Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.77682

An additive functional inequality in C^*-algebras

Siriluk Donganont

ABSTRACT: In this paper, we introduce an additive functional inequality

$$\left\|2g\left(\frac{\lambda u+y}{2}\right)-\lambda g(u)-g(y)\right\|\leq\|s(g(\lambda u+y)-\lambda g(u)-g(y))\| \tag{0.1}$$

for all $\lambda\in\mathbb{C}$, all unitary elements u in a unital C^*-algebra P and all $y\in P$, where $|s|<1$. Using both the direct method and the fixed point method, we establish the Hyers-Ulam stability of inequality (0.1) in unital C^*-algebras. Furthermore, we apply these results to the study of C^*-algebra homomorphisms and C^*-algebra derivations in unital C^*-algebras.

Key Words: Hyers-Ulam stability, fixed point method, additive functional inequality, C^*-algebra derivation in C^*-algebra, C^*-algebra homomorphism in C^*-algebra.

Contents

1. Introduction and preliminaries

The stability problem of functional equations originated from a question posed by Ulam [19] concerning the stability of group homomorphisms. Hyers [8] provided the first affirmative partial answer to Ulam's question in the context of Banach spaces. Hyers' theorem was later generalized by Aoki [2] for additive mappings, and by Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A further generalization of Rassias' theorem was obtained by Găvruta [6], who replaced the unbounded Cauchy difference with a general control function, following the spirit of Rassias' approach.

Park [14,15] introduced additive ρ-functional inequalities and proved the Hyers-Ulam stability of these inequalities in both Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations, functional inequalities, and differential equations have been extensively studied by numerous authors (see [1,5,7,10,12,20]).

We now recall a fundamental result in fixed point theory.

Theorem 1.1 [3,4] *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either*

$$d(J^nx,J^{n+1}x)=\infty$$

Submitted July 06, 2025. Published August 12, 2025
2010 *Mathematics Subject Classification:* Primary 47B47, 17B40, 47H10, 46L05, 39B62, 39B52.

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* w *of* J;

(3) w *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, w) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [9] were the first to apply the stability theory of functional equations to prove new fixed point theorems with practical applications. Using fixed point methods, the stability problems of various functional equations have since been extensively studied by numerous authors (see [16,17]).

Let P and Q be unital C^*-algebras. A $\mathbb{C}$-linear mapping $g : P \to P$ is a C^*-algebra derivation if $g : A \to A$ satisfies

$$g(xy) = g(x)y + xg(y), \qquad g(x^*) = g(x)^*$$

for all $x, y \in P$, and a $\mathbb{C}$-linear mapping $h : P \to Q$ is a C^*-algebra homomorphism if $h : P \to Q$ satisfies

$$h(xy) = h(x)h(y), \qquad h(x^*) = h(x)^*$$

for all $x, y \in P$.

In this paper, we solve the additive functional inequality (0.1) and prove the Hyers-Ulam stability of the functional inequality (0.1) in unital C^*-algebras by using the direct method and by the fixed point method. Furthermore, we investigate C^*-algebra derivations and C^*-algebra homomorphisms in unital C^*-algebras associated to the additive functional inequality (0.1).

Throughout this paper, assume that P is a unital C^*-algebra with unitary group $U(P) := \{u \in P | u^*u = uu^* = e\}$ and Q is a unital C^*-algebra and that s is a fixed nonzero complex number with $|s| < 1$.

2. Hyers-Ulam stability of the functional inequality (0.1): direct method

In this section, we solve and investigate the functional inequality (0.1) in unital C^*-algebras.

Lemma 2.1 *Assume that a mapping* $g : P \to Q$ *satisfies*

$$\left\| 2g\left(\frac{\lambda u + y}{2}\right) - \lambda g(u) - g(y) \right\| \leq \| s(g(\lambda u + y) - \lambda g(u) - g(y)) \| \tag{2.1}$$

for all $\lambda \in \mathbb{C}$, *all* $u \in U(P)$ *and all* $y \in P$. *Then the mapping* $g : P \to Q$ *is* $\mathbb{C}$*-linear.*

Proof: Let $\lambda = 0$ in (2.1), we get $2g\left(\frac{y}{2}\right) = g(y)$ for all $y \in P$ and so we get $g(0) = 0$.

$$\begin{aligned} \| g(\lambda u + y) - \lambda g(u) - g(y) \| &= \left\| 2g\left(\frac{\lambda u + y}{2}\right) - \lambda g(u) - g(y) \right\| \\ &\leq \| s(g(\lambda u + y) - \lambda g(u) - g(y)) \| \end{aligned}$$

and so

$$g(\lambda u + y) = \lambda g(u) + g(y) \tag{2.2}$$

for all $\lambda \in \mathbb{C}$, all $u \in U(P)$ and all $y \in P$, since $|s| < 1$

Let $y = 0$ in (2.2), we get $g(\lambda u) = \lambda g(u)$ for all $\lambda \in \mathbb{C}$ and all $u \in U(P)$.

