Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.77686

Li-Yorke chaos of C_0-semigroups under the weak topology

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ABSTRACT: In this paper, we introduce and investigate the concept of Li-Yorke chaos for C_0-semigroups in Banach spaces endowed with the weak topology. We establish that, unlike Li-Yorke chaos under the norm topology, weak Li-Yorke chaos can occur in finite-dimensional spaces. Several fundamental properties of such semigroups are presented. Furthermore, we derive necessary and sufficient conditions for a C_0-semigroup to exhibit Li-Yorke chaos in the weak topology.

Key Words: Li-Yorke chaos, Weak Li-Yorke chaos, irregular vectors, semi-irregular vectors, C_0-semigroup.

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1. Introduction and preliminaries

In the present paper, a topological dynamical system is a pair $(X, (T_t)_{t\geq 0})$, where X is a Banach space, X^* is the dual of X(X^* is the natural object to characterize proximity and distance in the weak topology.) and $(T_t)_{t\geq 0}$ is a strongly continuous semigroup (shortly C_0-semigroup). The infinitesimal generator of a C_0-semigroup $(T_t)_{t\geq 0}$ on X is defined by:

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \text{ for all } x \in D(A).$$

Recall that the orbit of a vector $x \in X$ under the action of a C_0-semigroup $(T_t)_{t\geq 0}$ on X is defined by

$$Orb((T_t)_{t\geq 0}, x) = \{T_t(x), t \geq 0\}.$$

When studying these orbits, we can reveal one of the most popular notions in the topological dynamics, which is hypercyclicity. It was first introduced for single operators by Bayart [6] and after that, Desh et al. studied this concept for C_0-semigroups in [15]. A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be hypercyclic if there exists a vector $x \in X$ such that its orbit under $(T_t)_{t\geq 0}$ is dense in X.
In the context of infinite separable Banach spaces, the notion of hypercyclicity is equivalent with the notion of topological transitivity which was introduced for single operators by Birkhoff in [10] and studied for C_0-semigroups in [15]. A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be topologically transitive if for every pair of non empty open sets U and V there exists $t_0 > 0$ such that

$$T_{t_0}(U) \cap V \neq \emptyset.$$

Li and Yorke were the first ones to connect the term "chaos" with a map in [18]. Following this, the author in [17] extended this notion to C_0-semigroups. Recall that a C_0-semigroup $(T_t)_{t\geq 0}$ on X is said to be Li-Yorke chaotic if and only if there exists an uncountable scrambled set $\Omega \subset X$ such that for every couple $(x, y) \in \Omega^2$ such that $x \neq y$, we have

$$\liminf_{t\to+\infty} \|T_t(x) - T_t(y)\| = 0 \text{ and } \limsup_{t\to+\infty} \|T_t(x) - T_t(y)\| > 0.$$

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2010 *Mathematics Subject Classification:* 47D06, 37B20, 47A16.
Submitted July 06, 2025. Published September 22, 2025

Beauzamy in [8] introduced the concept of irregular vectors which has a deep connection with Li-Yorke chaos. According to [24], a vector $x \in X$ is said to be irregular for $(T_t)_{t\geq 0}$ if

$$\liminf_{t\to+\infty} \|T_t(x)\| = 0 \text{ and } \limsup_{t\to+\infty} \|T_t(x)\| = +\infty.$$

A vector $x \in X$ is said to be semi-irregular for $(T_t)_{t\geq 0}$ if

$$\liminf_{t\to+\infty} \|T_t(x)\| = 0 \text{ and } \limsup_{t\to+\infty} \|T_t(x)\| > 0.$$

