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Spatial Modelling in Multiset Topological Spaces

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ABSTRACT: In this article, we give a theoretical description of spatial modelling of multiset topological relations. We have investigated the 4-intersection model to derive the topological relations between objects which are multisets.

Key Words: Multisets, spatial data, GIS, boundary.

Contents

1. Introduction

A geographical information system(GIS) is an integrated set of software and hardware tools used for the manipulation of digital spatial(geographic) and related attribute data. A relationship in a spatial data is a relationship between features in a Euclidean space. Topological relationships are a specific subset of the large variety of spatial relationship which are characterised by properties to be preserved under topological transformation.

In recent years, a lot of research has been done on topological relations in a crisp topological space. Among them the most popular are the Egenhofer 4intersection [1] and 9 intersection [2] model. However, there are several other ways of modelling topological spatial relations in GIS. Kainz et. al [3] instigated topological relations based on lattice theory where they reduced some complexities in spatial queries. Randell et al [4] investigated on the basis of logic where eight relations are identified based on their RCC theory. Recently, topological relation was extended to fuzzy domains. Fischer [16] presented a good example for need of studying fuzzy topological spatial relations. Schneider [19,20,21] gave the definitions of fuzzy spatial concepts like fuzzy point, fuzzy line and fuzzy region. Clementi [5,6,7,9] utilised the 9-intersection model to study relations by introducing a broad boundary algebraically. The egg-yolk model for fuzzy relations was proposed based on RCC theory. In recent year, spatial modelling for soft sets is also investigated in [18]. However various development in this area can be seen in [8,9,10,11,12,13,14,15]

The topological structures of multisets have been introduced by Girish et.al. [22].After the introduction of a topology on multisets many researcher have put their effort on studying many topological properties of multisets [23,24,25,26,28,**?**,29]

In our study we aim to give topological relation between objects which are multisets. We extend the popular 4-intersection model of crisp topological space. Topological notions uses the concept of continuity, closure, interior, boundary, exterior etc . which are defined in terms of neighbourhood relation. So in this chapter we first introduce the concept of multiset boundary and its properties in a multiset topological

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space.

A critical comparison of the 4 intersection and 9 intersection model introduced by Egenhofer et al in a crisp topological spcace is explained nicely in [6] as follows:

4-Intersection

Binary topological relations between two objects, A and B, are defined in terms of the four intersections of A's boundary (∂A) and interior (A^o) with the boundary (∂B) and interior (B^o) of B (Egenhofer and Franzosa 1991). This model is concisely represented by a 2×2-matrix, called the 4-intersection.

$$i_4(A,B) = \begin{bmatrix} A^o \cap B^o & A^o \cap \partial B \\ \partial A \cap B^o & \partial A \cap \partial B \end{bmatrix}$$

Topological invariants of these four intersections, i.e., properties that are preserved under topological transformations, are used to categorize topological relations. Examples of topological invariants, applicable to the 4-intersection, are the content (i.e., emptiness or non-emptiness) of a set, the dimension, and the number of separations (Franzosa and Egenhofer 1992).

9-Intersection

The 4-intersection model is extended by considering the location of each interior and boundary with respect to the other object's exterior; therefore, the binary topological relation between two objects, A and B, in $\mathbb{R}^2$ is based upon the intersection of A's interior (A^o), boundary (∂A), and exterior (A^-) with B's interior (B^o), boundary (∂B), and exterior (B^-). The nine intersections between the six object parts describe a topological relation and can be concisely represented by a 3×3-matrix i.e , called the 9-intersection.

$$i_9(A,B) = \begin{bmatrix} A^o \cap B^o & A^o \cap \partial B & A^o \cap B^- \\ \partial A \cap B^o & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B & A^{\cap} \partial B & A^- \cap B^- \end{bmatrix}$$

In analogy to the 4-intersection, each intersection will be characterized by a value empty (ϕ) or non-empty ($\neg\phi$), which allows one to distinguish $2^9 = 512$ different configurations. Only a small subset of them can be realized between two object in $\mathbb{R}^2$.