Since each $x \in P$ is a finite linear combination of unitary elements (see [11]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(P)$),

$$\begin{aligned} g(\lambda x + y) &= g(\lambda \sum_{j=1}^m \lambda_j u_j + y) = g(\sum_{j=1}^m \lambda\lambda_j u_j + y) = g(\lambda\lambda_1 u_1 + \sum_{j=2}^m \lambda\lambda_j u_j + y) \\ &= \lambda\lambda_1 g(u_1) + g(\sum_{j=2}^m \lambda\lambda_j u_j + y) \\ &\vdots \\ &= \lambda\lambda_1 g(u_1) + \lambda\lambda_2 g(u_2) + \cdots + g(\lambda\lambda_m u_m + y) \\ &= \lambda\lambda_1 g(u_1) + \lambda\lambda_2 g(u_2) + \cdots + \lambda\lambda_m g(u_m) + g(y) \\ &= \lambda(\lambda_1 g(u_1) + \lambda_2 g(u_2) + \cdots + \lambda_m g(u_m)) + g(y) \\ &= \lambda(\lambda_1 g(u_1) + \lambda_2 g(u_2) + \cdots + \lambda_{m-1} g(u_{m-1}) + g(\lambda_m u_m)) + g(y) \\ &= \lambda(\lambda_1 g(u_1) + \lambda_2 g(u_2) + \cdots + g(\lambda_{m-1} u_{m-1} + \lambda_m u_m)) + g(y) \\ &\vdots \\ &= \lambda(\lambda_1 g(u_1) + g(\lambda_2 u_2 + \cdots + \lambda_m u_m)) + g(y) \\ &= \lambda(g(\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_{m-1} u_{m-1} + \lambda_m u_m)) + g(y) \\ &= \lambda g(x) + g(y) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $y \in P$. So the mapping $g : P \to Q$ is $\mathbb{C}$-linear. □

Now we prove the Hyers-Ulam stability of the additive functional inequality (0.1) in unital C^*-algebras.

Theorem 2.1 *Let $\varphi : P \times P \to [0, \infty)$ be a function such that*

$$\Phi(u, y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.3}$$

for all $u \in U(P)$ and all $y \in P$. Assume that a mapping $g : P \to Q$ satisfies

$$\left\|2g\left(\frac{\lambda u + y}{2}\right) - \lambda g(u) - g(y)\right\| \le \|s(g(\lambda u + y) - \lambda g(u) - g(y))\| + \varphi(u, y) \tag{2.4}$$

for all $\lambda \in \mathbb{C}$, all $u \in U(P)$ and all $y \in P$. Then there exists a unique $\mathbb{C}$-linear mapping $G : P \to Q$ such that

$$\|g(x) - G(x)\| \le \Phi(u, y) \tag{2.5}$$

for all $u \in U(P)$ and all $y \in P$.

Proof: Let $\lambda = 0$ in (2.4), we get

$$\left\|g(y) - 2g\left(\frac{y}{2}\right)\right\| \le \varphi(u, y) \tag{2.6}$$

for all $u \in U(P)$ and all $y \in P$.

Similarly, we can show that

$$\left\|2^j g\left(\frac{y}{2^j}\right) - 2^{j+1} g\left(\frac{y}{2^{j+1}}\right)\right\| \le 2^j \varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right)$$

for all $u \in U(P)$, all $y \in P$ and each positive integer j. Thus

$$\begin{aligned} \left\|2^l g\left(\frac{y}{2^l}\right) - 2^m g\left(\frac{y}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j g\left(\frac{y}{2^j}\right) - 2^{j+1} g\left(\frac{y}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right) \end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$, all $u \in U(P)$ and all $y \in P$. It follows from (2.7) that the sequence $\{2^k g(\frac{y}{2^k})\}$ is Cauchy for all $y \in P$. Since Q is complete, the sequence $\{2^k g(\frac{y}{2^k})\}$ converges. So one can define the mapping $G : P \to Q$ by

$$G(y) := \lim_{k\to\infty} 2^k g\left(\frac{y}{2^k}\right)$$

for all $y \in P$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.5).

It folllows from (2.3) and (2.4) that

$$\begin{aligned} &\left\|2G\left(\frac{\lambda u + y}{2}\right) - \lambda G(u) - G(y)\right\| = \lim_{n\to\infty} 2^n \left\|2g\left(\frac{\lambda u + y}{2^{n+1}}\right) - \lambda g\left(\frac{u}{2^n}\right) - g\left(\frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 2^n \left\|s\left(g\left(\frac{\lambda u + y}{2^n}\right) - \lambda g\left(\frac{u}{2^n}\right) - g\left(\frac{y}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{u}{2^n}, \frac{y}{2^n}\right) \\ &= \|s(G(\lambda u + y) - \lambda G(u) - G(y))\| \end{aligned}$$

for all $\lambda \in \mathbb{C}$, all $u \in U(P)$ and all $y \in P$. So

$$\left\|2G\left(\frac{\lambda u + y}{2}\right) - \lambda G(u) - G(y)\right\| \leq \|s(G(\lambda u + y) - \lambda G(u) - G(y))\|$$

for all $\lambda \in \mathbb{C}$, all $u \in U(P)$ and all $y \in P$. By Lemma 2.1, the mapping $G : P \to Q$ is $\mathbb{C}$-linear.