Now turning to the topological dynamics under the weak topology. The study of weak orbits started in 1996 with the work of J.van Nerveen in [23]. The concept of weak hypercyclicity for single operators was introduced by Chan and Sanders in [11] by considering the density under the weak topology instead of the norm topology. An operator T is said to be weakly hypercyclic if there is a vector $x \in X$ whose orbit is dense under the weak topology of X, i.e,

$$\overline{Orb(T,x)}^{w} = X.$$

Clearly, the norm hypercyclicity implies the weak hypercyclicity since the norm topology is stronger than the weak topology. However the converse does not hold in general (see [11]).
In [5], the authors introduced the notion of weak hypercyclicity for C_0-semigroups. Recall that a C_0-semigroup $(T_t)_{t\geq 0}$ is said to be weakly hypercyclic if there exists a vector x whose orbit under $(T_t)_{t\geq 0}$ is weakly dense in X. The vector x is called weakly hypercyclic vector for $(T_t)_{t\geq 0}$. The set of all weakly hypercyclic vectors for $(T_t)_{t\geq 0}$ is denoted by $WHC((T_t)_{t\geq 0})$.
Recently, in [25] Li-Yorke chaos was studied for 2×2 matrices where the authors gave some sufficient conditions for the existence of Li-Yorke chaos under the weak topology. For more information see [2,3,4, 7,16,18,21,25].
The aim of this paper is to study the Li-Yorke chaos under the weak topology for C_0-semigroups.
The reminder of this paper is organized as follows. In Section 2, we define Li-Yorke chaos for C_0-semigroups, semi-irregular vectors and irregular vectors for C_0-semigroups under the weak topology. In Section 3, we give some properties related to the Li-Yorke chaos of C_0-semigroups under the weak topology. Also, we study the links between these notions. Moreover, we give some necessary and sufficient conditions for a C_0-semigroup to be Li-Yorke chaotic under the weak topology.

2. Li-Yorke chaotic C_0-semigroup under the weak topology

In this section, we introduce the notions of strong and weak Li-Yorke chaos under the weak topology for C_0-semigroups.

Definition 2.1. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. A couple $(x,y) \in X^2$ is said to be strongly Li-Yorke pair for $(T_t)_{t\geq 0}$ under the weak topology if

$$\liminf_{t\to+\infty} |f(T_t(x) - T_t(y))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x) - T_t(y))| > 0,$$

for every $f \in X^* \setminus \{0\}$. The set Ω such that every pair of distinct points in Ω forms a strong Li-Yorke pair under the weak topology is said to be a strong scrambled set under the weak topology.

Remark 2.2. The exclusion of $f = 0$ is essential for a meaningful definition. If $f = 0$ we will have $\limsup_{t\to+\infty} |f(T_t(x) - T_t(y))| = 0$, which is rendering Li-Yorke chaos impossible.

Example 2.3. Let $X = L_1(\mathbb{R}^+)$ and $(T_t)_{t\geq 0}$ be a C_0-semigroup defined on X by

$$T_t\phi(x) = \frac{1+(x+t)^2}{1+x^2}\phi(x+t), x, t \in \mathbb{R}^+.$$

$(T_t)_{t\geq 0}$ admits a strong Li-Yorke pair under the weak topology since it is Li-yorke chaotic under the norm topology ([16, Proposition 7.34]).

The following definition presents a weak version of Li-Yorke pairs under the weak topology.

Definition 2.4. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. For a fixed $f \in X^* \setminus \{0\}$, a couple $(x, y) \in X^2$ is said to be a weakly Li-Yorke pair for $(T_t)_{t\geq 0}$ under the weak topology if

$$\liminf_{t\to+\infty} |f(T_t(x) - T_t(y))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x) - T_t(y))| > 0.$$

The set Ω such that every pair of distinct points in Ω forms a weakly Li-Yorke pair, is said to be a weak scrambled set under the weak topology.

Now, we will define Li-Yorke chaos under the weak topology.

Definition 2.5. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. $(T_t)_{t\geq 0}$ is said to be strongly (resp. weakly) Li-Yorke chaotic under the weak topology if the strong (resp. weak) scrambled set Ω is uncountable.

Remark 2.6.
1. Obviously, every Strongly Li-Yorke chaotic C_0-semigroup under the weak topology is weakly Li-Yorke chaotic under the weak topology. The converse does not hold in general as we will see in Example 3.8.