In the present study, the spatial modelling of multiset topological relation is investigated by using the 4-Intersection model(4-IM) described above. We first introduce boundary of a multiset and study its properties to utilize in the 4-IM for objects which are multiset.

2. Need of the Study

In [16] Peter Fischer, very clearly mentioned about the need of alternative set theories for uncertainity in spatial information. According to him, the theory of multisets are more appropriate for spatial modelling of habitats of forest or woodland. The use of spatial information like ArcInfo's spatial database Engine and the associted shapes database indicates it has become convenient to utilise multiset witin GIS. The following is an example of vegetation mapping given by Fischer, describing the need and use of multisets in spatial modelling:

In a region, one can identify two broad classes of vegetation i. e. grassland and woodland. Further examination shows that there are interesting habitats in the region (both the grassland and the woodland). Within the woodland we can distinguish a high canopy where certain birds nest. Other birds prefer to nest in lower trees and saplings. Some species of mammal roam on both the ground and the trees within the woodland, whereas still others roam across both the grassland and the woodland, and a third group of mammals and some birds favor the open grassland. We now have a number of different categories of habitat (five in all), some of which co-exist at the same geographic location and others of which are distinct. This is conveniently modeled as multisets.

On the other hand, in [22], R. Munro et. al describes the need of mining complex relationship in spatial data which involves multifeature colocation, self colocation, one to many relationship and so on. Moreover as already mentioned that in the real world there are many instances of enormous repetition.‘So, in such scenario abstraction using multisets is more accurate than that of using sets. Thus Wildberg observed that there can exist the following three possible relations between any two objects in the universe:

1. they are completely different,
2. they are same but separate,
3. they are coinciding and identical.

Lastly, in any information system one may encounter that the object counts in the universe of discourse is more than one. In such cases the universe of discourse can be replaced by multisets for further evaluation.

3. Multiset Boundary

Boundary of Multisets(M-boundary)

Let (M, τ) be a M-topological space. If N is a submset of M then the boundary of N denoted by ∂N is defined by

$$\partial N = cl(N) \cap cl(N^c)$$

3.1. Properties of M-boundary

1. $C_{\partial N}(x) \leq \ C_N(x)$ if N is closed.
2. $C_{\partial N}(x) = C_{clN-intN}(x)$.
3. $C_{\partial N \cap intN}(x) \neq 0$.

The following properties of multiset boundary differs from crisp boundary in the sense that equality does not hold always for M-boundary,

1. $C_{cl(N)}(x) \geq C_{int(N) \cup \partial N}(x)$
2. $C_{int(N)} \leq C_{N-\partial N}(x)$

Proof: 1.

$$\begin{aligned} C_{int(N)\cup \partial N}(x) &= C_{int(A)\cup(clN \cap (intN)^c)}(x) \\ &= C_{(intN \cup (intN)^c) \cap (intN \cup clN)}(x) \\ &\leq C_{M \cap clN}(x) \\ &= C_{clN}(x), \end{aligned}$$

for all $x \in X$
REMARK: The equality hold only if $intN$ is a whole submset of M. Also, $int(N) \uplus \partial N = clN$.

2.

$$\begin{aligned}C_{N-\partial N}(x) &= C_{N\cap(\partial N)^c}(x)\\ &= C_{N\cap(clN\cap(clN^c))^c}(x)\\ &= C_{N\cap((clN)^c\cup(clN^c)^c)}(x)\\ &= C_{N\cap(int(N^c)\cup intN)}(x)\\ &= C_{(N\cap int(N^c))\cup(N\cup intN)}(x)\\ &= C_{(N\cap int(N^c))\cup intN}(x)\\ &\geq C_{intN}(x),\end{aligned}$$

for all $x \in X$ □

Proposition 3.1 *$C_{\partial N\cap intN}(x) = 0$ iff $int(N)$ is a whole submset of M.*

Proof: Let $intN$ is a whole submset of M i.e. $C_{intN}(x) = C_M(x)$, for all $x \in \text{supp}(intN)$. Then $intN \cap (intN)^c = \phi$.