The proof of the uniqueness of the mapping G is similar to the proof of [14, Theorem 2.3]. □

Theorem 2.2 *Let $\varphi : P \times P \to [0, \infty)$ be a function such that*

$$\Psi(u, y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi\left(2^j u, 2^j y\right) < \infty \tag{2.8}$$

for all $u \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.4). Then there exists a unique $\mathbb{C}$-linear mapping $G : P \to Q$ such that

$$\|g(x) - G(x)\| \leq \Psi(u, y) \tag{2.9}$$

for all $u \in U(P)$ and all $y \in P$.

Proof: Let $\lambda = 0$ and replacing u by $2u$ in (2.4), we get

$$\left\|g(y) - 2g\left(\frac{y}{2}\right)\right\| \leq \varphi(2u, y)$$

and so

$$\left\|g(y) - \frac{1}{2}g(2y)\right\| \leq \frac{1}{2}\varphi(2u, 2y)$$

for all $u \in U(P)$ and all $y \in P$.

Similarly, we can show that

$$\left\|\frac{1}{2^j}g\left(2^jy\right)-\frac{1}{2^{j+1}}g\left(2^{j+1}y\right)\right\|\le\frac{1}{2^{j+1}}\varphi\left(2^{j+1}u,2^{j+1}y\right)$$

for all $u\in U(P)$, all $y\in P$ and each positive integer j. Thus

$$\begin{aligned}\|\frac{1}{2^l}g(2^ly)-\frac{1}{2^m}g(2^my)\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g\left(2^jy\right)-\frac{1}{2^{j+1}}g\left(2^{j+1}y\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi\left(2^ju,2^jy\right)\end{aligned}\tag{2.10}$$

for all nonnegative integers m and l with $m>l$, all $u\in U(P)$ and all $y\in P$. It follows from (2.10) that the sequence $\{\frac{1}{2^k}g(2^ky)\}$ is Cauchy for all $y\in P$. Since Q is complete, the sequence $\{\frac{1}{2^k}g(2^ky)\}$ converges. So one can define the mapping $G:P\to Q$ by

$$G(y):=\lim_{k\to\infty}\frac{1}{2^k}g\left(2^ky\right)$$

for all $y\in P$. Moreover, letting $l=0$ and passing to the limit $m\to\infty$ in (2.10), we get (2.9).

It folllows from (2.4) and (2.8) that

$$\begin{aligned}&\left\|2G\left(\frac{\lambda u+y}{2}\right)-\lambda G(u)-G(y)\right\|=\lim_{n\to\infty}\frac{1}{2^n}\left\|2g\left(2^{n-1}(\lambda u+y)\right)-\lambda g\left(2^nu\right)-g\left(2^ny\right)\right\|\\&\le\lim_{n\to\infty}\frac{1}{2^n}\left\|s\left(g\left(2^n(\lambda u+y)\right)-\lambda g\left(2^nu\right)-g\left(2^ny\right)\right)\right\|+\lim_{n\to\infty}\frac{1}{2^n}\varphi\left(2^nu,2^ny\right)\\&=\|s(G\left(\lambda u+y\right)-\lambda G(u)-G(y))\|\end{aligned}$$

for all $\lambda\in\mathbb{C}$, all $u\in U(P)$ and all $y\in P$. So

$$\left\|2G\left(\frac{\lambda u+y}{2}\right)-\lambda G(u)-G(y)\right\|\le\|s(G\left(\lambda u+y\right)-\lambda G(u)-G(y))\|$$

for all $\lambda\in\mathbb{C}$, all $u\in U(P)$ and all $y\in P$. By Lemma 2.1, the mapping $G:P\to Q$ is $\mathbb{C}$-linear.

The proof of the uniqueness of the mapping G is similar to the proof of [14, Theorem 2.3]. □

3. Hyers-Ulam stability of C^*-algebra derivations and C^*-algebra homomorphisms in C^*-algebras: direct method

Using the direct method, we prove the Hyers-Ulam stability of C^*-algebra homomorphisms in unital C^*-algebras associated to the additive functional inequality (2.1).

Theorem 3.1 *Let $\varphi:P^2\to[0,\infty)$ be a function such that*

$$\sum_{j=0}^{\infty}4^j\varphi\left(\frac{u}{2^j},\frac{y}{2^j}\right)<\infty\tag{3.1}$$

for all $u\in U(P)$ and all $y\in P$. Let $g:P\to Q$ be a mapping satisfying (2.4). If the mapping $g:P\to Q$ satisfies

$$\|g(uv)-g(u)g(v)\|\le\varphi(u,v),\tag{3.2}$$

$$\|g(u^*)-g(u)^*\|\le\varphi(u,u)\tag{3.3}$$

for all $u, v \in U(P)$, then there exists a unique C^*-algebra homomorphism $H : P \to Q$ such that

$$\|g(y) - H(y)\| \le \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right) \tag{3.4}$$

for all $u \in U(P)$ and all $y \in P$.