2. Every weakly hypercyclic C_0-semigroup is Li-Yorke chaotic under the weak topology. In fact, let x be a weakly hypercyclic. We can take the weak scrambled set $\{\alpha x, |\alpha| < 1\}$. Let $u = \alpha x$, $v = \beta x$ in S and $f \in X^* \setminus \{0\}$, we need to prove that

$$\liminf_{t\to+\infty} |f(T_t)(u - v))| = 0 \text{ and } \limsup_{t\to+\infty} |f(T_t(u - v))| > 0.$$

Since $|\alpha - \beta| > 0$, it suffices to show that:

$$\liminf_{t\to+\infty} |f(T_t x)| = 0.$$
$$\limsup_{t\to+\infty} |f(T_t x)| > 0.$$

- By the weak density we can easily deduce that $\liminf_{t\to+\infty} |f(T_t x)| = 0$
- Now, if $\limsup_{t\to+\infty} |f(T_t x)| = 0$. Then, $|f(T_t(x))| \leq M$ for large t. Choose y_0 with $|f(y_0)| > M$. Therefore the weakly open set $\{z : |f(z) - f(y_0)| < \delta\}$ does not meet the orbit of x which contradicts the weak density.

Thus $(T(t))_{t\geq 0}$ is Li-Yorke chaotic under the weak topology. However, the converse is not true as shown by Example 3.4.

Li-Yorke chaos in the norm and the weak topology are equivalent in the case of the translation C_0-semigroup on weighted Lebesgue spaces. To show that, we need the following definitions.
Let I be one of the intervals $]-\infty, \infty[$ or $[0, \infty[$.

Definition 2.7. An admissible weight function on I is a measurable function $\rho : I \to \mathbb{R}$ satisfying the following conditions:

i) $\rho(\tau) > 0$ for all $\tau \in I$;

ii) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{\omega t}\rho(t + \tau)$, for all $\tau \in I$ and all $t > 0$.

Definition 2.8. Let ρ be an admissible weight function on I. For $1 \leq p < \infty$, we define the space

$$L^p_\rho(I, \mathbb{R}^+) = \left\{ u : I \to \mathbb{R}^+ \text{ with } u \text{ measurable and } \int_I |u(\tau)|^p \rho(\tau) d\tau < \infty \right\}$$

equipped with the norm

$$\|u\|_p := \left(\int_I |u(\tau)|^p \rho(\tau) d\tau \right)^{1/p} < \infty.$$

Theorem 2.9. *Let ρ be an admissible weight function on I, $X = L^p_\rho(I, \mathbb{R}_+)$ and $(T_t)_{t\geq 0}$ be the translation C_0-semigroup on X defined by $T_t f(x) = f(x+t)$ for every $f \in X$ and $t \geq 0$. The following assertions are equivalent*

i) $(T_t)_{t\geq 0}$ *is Li-Yorke chaotic.*

ii) $(T_t)_{t\geq 0}$ *is strongly Li-Yorke chaotic under the weak topology.*

iii) $\liminf\limits_{k\to\infty} \rho(t_k) = 0$.

Proof:

In fact it suffices to show $ii) \Rightarrow iii)$ and $iii) \Rightarrow i)$.

$ii) \Rightarrow iii)$. Assume that $(T_t)_{t\geq 0}$ is strongly Li-Yorke chaotic under the weak topology and $\liminf\limits_{k\to\infty} \rho(t_k) > 0$. Then, there exist $C > 0$ and $T \in \mathbb{R}^+$ such that $\rho(x) \geq C$ for every $x \geq T$.

