates a comparison between the exact solution $G(\tau)$ and the approximate solution $G_{\text{app}}(\tau)$, showing agreement across the interval $[0, 1]$. The minimal deviation highlights the high accuracy and stability of the approximation method. Figure (7) illustrates a comparison between the absolute errors for two different methods across the interval $[0, 1]$. The minimal deviation between the methods demonstrates the high accuracy and stability of the S5KCPs method compared to the other method, highlighting the effectiveness of the approximation in capturing the exact solution's behavior.

Table 5: Comparison of absolute errors for example 2 with $\varphi = 2$.

τ	Exact Solution	Absolute Error (S5KCPs)	Method [43]
0.0	0.0000	6.93×10^{-18}	0.0
0.1	-0.0099	1.04×10^{-17}	2.83×10^{-5}
0.2	-0.0384	4.16×10^{-17}	2.36×10^{-5}
0.3	-0.0819	2.77×10^{-17}	2.02×10^{-5}
0.4	-0.1344	2.77×10^{-17}	8.19×10^{-5}
0.5	-0.1875	8.32×10^{-17}	1.81×10^{-5}
0.6	-0.2304	1.11×10^{-16}	1.73×10^{-5}
0.7	-0.2499	1.11×10^{-16}	1.65×10^{-5}
0.8	-0.2304	1.66×10^{-16}	1.91×10^{-5}
0.9	-0.1539	1.38×10^{-16}	2.36×10^{-5}
1.0	-9.02×10^{-17}	9.02×10^{-17}	2.53×10^{-5}

Example 3: Consider the following boundary-value problem [36]:

$$D^{3/2}G(\tau) + G(\tau) = \tau^5 - \tau^4 + \frac{128\tau^{3.5}}{7\sqrt{\pi}} - \frac{64\tau^{2.5}}{5\sqrt{\pi}}, \tag{6.5}$$

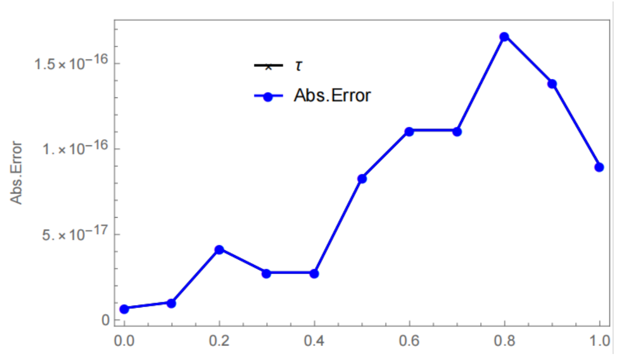

Figure 7: Absolute Errors for example 2 with $\varphi = 2$.

subject to the boundary conditions:

$$G(0) = 0 \quad \text{and} \quad G(1) = 1, \quad (0 \le \tau \le 1).$$

We solve the boundary-value problem using the introduced method , with the exact solution given by $G(\tau) = \tau^4(\tau - 1)$. The results are presented in Table (6) highlights the accuracy of the S5kCPs method, demonstrating its capacity to consistently yield very small absolute errors over the entire interval. This indicates that the S5kCPs method is a reliable and efficient technique for solving such problems. Furthermore, figures (8), (9) give comparison between exact and numerical solution and the absolute error.

Table 6: A comparison of the results for example 3 when $n = 5$

τ	Exact Solution $G(\tau)$	Absolute Error (S5kCPs)	Method [36]
0.0	0.0	1.31×10^{-16}	0.0
0.1	-0.00009	1.16×10^{-16}	2×10^{-6}
0.2	-0.00128	8.04×10^{-17}	3×10^{-6}
0.3	-0.00567	5.72×10^{-17}	1×10^{-6}
0.4	-0.01536	2.08×10^{-17}	0.0
0.5	-0.03125	6.93×10^{-18}	2×10^{-6}
0.6	-0.05184	4.16×10^{-17}	1×10^{-6}
0.7	-0.07203	6.93×10^{-17}	3×10^{-6}
0.8	-0.08192	9.71×10^{-17}	2×10^{-6}
0.9	-0.06561	1.52×10^{-16}	0×10^{-6}
1.0	-1.2837×10^{-16}	1.28×10^{-16}	2×10^{-6}

Example 4. The following fractional boundary value problem of the Bagley–Torvik equation [43,44]:

$$G''(\tau) + 0.5D^{0.5}G(\tau) + G(\tau) = 3 + \frac{4\tau^{3/2}}{3\sqrt{\pi}} + \tau^2, \tag{6.6}$$

with the boundary conditions:

$$G(0) = 1, \quad G(1) = 2.$$

Table (7) presents a comparison of the absolute errors for two methods (S5KCPs and another method) with the exact solution $G(\tau) = \tau^2 + 1$. The presented method consistently produces zero absolute errors for all values of τ, highlighting its exceptional accuracy. On the other hand, the other method exhibits nonzero errors, indicating that S5kCPs outperforms it for solving this problem. Finally, figures (10), (11) show the comparison between exact and S5kCPs solution and the absolute error.