Proof: By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ satisfying (3.4). The mapping $H : P \to Q$ is given by

$$H(y) := \lim_{k\to\infty} 2^k g\left(\frac{y}{2^k}\right)$$

for all $y \in P$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|H(uv) - H(u)H(v)\| &= \lim_{n\to\infty} 4^n \left\|g\left(\frac{uv}{4^n}\right) - g\left(\frac{u}{2^n}\right)g\left(\frac{v}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) = 0\end{aligned}$$

and so

$$H(uv) = H(u)H(v)$$

for all $u, v \in U(P)$.

Since $H : P \to Q$ is $\mathbb{C}$-linear and each $x, y \in P$ is a finite linear combination of unitary elements (see [11]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$, $y = \sum_{k=1}^{l} \mu_k v_k$ ($\lambda_j, \mu_k \in \mathbb{C}$, $u_j, v_k \in U(P)$),

$$\begin{aligned}H(xy) &= H(\sum_{j=1}^{m}\sum_{k=1}^{l} \lambda_j\mu_k u_j v_k) = \sum_{j=1}^{m}\sum_{k=1}^{l} \lambda_j\mu_k H(u_j v_k) = \sum_{j=1}^{m}\sum_{k=1}^{l} \lambda_j\mu_k H(u_j)H(v_k) \\ &= (\sum_{j=1}^{m} \lambda_j H(u_j))(\sum_{k=1}^{l} \mu_k H(v_k)) = H(\sum_{j=1}^{m} \lambda_j u_j)H(\sum_{k=1}^{l} \mu_k v_k) = H(x)H(y)\end{aligned}$$

for all $x, y \in P$. So the $\mathbb{C}$-linear mapping $H : P \to Q$ is multiplicative.

It follows from (3.1) and (3.3) that

$$\begin{aligned}\|H(u^*) - H(u)^*\| &= \lim_{n\to\infty} 2^n \left\|g\left(\frac{u^*}{2^n}\right) - g\left(\frac{u}{2^n}\right)^*\right\| \\ &\le \lim_{n\to\infty} 2^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}\right) \le \lim_{n\to\infty} 4^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}\right) = 0\end{aligned}$$

and so

$$H(u^*) = H(u)^*$$

for all $u \in U(P)$.

Since $H : P \to Q$ is $\mathbb{C}$-linear and each $x \in P$ is a finite linear combination of unitary elements (see [11]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(P)$),

$$\begin{aligned}H(x^*) &= H(\sum_{j=1}^{m} \overline{\lambda_j} u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j)^* = H(\sum_{j=1}^{m} \lambda_j u_j)^* \\ &= H(x)^*\end{aligned}$$

for all $x \in P$. So the $\mathbb{C}$-linear mapping $H : P \to Q$ is involutive. Thus the $\mathbb{C}$-linear mapping $H : P \to Q$ is a C^*-algebra homomorphism. $\square$

Corollary 3.1 *Let $r > 2$ and θ be nonnegative real numbers and $g : P \to Q$ be a mapping satisfying*

$$\left\| 2g\left(\frac{\lambda u + y}{2}\right) - \lambda g(u) - g(y) \right\| \leq \|s(g(\lambda u + y) - \lambda g(u) - g(y))\| + \theta(1 + \|y\|^r) \tag{3.5}$$

for all $\lambda \in \mathbb{C}$, all $u \in U(P)$ and all $y \in P$. If the mapping $g : P \to Q$ satisfies

$$\|g(uv) - g(u)g(v)\| \leq 2\theta, \tag{3.6}$$

$$\|g(u^*) - g(u)^*\| \leq 2\theta, \tag{3.7}$$

for all $u, v \in U(P)$, then there exists a unique C^*-algebra homomorphism $H : P \to Q$ such that

$$\|g(y) - H(y)\| \leq \frac{2^r\theta}{2^r - 2}(1 + \|y\|^r) \tag{3.8}$$

for all $y \in P$.

Proof: The result follows from Theorem 3.1 by taking $\varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right) = \frac{\theta}{2^{rj}}(1 + \|y\|^r)$ for all $u \in U(P)$ and all $y \in P$. □

Theorem 3.2 *Let $\varphi : P^2 \to [0, \infty)$ be a function and $g : P \to Q$ be a mapping satisfying (2.8), (3.2) and (3.3). Then there exists a unique C^*-algebra homomorphism $H : P \to Q$ satisfying satisfying*

$$\|g(x) - D(x)\| \leq \Psi(u, y) \tag{3.9}$$

for all $y \in P$ and all $u \in U(P)$.