Let $f \in X$, we have

$$\int_T^{+\infty} |f(s)|^p.Cds \leq \int_0^{+\infty} |f(s)|^p \rho(s) ds < +\infty,$$

it follows,

$$\lim_{t\to+\infty} \int_t^{+\infty} |f(s)|^p ds = 0.$$

In parallel, we have

$$\begin{aligned}
\|T_t f(x)\| &= (\int_0^{+\infty} |T_t f(s)|^p \rho(s) ds)^{\frac{1}{p}} \\
&= (\int_0^{+\infty} |f(s+t)|^p \rho(s) ds)^{\frac{1}{p}} \\
&= (\int_t^{+\infty} |f(s)|^p \rho(s-t) ds)^{\frac{1}{p}} \\
&\leq (\int_t^{+\infty} |f(s)|^p M ds)^{\frac{1}{p}}.
\end{aligned}$$

This means that,

$$\lim_{t\to+\infty} \|T_t f(x)\| = 0, \forall f \in X.$$

It follows that,

$$\lim_{t\to+\infty} |f^* T_t f(x)| = 0, \forall f \in X \text{ and } \forall f^* \in X^*.$$

Thus, $(T_t)_{t\geq 0}$ is not strongly Li-yorke chaotic under the weak topology which is contradicts the hypothesis.

$iii) \Rightarrow i)$. Assume that $\liminf\limits_{k\to\infty} \rho(t_k) = 0$. By [15, Theorem 4.7], $(T_t)_{t\geq 0}$ is hypercyclic and hence it is Li-yorke chaotic. □

Now, we will define irregular and semi-irregular vectors under the weak topology.

Definition 2.10. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X.

i) A vector $x \in X$ is said to be strongly irregular for $(T_t)_{t\geq 0}$ under the weak topology if

$$\liminf_{t\to+\infty} |f(T_t(x))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x))| = +\infty,$$

for every $f \in X^* \setminus \{0\}$

ii) A vector $x \in X$ is said to be strongly semi-irregular for $(T_t)_{t\geq 0}$ under the weak topology if

$$\liminf_{t\to+\infty} |f(T_t(x))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x))| > 0,$$

for every $f \in X^* \setminus \{0\}$.

Definition 2.11. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X and $f \in X^* \setminus \{0\}$ be given.

i) A vector $x \in X$ is said to be weakly irregular for $(T_t)_{t\geq 0}$ under the weak topology if

$$\liminf_{t\to+\infty} |f(T_t(x))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x))| = +\infty.$$

ii) A vector $x \in X$ is said to be weakly semi-irregular for $(T_t)_{t\geq 0}$ under the weak topology if

$$\liminf_{t\to+\infty} |f(T_t(x))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x))| > 0.$$

The existence of (semi)- irregular vectors ensures Li-Yorke chaos in the weak topology; we prove this in the next section.

3. Properties of Li-Yorke chaotic C_0-semigroup under the weak topology

In this section, our focus turns to the properties related to Li-Yorke chaos under the weak topology.

Theorem 3.1. *The set of all weakly irregular vectors for $(T_t)_{t\geq 0}$ under the weak topology is norm dense in the set of all weakly semi-irregular vectors for $(T_t)_{t\geq 0}$ under the weak topology.*

Proof: Let $x \in X$ be a weakly semi-irregular vector for $(T_t)_{t\geq 0}$ under the weak topology and set

$$X_0 = \overline{span}(Orb((T_t)_{t\geq 0}, x)).$$

It is clear that X_0 is closed, and T_t-invariant for every $t \geq 0$. Let $(S_t)_{t\geq 0}$ be the C_0-semigroup $(T_t)_{t\geq 0}$ restricted to X_0. Then, for some $y \in X_0$, $C > 0$ and $f \in X^* \setminus \{0\}$ we have,

$$\sup_{t\in\mathbb{R}^+} |f(S_t)(y)| > C|f(y)|.$$

In fact, assume that for every $y \in X_0$ and every $C > 0$ we have

$$\sup_{t\in\mathbb{R}^+} |f(S_t)(y)| \leq C|f(y)|.$$

Now since y is weakly semi-irregular under the weak topology, we have

$$\liminf_{t\to+\infty} |f(S_t(y))| = 0, \text{ and } \limsup_{t\to+\infty} |f(S_t(y))| > 0.$$

Hence, there exists $\delta > 0$ such that $\limsup_{t\to+\infty} |f(S_t(y))| > \delta$. Thus, there exist $t, s \in \mathbb{R}^+$ such that $|f(S_t(y))| \leq \epsilon$ and $|f(S_s(y))| > \delta + \epsilon$. Without loss of generality assume that $s \geq t$.
Therefore,

$$\epsilon + \delta < |f(S_s)(y)| = |f(S_{s-t+t})(y)| \leq C|f(S_t)(y)| \leq C\epsilon.$$