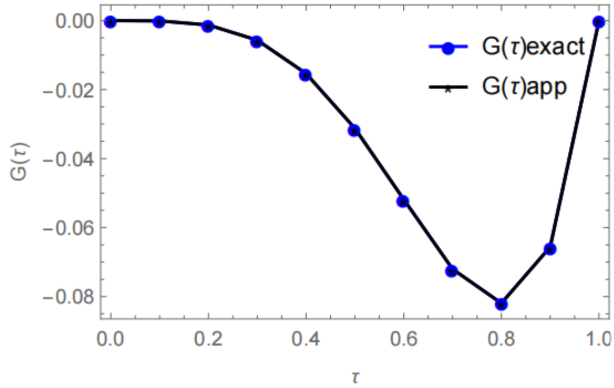

Figure 8: comparison between exact and numerical solution for example 3.

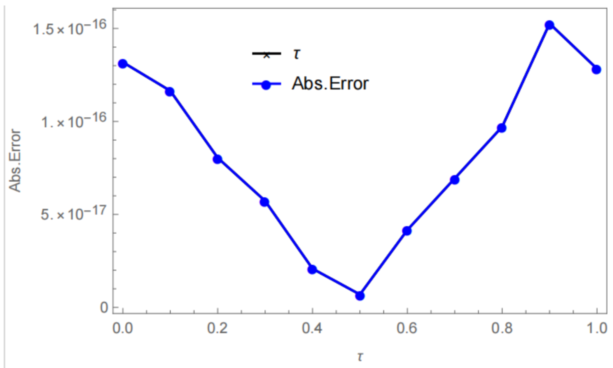

Figure 9: comparison between absolute error and the values of τ for example 3.

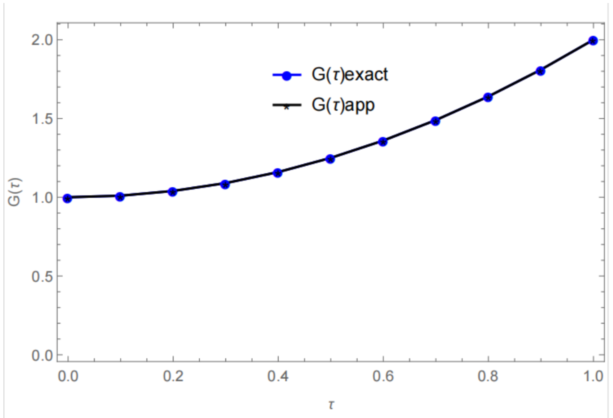

Figure 10: Numerical and exact solution for example 4.

Table 7: A comparison of the results for example 4 when $n = 8$

τ	Exact Solution $G(\tau)$	Absolute Error (S5KCPs)	Absolute Error [44]
0.0	1.0	0.0	0.0
0.1	1.01	2.22×10^{-16}	1.93×10^{-12}
0.2	1.04	0.0	3.16×10^{-11}
0.3	1.09	0.0	3.67×10^{-10}
0.4	1.16	2.22×10^{-16}	3.66×10^{-9}
0.5	1.25	0.0	3.30×10^{-9}
0.6	1.36	0.0	2.74×10^{-9}
0.7	1.49	0.0	2.09×10^{-10}
0.8	1.64	0.0	1.40×10^{-11}
0.9	1.81	0.0	7.00×10^{-12}
1.0	2.0	0.0	0.0

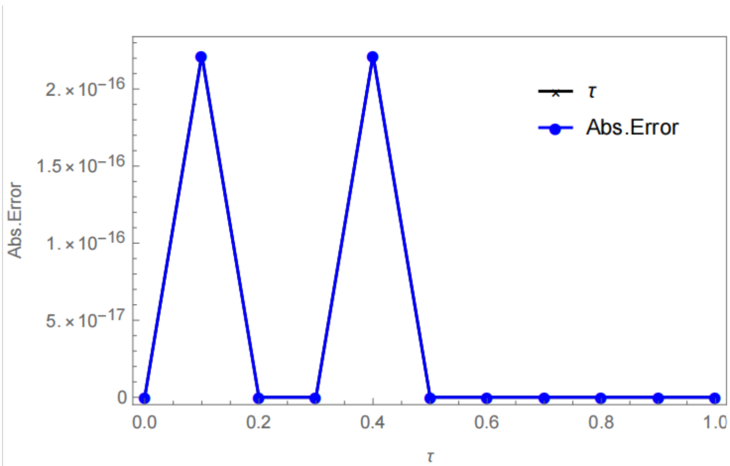

Figure 11: Absolute errors for example 4.

Conclusion

In this work, we have presented an efficient approach using S5kCPs via matrix computational tau method for solving FDEs. The proposed method was applied to a series of test cases, showcasing its accuracy and computational efficiency. The comparison between the numerical solutions and exact solutions demonstrated excellent agreement, reinforcing the reliability of the method. Through detailed error analysis, we established that the method converges rapidly with increasing terms, highlighting its robustness in dealing with both linear and nonlinear fractional-order problems. This approach proves to be an effective and versatile tool for solving a wide range of fractional differential equations, offering potential for use in diverse scientific and engineering fields. Future work can extend this technique to handle more complex boundary conditions and higher-order fractional derivatives, further expanding its applicability.

Data Availability

Not applicable.

Conflict of interest

The authors declare that they have no conflicts of interest.

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Authors' contributions

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