Proof: By Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ satisfying (3.9). The mapping $H : P \to Q$ is given by

$$H(y) := \lim_{k\to\infty} \frac{1}{2^k} g\left(2^k y\right)$$

for all $y \in P$.

It follows from (2.8) and (3.2) that

$$\begin{aligned} \|H(uv) - H(u)H(v)\| &= \lim_{n\to\infty} \frac{1}{4^n} \|g(4^n uv) - g(2^n u)\, g(2^n v)\| \\ &\leq \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n u, 2^n v) \leq \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n u, 2^n v) = 0 \end{aligned}$$

and so $H(uv) = H(u)H(v)$ for all $u, v \in U(P)$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.2 *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to Q$ be a mapping satisfying (3.5), (3.6) and (3.7). Then there exists a unique C^*-algebra homomorphism $H : P \to Q$ such that*

$$\|g(x) - H(x)\| \leq \frac{2^r\theta}{2 - 2^r}(1 + \|y\|^r) \tag{3.10}$$

for all $y \in P$.

Proof: The result follows from Theorem 3.2 by taking $\varphi(2^j u, 2^j y) = 2^{rj}\theta(1 + \|y\|^r)$ for all $u \in U(P)$ and all $y \in P$. □

Now, we prove the Hyers-Ulam stability of C^*-algebra derivations in unital C^*-algebras associated to the additive functional inequality (2.1) by using the direct method.

Theorem 3.3 *Let $\varphi : P^2 \to [0,\infty)$ be a function satisfying (3.1). If the mapping $g : P \to P$ satisfies (3.3) and*

$$\|g(uv) - g(u)v - ug(v)\| \leq \varphi(u,v), \tag{3.11}$$

then there exists a unique C^*-algebra derivation $D : P \to P$ such that

$$\|g(y) - D(y)\| \leq \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right) \tag{3.12}$$

for all $u \in U(P)$ and all $y \in P$.

Proof: By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $D : P \to P$ satisfying (3.12). The mapping $D : P \to P$ is given by

$$D(y) := \lim_{k\to\infty} 2^k g\left(\frac{y}{2^k}\right)$$

for all $y \in P$.

It follows from (3.1) and (3.11) that

$$\begin{aligned} \|D(uv) - D(u)v - uD(v)\| &= \lim_{n\to\infty} 4^n \left\|g\left(\frac{uv}{4^n}\right) - g\left(\frac{u}{2^n}\right)\frac{v}{2^n} - \frac{u}{2^n} g\left(\frac{v}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) = 0 \end{aligned}$$

and so

$$D(uv) = D(u)v + uD(v)$$

for all $u, v \in U(P)$.

Since $D : P \to P$ is $\mathbb{C}$-linear and each $x, y \in P$ is a finite linear combination of unitary elements (see [11]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$, $y = \sum_{k=1}^{l} \mu_k v_k$ ($\lambda_j, \mu_k \in \mathbb{C}$, $u_j, v_k \in U(P)$),

$$\begin{aligned} D(xy) &= D(\sum_{j=1}^{m}\sum_{k=1}^{l} \lambda_j\mu_k u_j v_k) = \sum_{j=1}^{m}\sum_{k=1}^{l} \lambda_j\mu_k D(u_j v_k) = \sum_{j=1}^{m}\sum_{k=1}^{l} \lambda_j\mu_k (D(u_j)v_k + u_j D(v_k)) \\ &= (\sum_{j=1}^{m} \lambda_j D(u_j))(\sum_{k=1}^{l} \mu_k v_k) + (\sum_{j=1}^{m} \lambda_j u_j)(\sum_{k=1}^{l} \mu_k D(v_k)) \\ &= D(\sum_{j=1}^{m} \lambda_j u_j) y + xD(\sum_{k=1}^{l} \mu_k v_k) = D(x)y + xD(y) \end{aligned}$$

for all $x, y \in P$.

It follows from (3.1) and (3.3) that

$$\begin{aligned} \|D(u^*) - D(u)^*\| &= \lim_{n\to\infty} 2^n \left\|g\left(\frac{u^*}{2^n}\right) - g\left(\frac{u}{2^n}\right)^*\right\| \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}\right) \leq \lim_{n\to\infty} 4^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}\right) = 0 \end{aligned}$$

and so

$$D(u^*) = D(u)^*$$

for all $u \in U(P)$.

Since $D : P \to P$ is $\mathbb{C}$-linear and each $x \in P$ is a finite linear combination of unitary elements (see [11]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(P)$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = H(\sum_{j=1}^m \lambda_j u_j)^* \\ &= H(x)^* \end{aligned}$$

for all $x \in P$. So the $\mathbb{C}$-linear mapping $D : P \to P$ is involutive. Thus the $\mathbb{C}$-linear mapping $D : P \to P$ is a C^*-algebra derivation. □

Corollary 3.3 *Let $r > 2$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying (3.5). If the mapping $g : P \to P$ satisfies (3.7) and*

$$\|g(uv) - g(u)v - ug(v)\| \le 2\theta, \tag{3.13}$$

for all $u, v \in U(P)$, then there exists a unique C^*-algebra derivation $D : P \to P$ such that

$$\|g(y) - D(y)\| \le \frac{2^r\theta}{2^r - 2}(1 + \|y\|^r) \tag{3.14}$$

for all $y \in P$.