Contradiction(taking $\epsilon \to 0$ we will have $\delta < 0$).
Which means that there exists $y \in X_0$ such that

$$\limsup_{t\in\mathbb{R}^+} |f(S_t)(y)| = \infty.$$

Put

$$Y_0 = \{y \in X_0; \sup_{t\in\mathbb{R}^+} |f(S_t)(y)| > C|f(y)|\}.$$

Let $z \in X_0 \setminus Y_0$, we have then

$$\sup_{t\in\mathbb{R}^+} |f(S_t)(z)| < \infty.$$

and for $y \in X_0$ and $\lambda \neq 0$ we have,

$$\limsup_{t\to+\infty} |f(S_t)(z + \lambda y)| = \infty.$$

Thus, $z \in \overline{Y_0}$. It follows that Y_0 is dense in X_0.
On the other hand put

$$Z_0 = \{y \in X_0; \liminf_{t\to+\infty} |f(S_t)(y)| = 0\}.$$

Z_0 is clearly dense.
Thus, $Y_0 \cap Z_0$ formed by weakly irregular vectors under the weak topology is dense in X_0. □

The next result is a direct consequence of the previous theorem.

Corollary 3.2. *Let $(T_t)_{t\geq0}$ be a C_0-semigroup on X. Then every neighborhood of a weakly semi-irregular vector for $(T_t)_{t\geq0}$ contains a weakly irregular vector for $(T_t)_{t\geq0}$ under the weak topology.*

The following result study the connection between Li-Yorke chaos and (semi)-irregular vectors under the weak topology.

Theorem 3.3. *Let $(T_t)_{t\geq0}$ be a C_0-semigroup on X. The following assertions are equivalent*

i) $(T_t)_{t\geq0}$ *is weakly Li-Yorke chaotic under the weak topology.*

ii) $(T_t)_{t\geq0}$ *admits a weakly Li-Yorke pair under the weak topology.*

iii) $(T_t)_{t\geq0}$ *admits a weakly semi-irregular vector under the weak topology.*

iv) $(T_t)_{t\geq0}$ *admits a weakly irregular vector under the weak topology.*

Proof:

i) $\Rightarrow$ *ii)* Obvious.

$ii) \Rightarrow iii)$ Suppose that $(T_t)_{t\geq 0}$ admits a weak Li-Yorke pair $(x, y) \in X^2$ under the weak topology. Then, for a fixed $f \in X^* \setminus \{0\}$, there exists $t > 0$ such that

$$\liminf_{t\to+\infty} |f(T_t(x) - T_t(y))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(T_t(x) - T_t(y))| > 0.$$

We put $z = x - y$ we obtain z is weakly semi-irregular vector under the weak topology.

$iii) \Rightarrow iv)$ Directly by Corollary 3.2.

$iv) \Rightarrow i)$ Suppose that x is a weakly irregular vector for $(T_t)_{t\geq 0}$ under the weak topology, then $\Omega = span\{x\}$ is an uncountable strong scrambled set under the weak topology for $(T_t)_{t\geq 0}$.

□

Example 3.4. Let $X = \prod_{n=1}^{+\infty} L^2([0, 2\pi])$ endowed with the norm

$$\|(f_n)\|_X = \sup_{n\in\mathbb{N}} \|f_n\|_{L^2}$$

and $(T_t)_{t\geq 0}$ a one parameter family on X defined by

$$T_t f(x) = f(x + t).$$

Clearly $(T_t)_{t\geq 0}$ is C_0-semigroup on X. Moreover, it is strongly Li-Yorke chaotic under the weak topology. In fact, let $(f_n)_{n\in\mathbb{N}}$ defined by $f_n(x) = sin(nx)((-1)^n + 1) + c_n$ for every $n \in \mathbb{N}$ where

$$c_n = \begin{cases} 0 & \text{if n is even} \\ 1 & \text{if n is odd} \end{cases}$$