Now,

$$\begin{aligned}C_{\partial N\cap intN} &= C_{(clN\cap clN^c)\cap intN}(x)\\ &= C_{(clN\cap(intN)^c)\cap intN}(x)\\ &= C_{clN\cap((intN)^c\cap intN)}(x)\\ &= C_{clN\cap\phi}(x)\\ &= 0,\end{aligned}$$

for all $x \in X$.

Conversely, let $intN$ is not a whole submset of M i.e $C_{intN} \leq C_M(x)$, for every $x \in intN$.

Then,
$C_{(intN)^c} \leq C_M(x)$, for every $x \in intN$.

$$\begin{aligned}C_{\partial N\cap intN}(x) &= C_{(clN\cap clN^c)\cap intN}\\ &= C_{clN\cap(intN)^c\cap intN}\\ &\neq 0.\end{aligned}$$

□

Remark 3.1 *The mutually exclusive condition for interior and boundary is satisfied if and only if interior is a whole submset of the universal multiset.*

Our aim here is to apply the 4-Intesection Model(4-IM) to evaluate the spatial relation between multiset objects for which the mutually exclusive condition for interior and boundary must be satisfied. Moreover, $C_{cl(N)}(x) = C_{int(N)\cup\partial N}(x)$ must hold for spatial regions. But, the above results show that these condition do not always hold for multisets. But if we consider the multiset topological space in such a way that the open msets have full multiplicity for all the elements in the space, then the following two conditions are satisfied,
(1) $C_{\partial N\cap intN}(x) = 0$.

(2) $C_{cl(N)}(x) = C_{int(N)\cup\partial N}(x)$.

This is because, after this assumption the interior of any submset of the space is a whole submset. So, assuming full multiplicity conditions for open msets, the 4-IM model can be applied for submsets of a multiset topological space.

Separation in a Multiset Topological Space

Let (M, τ) be a M-topological space and $N \subset M$. A separation of N is a pair N_1 and N_2 of non-empty submsets of M which satisfy the following,

1. $N_1 \cup N_2 = N$
2. neither contains accumulation point of other. In other words M_1 and M_2 are separated iff both the following conditions hold

$$C_{M_1\cap cl(M_2)} = 0 \ \forall x \epsilon X$$
$$C_{M_2\cap cl(M_1)} = 0 \ \forall x \epsilon X$$

Theorem 3.1 *Let $N \subset M$ and $clN \neq M$. Then $intN$ and $M - clN$ form a M-separation of $M - \partial N$, and thus ∂N separates M.*

Proof: Clearly, $intN$ and $M - clN$ are non empty and they are mutually disjoint open msets. Since $intN$ is a whole submset of M, we have $cl(N) = int(N) \cup \partial N$ which implies that $M - \partial N = intN \cup (M - clN)$. Thus $intN$ and $(M - clN)$ form a separation of $M - \partial N$. □

4. Theoritical Description of Topological Spatial Relations

In our model, we consider a multiset topological space and its subsets to obtain the topological relations which is based on the intersection between multiset boundary and mutiset interior. Thus if M and N are any two submsets, then our aim is to determine $\partial M \cap \partial N$, $\partial M \cap intN$, $\partial N \cap intM$, $intM \cap intN$.

We use ○ for empty intersection and ⊗ for non-empty intersection. The sixteen possibilities from these combinations can be summarized in the Table 1.