Proof: The result follows from Theorem 3.3 by taking $\varphi\left(\frac{u}{2^j}, \frac{y}{2^j}\right) = \frac{\theta}{2^{rj}}(1 + \|y\|^r)$ for all $u \in U(P)$ and all $y \in P$. □

Theorem 3.4 *Let $\varphi : P^2 \to [0, \infty)$ be a function and $g : P \to P$ be a mapping satisfying (2.8), (3.11) and (3.3). Then there exists a unique C^*-algebra derivation $D : P \to P$ satisfying*

$$\|g(x) - D(x)\| \le \Psi(u, y) \tag{3.15}$$

for all $u \in U(P)$ and all $y \in P$, where $\Psi(u, y)$ is given in the statement of Theorem 2.2.

Proof: By Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $D : P \to P$ satisfying (3.15). The mapping $D : P \to P$ is given by

$$D(y) := \lim_{k\to\infty} \frac{1}{2^k} g\left(2^k y\right)$$

for all $y \in P$.

It follows from (2.8) and (3.11) that

$$\begin{aligned} \|D(uv) - D(u)v - uD(v)\| &= \lim_{n\to\infty} \frac{1}{4^n} \|g\left(4^n uv\right) - g\left(2^n u\right)\left(2^n v\right) - (2^n u)g(2^n v)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n} \varphi\left(2^n u, 2^n v\right) \le \lim_{n\to\infty} \frac{1}{2^n} \varphi\left(2^n u, 2^n v\right) = 0 \end{aligned}$$

and so $D(uv) = D(u)v + uD(v)$ for all $u, v \in U(P)$.

The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 3.4 *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying (3.5), (3.7) and (3.13). Then there exists a unique C^*-algebra derivation $D : P \to P$ such that*

$$\|g(x) - D(x)\| \le \frac{2^r\theta}{2 - 2^r}(1 + \|y\|^r) \tag{3.16}$$

for all $y \in P$.

Proof: The result follows from Theorem 3.4 by taking $\varphi(2^j u, 2^j y) = 2^{rj}\theta(1 + \|y\|^r)$ for all $u \in U(P)$ and all $y \in P$. □

4. Hyers-Ulam stability of the functional inequality (0.1): fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of the additive functional inequality (0.1) in unital C^*-algebras.

Theorem 4.1 *Let $\varphi : P^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{u}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(u,y)$$

for all $u \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.4). Then there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ such that

$$\|g(y) - H(y)\| \leq \frac{1}{1-L}\varphi(u,y) \tag{4.1}$$

for all $u \in U(P)$ and all $y \in P$.

Proof: Consider the set

$$S := \{g : P \to Q\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ \ : \ \|g(y) - h(y)\| \leq \mu\varphi(u,y), \quad \forall u \in U(P), \forall y \in P\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [13]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(y) := 2g\left(\frac{y}{2}\right)$$

for all $y \in P$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(y) - h(y)\| \leq \varepsilon\varphi(u,y)$$

for all $u \in U(P)$ and all $y \in P$. Since

$$\left\|2g\left(\frac{y}{2}\right) - 2h\left(\frac{y}{2}\right)\right\| \leq 2\varepsilon\varphi\left(\frac{u}{2}, \frac{y}{2}\right) \leq 2\varepsilon\frac{L}{2}\varphi(u,y) = L\varepsilon\varphi(u,y)$$

for all $u \in U(P)$ and all $y \in P$, $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.6) that

$$\left\|g(y) - 2g\left(\frac{y}{2}\right)\right\| \leq \varphi(u,y)$$

for all $u \in U(P)$ and all $y \in P$. So $d(g, Jg) \leq 1$.

By Theorem 1.1, there exists a mapping $H : P \to P$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H(y) = 2H\left(\frac{y}{2}\right) \tag{4.2}$$

for all $y \in P$. The mapping H is a unique fixed point of J. This implies that H is a unique mapping satisfying (4.2) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(y) - H(y)\| \leq \mu\varphi(u,y)$$

for all $u \in U(P)$ and all $y \in P$;

(2) $d(J^l g, H) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l g\left(\frac{y}{2^l}\right) = H(y)$$

for all $u \in U(P)$ and all $y \in P$;

(3) $d(g, H) \le \frac{1}{1-L} d(g, Jg)$, which implies

$$\|g(x) - H(x)\| \le \frac{1}{1-L}\varphi(u, y)$$

for all $u \in U(P)$ and all $y \in P$. Thus we get the inequality (4.1).