$(f_n)_{n\in\mathbb{N}}$ is a semi- irregular vector on X under weak topology for $(T_t)_{t\geq 0}$. Indeed, we have

$$\begin{aligned} T_n f_n(x) &= sin(n(x + n))((-1)^n + 1) + c_n \\ &= \begin{cases} 2sin(n(x + n)) & \text{if n is even} \\ 1 & \text{if n is odd} \end{cases} \end{aligned}$$

Since $sin(n(x + n))$ converges weakly to 0 we have

$$\lim_{n\to+\infty} |f^*(T_n f_n(x))| = \lim_{n\to+\infty} 2|f^*(sin(n(x + n))| = 0, \text{ for every} f^* \in X^*,$$

if n is even. On the other hand, we have

$$\lim_{n\to+\infty} |f^*(T_n f_n(x))| = 1,$$

if n is odd.
Therefore, by Theorem 3.3 $(T_t)_{t\geq 0}$ is strongly Li-Yorke chaotic under the weak topology, but it is not Li-yorke chaotic under the norm topology nor weakly hypercyclic since all orbits under $(T_t)_{t\geq 0}$ are bounded.

Proposition 3.5. *Weak Li-Yorke chaos under the weak topology is preserved under conjugacy.*

Proof: Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be two C_0-semigroups on X and Y respectively, $\phi : X \to Y$ is an isomorphism such that $\phi \circ T_t = S_t \circ \phi$ for every $t \geq 0$ and f be fixed in $X^* \setminus \{0\}$.
Assume that $(S_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology, by Theorem 3.3 $(S_t)_{t\geq 0}$ admits a weakly semi-irregular vector under the weak topology $y \in Y$. Now, since ϕ is isomorphism there exists

$x \in X$ such that $y = \phi(x)$.
On the other hand,

$$\liminf_{t\to+\infty} |f(S_t(\phi(x)))| = 0,$$

and

$$\limsup_{t\to+\infty} |f(S_t(\phi(x)))| > 0.$$

Then, we have

$$\lim_{t\to+\infty} |f \circ \phi(T_t(x))| = 0,$$

and

$$\lim_{t\to+\infty} |f \circ \phi(T_t(x))| > 0.$$

Thus, $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology. □

While it is known that Li-Yorke chaos in the norm topology is preserved between C_0-semigroups and their operators T_{t_0} for every $t_0 \in \mathbb{R}^+$ [13, Theorem 3.2], this property fails for Li-Yorke chaos under the weak topology as it is shown by the following example

Example 3.6. Let $X = \prod_{n=1}^{+\infty} L^2([0, 2\pi])$ where each component $L^2([0, 2\pi])$ is periodic with period 2π, endowed with the norm

$$\|(f_n)\|_X = \sup_{n\in\mathbb{N}} \|f_n\|_{L^2}$$

and $(T_t)_{t\geq 0}$ a C_0-semigroup on X defined by

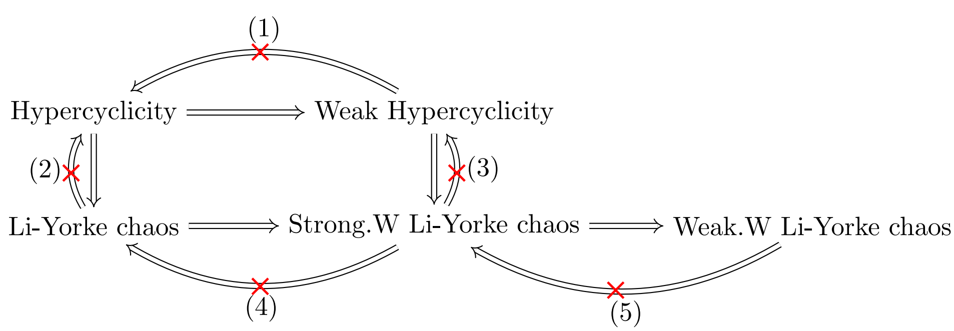
$$T_t f(x) = f(x + t).$$

Clearly $(T_t)_{t\geq 0}$ is a strongly Li-Yorke chaotic C_0-semigroup under the weak topology Example 3.4. However, if we take $t_0 = 2\pi$

$$T_{t_0} f = f,$$

The operator T_{t_0} acts as the identity operator which obviously cannot be strongly Li-yorke chaotic under the weak topology.