Relations	$\partial \cap \partial$	$int \cap int$	$\partial \cap int$	$int \cap \partial$
mr_0	○	○	○	○
mr_1	⊗	○	○	○
mr_2	○	⊗	○	○
mr_3	⊗	⊗	○	○
mr_4	○	○	⊗	○
mr_5	⊗	○	⊗	○
mr_6	○	⊗	⊗	○
mr_7	⊗	⊗	⊗	○
mr_8	○	○	○	⊗
mr_9	⊗	○	○	⊗
mr_{10}	○	⊗	○	⊗
mr_{11}	⊗	⊗	○	⊗
mr_{12}	○	○	⊗	⊗
mr_{13}	⊗	○	⊗	⊗
mr_{14}	○	⊗	⊗	⊗
mr_{15}	⊗	⊗	⊗	⊗

Table 1: Multiset topological spatial relations.

So, these are the spatial relations possible between any pair of subsets of a topological space. Now, our aim is to model topological relations between spatial regions. We define spatial multiset region as follows:

Spatial Multiset Region

Let (M, τ) be a M-connected topological space. A spatial multiset region in M is a non-empty submset N such that,

1. $int(N)$ is M-connected.
2. $cl(int(N)) = N$.

Proposition 4.1 *If N is any spatial region in M, then $\partial N \neq \emptyset$.*

Proof: By definition of spatial region $intN$ is a whole submset of M and $clN = N \neq M$. Thus from Theorem 1 it follows the $intN$ and $M - N$ form a separation of $M - \partial N$. If $\partial N = \emptyset$, then the two msets will form a separation of M, which is a contradiction, since M is M-connected. Hence, $\partial N \neq \emptyset$. □

Proposition 4.2 *The spatial relations mr_2, mr_4, mr_5, mr_8, mr_9, mr_{12}, mr_{13} does not exist for spatial multiset region.*

Proof: In the spatial multiset region mr_4, mr_5, mr_8, mr_9, mr_{12}, mr_{13}, it has been observed that whenever boundary-interior or interior-boundary is non-empty, then the interior-interior is empty. Our aim is to prove that such a combination cannot occur in a spatial multiset region.
Let us consider two spatial multiset regions M and N for which $\partial M \cap intN \neq \emptyset$. To show that $intM \cap intN \neq \emptyset$. We have, $intM \cup \partial M = clM$. So, $intM \cup \partial(intM) = cl(intM)$. Also $clM = M = cl(intM)$.

Therefore, $intM \cup \partial M = intM \cup \partial(intM)$. Moreover, $intM \cap \partial(intM) = \emptyset$ and $M \cap \partial M = \emptyset$. So, $\partial M = \partial(intM)$. Since, $\partial M \cap intN \neq \emptyset$, let $m/x \in \partial M \cap intN$ i.e. $m/x \in \partial M$ and $m/x \in intN$. But, $intN$ is an open mset containing m/x, so it follows $intM \cap intN \neq \emptyset$. Thus when the boundary-interior is non-empty then the interior-interior must also be non-empty.

For the spatial relation mr_2, let us assume $\partial M \cap \partial N$ and $intM \cap intN \neq \emptyset$. We aim to show that if $\partial M \cap intN = \emptyset$ then $\partial N \cap intM \neq \emptyset$. Since, $clN = N = intN \cup \partial N$, we have,

$$\begin{aligned}\partial M \cap N =& \partial M \cap (intN \cup \partial N)\\ =& (\partial M \cap intN) \cup (\partial M \cap \partial N)\\ =& \emptyset\end{aligned}$$

Therefore, $N \subset \mathcal{M} - \partial M$, where $\mathcal{M}$ is the universal multiset where the topology is defined.

It follows $intM$ and $\mathcal{M} - M$ form a separation of $\mathcal{M} - \partial M$. Since N is M-connected it implies $N \subset intM$ or $N \subset \mathcal{M} - M$. But $intM \cap intN \neq \emptyset$, it implies $N \subset intM$ and so $\partial N \subset intM$. Hence, clearly $\partial N \cap intM \neq \emptyset$. □

5. Conclusion

In the present study, we have given a theoritical analysis of multiset spatial relations by using the 4-Intersection model and some valid assumptions. We are only interested whether the intersection is empty or non-empty. The non-empty intersection can have some values based on the multiplicity of a multiset. We keep it as study for future perspective.

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