The rest of the proof is the same as in the proof of Theorem 2.1. □

Theorem 4.2 *Let $\varphi : P^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(u, y) \le 2L\varphi\left(\frac{u}{2}, \frac{y}{2}\right) \tag{4.3}$$

for all $u \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.4). Then there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ such that

$$\|g(y) - H(y)\| \le \frac{L}{1-L}\varphi(u, y) \tag{4.4}$$

for all $u \in U(P)$ and all $y \in P$.

Proof: Let S and d be given in the proof of Theorem 4.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(y) := \frac{1}{2} g(2y)$$

for all $y \in P$.

Let $\lambda = 0$ and replacing u and y by $2u$ and $2y$ in (2.4), we get

$$\left\|g(y) - \frac{1}{2} g(2y)\right\| \le \frac{1}{2}\varphi(2u, 2y) \le \frac{2L}{2}\varphi(u, y) = L\varphi(u, y)$$

for all $u \in U(P)$ and all $y \in P$. So $d(g, Jg) \le L$.

The rest of the proof is simialr to the proofs of Theorems 2.2 and 4.1. □

5. Hyers-Ulam stability of C^*-algebra derivations and C^*-algebra homomorphisms in C^*-algebras: fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of C^*-algebra homomorphisms in unital C^*-algebras associated to the additive functional inequality (2.1).

Theorem 5.1 *Let $\varphi : P^2 \to [0, \infty)$ be a function such that*

$$\varphi\left(\frac{u}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(u, y) \le \frac{L}{2}\varphi(u, y) \tag{5.1}$$

for all $u \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.4). If the mapping $g : P \to Q$ satisfies (3.2) and (3.3), then there exists a unique homomorphism $H : P \to Q$ satisfying (4.1).

Proof: It follows from (5.1) and (3.2) that

$$\begin{aligned}\|H(uv)-H(u)H(v)\| &= \lim_{n\to\infty} 4^n \left\|g\left(\frac{uv}{4^n}\right)-g\left(\frac{u}{2^n}\right)g\left(\frac{v}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 4^n\varphi\left(\frac{u}{2^n},\frac{v}{2^n}\right)\leq \lim_{n\to\infty} 4^n\frac{L^n}{4^n}\varphi(u,v)=0\end{aligned}$$

and so

$$H(uv)=H(u)H(v)$$

for all $u,v\in U(P)$.

The rest of the proof is similar to the proofs of Theorems 2.1, 3.1 and 4.1. □

Corollary 5.1 *Let $r>2$ and θ be nonnegative real numbers and $g:P\to Q$ be a mapping satisfying (3.5), (3.6) and (3.7). Then there exists a unique C^*-algebra homomorphism $H:P\to Q$ satisfying (3.8).*

Proof: The result follows from Theorem 5.1 by taking $L=2^{1-r}$ and $\varphi\left(\frac{u}{2},\frac{y}{2}\right)=\frac{\theta}{2^r}(1+\|y\|^r)$ for all $u\in U(P)$ and all $y\in P$. □

Theorem 5.2 *Let $\varphi:P^2\to[0,\infty)$ be a function and $g:P\to Q$ be a mapping satisfying (2.4), (3.2), (3.3) and (4.3). Then there exists a unique C^*-algebra homomorphism $H:P\to Q$ satisfying (4.4).*

Proof: It follows from (4.3) and (3.2) that

$$\begin{aligned}\|H(uv)-H(u)H(v)\| &= \lim_{n\to\infty}\frac{1}{4^n}\|g(4^nuv)-g(2^nu)g(2^nv)\| \\ &\leq \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nu,2^nv)\leq \lim_{n\to\infty}\frac{2^nL^n}{4^n}\varphi(u,v)=0\end{aligned}$$

and so $H(uv)=H(u)H(v)$ for all $u,v\in U(P)$.

The rest of the proof is similar to the proofs of Theorems 3.1, 3.2 and 5.1. □

Corollary 5.2 *Let $r<1$ and θ be nonnegative real numbers and $g:P\to Q$ be a mapping satisfying (3.5), (3.6) and (3.7). Then there exists a unique C^*-algebra homomorphism $H:P\to Q$ satisfying (3.10).*

Proof: The result follows from Theorem 5.2 by taking $L=2^{r-1}$ and $\varphi(2u,2y)=2^r\theta(1+\|y\|^r)$ for all $u\in U(P)$ and all $y\in P$. □

Now, we prove the Hyers-Ulam stability of C^*-algebra derivations in unital C^*-algebras associated to the additive functional inequality (2.1) by using the fixed point method.

Theorem 5.3 *Let $\varphi:P^2\to[0,\infty)$ be a function satisfying (5.1). If the mapping $g:P\to P$ satisfies (2.4), (3.3) and (3.11), then there exists a unique C^*-algebra derivation $D:P\to P$ such that*

$$\|g(y)-D(y)\|\leq\frac{1}{1-L}\varphi(u,y)$$

for all $u\in U(P)$ and all $y\in P$.