Remark 3.7. The next diagram summary the links between the notion of Li-Yorke chaos and hypercyclicity under the norm and weak topologies. We used the notation Strong.W (Weak.W) to refer to strong (weak) Li-Yorke chaos under the weak topology.



(1):Weak Hypercyclicity $\not\Rightarrow$ hypercyclicity: see [5, Example 3.3].
(2):Li-Yorke chaos $\not\Rightarrow$ Hypercyclicity: see [21, Example 3.4]
(3) and (4): Strong.W Li-yorke chaos $\not\Rightarrow$ Li-Yorke chaos (weak hypercyclicity): There exist a C_0-semigroup that is strongly li-yorke chaotic under the weak topology. However, it is not li-yorke chaotic under the norm topology nor weakly hypercyclic (See Example 3.4).

(5):Weak.W Li-yorke chaos $\not\Rightarrow$ Strong.W Li-yorke chaos: In finite dimensional space the weak and norm topology coincide and as it is well known Li-yorke chaos does not exists on finite dimensional spaces [1, Corollary 2.5], which means that strong Li-yorke chaos under the weak topology also cannot exists on finite dimensional spaces. However, There exists a C_0-semigroup that is weakly Li-yorke chaotic under the weak topology on a finite dimensional space as it shown by the following example:

Example 3.8. Let $X = \mathbb{C}^n$ and A a diagonal matrix of $n \times n$ dimension.

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$. Let $r \in \mathbb{N}$ with $1 < r < n$ such that:

- $\lambda_r = i\theta$ for some $\theta \in \mathbb{R}$ (so $\alpha = |e^{\lambda_r}| = 1$),
- $\lambda_{r+1} = \lambda_r + i\pi$ (so $e^{\lambda_{r+1}} = -e^{\lambda_r}$).

Let $(T_t)_{t\geq 0}$ be the C_0-semigroup defined as $T_t(x) = e^{tA}x$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$.
$(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology.
Indeed, let z be a vector in $\mathbb{R}^n$ defined by

$$z = (z_i)_{0\leq i\leq n} = \begin{cases} 1 \text{ if } & 1 \leq i \leq r+1 \\ 0 \text{ if } & \text{othewise} \end{cases}$$

Then, for any $t \in \mathbb{R}^+$, we have:

$$T_t(z) = \left(e^{\lambda_1 t}, \dots, e^{\lambda_{r-1} t}, e^{\lambda_r t}, e^{\lambda_{r+1} t}, 0, \dots, 0\right)$$

Define the fixed linear functional $f(x) = x_r + x_{r+1}$ for $x = (x_i)_{0\leq i\leq n} \in \mathbb{C}^n$. Then, for $t = m \in \mathbb{N}$

$$\begin{aligned} |\langle T_m(z), f\rangle| &= |e^{\lambda_r m} + e^{\lambda_{r+1} m}| \\ &= |\alpha^m + (-1)^m \alpha^m| \\ &= \begin{cases} 0 \text{ if } & m \text{ } is \text{ } odd \\ 2 \text{ if } & m \text{ } is \text{ } even \end{cases} \end{aligned}$$

Thus, z is a weakly irregular vector under the weak topology for $(T_t)_{t\geq 0}$. It follows that, $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology.

In the following result, we give necessary and sufficient condition for the existence of Li-Yorke chaos under the weak topology.

Definition 3.9. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If there exists a subset $X_0 \subset X$ such that for a fixed $f \in X^* \setminus \{0\}$ we have:

i) For every $x \in X_0$, there exists a increasing sequence $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ such that

$$\lim_{n\to\infty} f(T_{t_n}(x)) = 0.$$

ii) The exists $C > 0$ $\sup_{t\in\mathbb{R}^+} |f(T_t(y))| > C|f(y)|$, for every $y \in Y := \overline{span}(X_0)$.