Proof: It follows from (5.1) and (3.11) that

$$\begin{aligned}\|D(uv)-D(u)v-uD(v)\| &= \lim_{n\to\infty}4^n\left\|g\left(\frac{uv}{4^n}\right)-g\left(\frac{u}{2^n}\right)\frac{v}{2^n}-\frac{u}{2^n}g\left(\frac{v}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty}4^n\varphi\left(\frac{u}{2^n},\frac{v}{2^n}\right)\leq\lim_{n\to\infty}4^n\frac{L^n}{4^n}\varphi(u,y)=0\end{aligned}$$

and so

$$D(uv) = D(u)v + uD(v)$$

for all $u, v \in U(P)$.

The rest of the proof is similar to the proofs of Theorems 3.3 and 4.1. □

Corollary 5.3 *Let $r > 2$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying (3.5), (3.7) and (3.13). Then there exists a unique C^*-algebra derivation $D : P \to P$ satisfying (3.14).*

Proof: The result follows from Theorem 5.3 by taking $L = 2^{1-r}$ and $\varphi\left(\frac{u}{2}, \frac{y}{2}\right) = \frac{\theta}{2^r}(1 + \|y\|^r)$ for all $u \in U(P)$ and all $y \in P$. □

Theorem 5.4 *Let $\varphi : P^2 \to [0, \infty)$ be a function and $g : P \to P$ be a mapping satisfying (2.4), (3.3), (3.11) and (4.3). Then there exists a unique C^*-algebra derivation $D : P \to P$ such that*

$$\|g(y) - D(y)\| \leq \frac{L}{1-L}\varphi(u, y)$$

for all $u \in U(P)$ and all $y \in P$.

Proof: It follows from (3.11) and (4.3) that

$$\begin{aligned}\|D(uv) - D(u)v - uD(v)\| &= \lim_{n\to\infty} \frac{1}{4^n} \|g(4^n uv) - g(2^n u)(2^n v) - (2^n u)g(2^n v)\| \\ &\leq \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n u, 2^n v) \leq \lim_{n\to\infty} \frac{2^n L^n}{4^n}\varphi(u, v) = 0\end{aligned}$$

and so $D(uv) = D(u)v + uD(v)$ for all $u, v \in U(P)$.

The rest of the proof is similar to the proofs of Theorems 3.3 and 4.2. □

Corollary 5.4 *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying (3.5), (3.13) and (3.7). Then there exists a unique C^*-algebra derivation $D : P \to P$ satisfying (3.16).*

Proof: The result follows from Theorem 5.4 by taking $L = 2^{r-1}$ and $\varphi(2u, 2y) = 2^r\theta(1 + \|y\|^r)$ for all $u \in U(P)$ and all $y \in P$. □

6. Conclusion

We introduced the additive functional inequality (0.1)and, using both the direct method and the fixed point method, proved its Hyers-Ulam stability in unital C^*-algebras. Furthermore, we applied these results to the study of C^*-algebra homomorphisms and derivations in unital C^*-algebras.

Declarations

Availablity of data and materials

Not applicable.

Human and animal rights

I would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest

The author declares that he has no competing interests.

Acknowledgments

This research was supported by the University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026).

7. Bibliography

References

1. M. Ahmad, A. Zada, *Ulam's stability results of conformable neutral fractional differential equations Ψ-Hilfer fractional integrodifferential equations*, Bol. Soc. Parana. Mat. (3) **41** (2023), 1–13.

2. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

3. L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Art. ID 4.

4. J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

5. S. Donganont, C. Park, *Combined system of additive functional equations in Banach algebras*, Open Mathematics **22** (2024), no. 1, 20230177.

6. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

7. F. Haddouchi, *Existence and Ulam-Hyers stability results for a class of fractional integro-differential equations involving nonlocal fractional integro-differential boundary conditions*, Bol. Soc. Parana. Mat. (3) **42** (2024), 1–19.

8. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

9. G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Int. J. Math. Math. Sci. **19** (1996), 219–228.

10. S. Jung, D. Popa, M. Th. Rassias, *On the stability of the linear functional equation in a single variable on complete metric spaces*, J. Global Optim. **59** (2014), 13–16.

11. R. V. Kadison, J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras: Elementary Theory*, Academic Press, New York, 1983.

12. Y. Lee, S. Jung, M. Th. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), 43–61.

13. D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

14. C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

15. C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

16. C. Park, *Fixed point method for set-valued functional equations*, J. Fixed Point Theory Appl. **19** (2017), 2297–2308.

17. V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

18. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

19. S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

20. Z. Zamani, B. Yousefi, H. Azadi Kenary, *Fuzzy Hyers-Ulam-Rassias stability for generalized additive functional equations*, Bol. Soc. Parana. Mat. (3) **40** (2022), 1–14.

Siriluk Donganont,
School of science,
University of Phayao,
Thailand.
E-mail address: siriluk.pa@up.ac.th; Orcid ID: https://orcid.org/0000-0003-0594-9082