Then $(T_t)_{t\geq 0}$ satisfies the weak Li-Yorke Chaos Criterion under the weak topology.

Theorem 3.10. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology if and only if it satisfies the weak Li-Yorke Chaos Criterion under the weak topology.*

Proof: Assume that $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology, by Theorem 3.3 $(T_t)_{t\geq 0}$ admits a weakly semi-irregular x under the weak topology. Put $X_0 = \{x\}$, then $(T_t)_{t\geq 0}$ the weak Li-Yorke Chaos Criterion under the weak topology.
Conversely, Assume that $(T_t)_{t\geq 0}$ satisfies the weak Li-Yorke Chaos Criterion under the weak topology. Let $X_0 \subset X$ and $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ satisfying i) and ii) of the criterion. If there exists $x \in X_0$ such that x is weakly semi irregular under the weak topology, then $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology.
If not, by i) we have for every $x \in X_0$ and a fixed $f \in X^* \setminus \{0\}$ we have

$$\lim_{n\to\infty} f(T_{t_n}(x)) = 0.$$

Let

$$Y_0 = \overline{\{x \in X; \lim_{n\to\infty} f(T_{t_n}(x)) = 0\}}.$$

Y_0 is closed and T_t-invariant for every $t \geq 0$. Let $(S_t)_{t\geq 0}$ be the restriction of $(T_t)_{t\geq 0}$ to Y_0. Since y from ii) is in Y_0 we have

$$\sup_{t\in\mathbb{R}^+} |f(S_t(y))| > C|f(y)|.$$

Put

$$Z_0 = \{y \in Y_0; \sup_{t\in\mathbb{R}^+} |f(S_t)(y)| = +\infty\},$$

and

$$Z_1 = \{y \in Y_0; \inf_{t\in\mathbb{R}^+} |f(S_t)(y)| = 0\}.$$

Z_1 is clearly dense in Y_0, also we have Z_0 is clearly dense. Thus, $z \in Z_0 \cap Z_1$ is a weakly irregular vector under the weak topology. Hence $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology. □

Theorem 3.11. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Let $f \in X^* \setminus \{0\}$, if there exists a weakly dense set X_0 such that*

$$\liminf_{t\to+\infty} |f(T_t(x))| = 0,$$

for every $x \in X_0$, then the following assertions are equivalent:

i) $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology.

ii) $(T_t)_{t\geq 0}$ admits a weakly unbounded orbit.

Proof:

i) ⇒ *ii)* Assume that $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology. Then, by Theorem 3.3 $(T_t)_{t\geq 0}$ admits a weakly irregular vector under the weak topology x. By definition, for a fixed $f \in X^* \setminus \{0\}$ we have

$$\limsup_{t\to\infty} |fT_t(x)| = +\infty.$$

It follows that, for every $y \in Orb((T_t)_{t\geq 0}, x)$ we have

$$\limsup_{t\to\infty} |fT_t(y)| = +\infty.$$

ii) ⇒ *i)* Now, assume that $(T_t)_{t\geq 0}$ admits a weakly unbounded orbit. This means that there exists $x \in X$ such that $Orb((T_t)_{t\geq 0}, x)$ is weakly unbounded. Therefore, for some $f \in X^* \setminus \{0\}$ we have :

$$\sup_{t\in\mathbb{R}^+} |fT_t(x)| = +\infty.$$

Thus, we have

$$\limsup_{t\to\infty} |f(T_t(x))| = +\infty.$$

Therefore, by the weak Li-Yorke chaos critertion $(T_t)_{t\geq 0}$ is weakly Li-Yorke chaotic under the weak topology.

□

Acknowledgments

We are grateful to the anonymous referees for their careful reading and constructive comments.

FUNDING

Not applicable.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

AUTHOR CONTRIBUTIONS

All authors are contributed equally in the paper.

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