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Weak solution for perturbed fractional p-Laplacian system via Young measures

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ABSTRACT: In this work, we demonstrate the existence of weak solutions to a class of fractional p-Laplacian problems in degenerate form. Under appropriate assumptions concerning the main functions, the existence of weak solutions is obtained by applying the Galerkin method combined with the theory of Young measures.

Key Words: fractional p-Laplacian systems, weak solution, Young measure.

Contents

1. Introduction

The study of problems involving fractional and nonlocal operators has received a lot of interest lately. Specifically, we refer to Di Nezza and *al.* [15] for a full introduction to the study of the fractional Sobolev spaces and the fractional p-Laplacian operators. In this paper, we investigate the existence of weak solutions for the fractional p-Laplacian problem namely

$$\begin{cases} (-\Delta)_p^s(u-\Theta(u)) + H(x,u) = f(x,u) \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$ $(N \geq 3)$, $p \geq 2$, $N > ps$ where s is a fixed real between 0 and 1, Θ is real function, H and f are a Carathéodory functions assumed to satisfy some conditions (see below). Here, $(-\Delta)_p^s$ is the fractional p-Laplacian operator which will be detailled in Section 2. Note that this type of problem was studied in [19] by using the variational method. In this study, the authors consider that H is a real function and $f \in L^\infty(\Omega)$. In the specific case where $\Theta = 0$ and $H \equiv 0$, we obtain the fractional p-Laplacian system of the form

$$\begin{cases} (-\Delta)_p^s(u) = f(x,u) \text{ in } \Omega, \\ \\ u = 0 \text{ on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.2}$$

The problem (1.2) has been treated in several papers. For example, Qui and Xiang [17] proved the existence of nonnegative solutions by using Leray-Schauder's nonlinear alternative. Recently, Balaadich and Azroul [3] studied this problem by the theory of Young measures. They also address other nonlinear problems thanks to this method (see [2,4,5]).
When $p = 2$, problem (1.2) reduces to the fractional Laplacian problem

$$(-\Delta)^s u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \mathbb{R}^N \backslash \Omega.$$

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In [20], the authors get the existence of nontrivial weak solutions of this problem by using the moutain pass theorem. See [12,13] for more details and results.
In the present paper, we study the existence of weak solutions for the problem (1.1) involving nonlocal fractional operators by using the tools of Young measures. To the best of our knowledge, this problem has never been studied in this framework. Young measures theory have applications in the calculus of variations, especially models from material science, and the study of nonlinear partial differential equations, as well as in various optimization.
Our study problem (1.1) arises in continuum mechanics, population dynamics, and many other different applications. More precisely, we can cite for example, the following parabolic problem models the flow of fluid through porous media:

$$\frac{\partial \theta}{\partial t} - \operatorname{div}\Big(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\Big) = 0,$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.
This paper is divided into three sections. In Section 2, we gives some preliminaries on fractional Sobolev spaces and some basic tools to prove Theorem 2.1. In Section 3, we prove the existence result by passage to the limit in the Galerkin approximating equations.

2. Preliminaries

In this section, we will recall some notations and definitions, and we will state some results that will be used in this work.
Let Ω be a bounded open domain of $\mathbb{R}^N$ and let $0 < s < 1 < p < \infty$ be a real number. We consider the fractional critical exponent defined by

$$p_s^* = \begin{cases} \dfrac{Np}{N - sp} & \text{if} \quad sp < N, \\ \infty & \text{if} \quad sp \geq N. \end{cases}$$

The fractional p-Laplacian operator $(-\Delta)_p^s$ can be defined as follows :

$$(-\Delta)_p^s u(x) = 2 \lim_{\pi \longrightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\pi(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \;\; x \in \mathbb{R}^N,$$

where $B_\pi(x) = \{x \in \mathbb{R}^N : \; |x - y| < \pi\}$.
In the following, we denote $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$ where

$$\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N} \text{ and } \mathcal{C}(\Omega) = \mathbb{R}^N \setminus \Omega.$$

W is a linear space of measurable functions from $\mathbb{R}^N$ to $\mathbb{R}^m$ such that the restriction to Ω of any function $u \in W$ belongs to $L^p(\Omega; \mathbb{R}^m)$ and

$$\int\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \Big[\int\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy\Big]^{1/p}.$$

We will work in the closed linear subspace

$$W_0 = \Big\{u \in W : \; u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\Big\}.$$

In this space, we may also use the norm

$$\|u\|_{W_0} := \Big[\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \Big]^{1/p}.$$

Then $\Big(W_0, \|\cdot\|_{W_0}\Big)$ is a uniformly convex Banach space (see [22]). Moreover, the following Poincare's inequality from [8] will be used below: there exists $C_\omega > 0$ such that

$$\|h\|_{L^\omega(\Omega;\mathbb{R}^m)} \leq C_\omega \|h\|_{W_0} \text{ for all } h \in W_0 \text{ and } \omega \in [1, p_s^*]. \tag{2.1}$$

Lemma 2.1 *([10]) The space $C_0^\infty(\Omega;\mathbb{R}^m)$ of infinitely differentiable functions with compact support on Ω is dense in W_0.*

Lemma 2.2 *([15]) The following embedding $W_0 \hookrightarrow L^r(\Omega;\mathbb{R}^m)$ is compact for all $r \in [1, p_s^*)$ and continuous for all $r \in [1, p_s^*]$.*

The dual space of $\Big(W_0, \|\cdot\|_{W_0}\Big)$ is denoted by $\Big(W_0^*, \|\cdot\|_{W_0^*}\Big)$

Lemma 2.3 *[1] For $\xi,\ \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have :*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi-\eta).$$

Lemma 2.4 *For $a \geq 0,\ b \geq 0$ and $1 \leq p < +\infty$, we have :*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

As mentioned in the introduction, we will use the concept of Young measure. Here, we give a brief review of Young measure and some properties needed in this paper (see [6,9,11]).
In the following $\mathcal{C}_0(\mathbb{R}^m)$ denote the space of continuous real-valued functions on $\mathbb{R}^m$ with compact support with respect to the $\|\cdot\|_\infty$-norm. Its dual $\mathcal{M}(\mathbb{R}^m)$ is the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle \nu, \phi \rangle = \int_{\mathbb{R}^m} \phi(\lambda) d\nu(\lambda)$$

for $\nu : \Omega \longrightarrow \mathcal{M}(\mathbb{R}^m)$ and $\phi \in \mathcal{C}_0(\mathbb{R}^m)$.
Note that $\langle \nu, id \rangle = \displaystyle\int_{\mathbb{R}^m} \lambda d\nu(\lambda)$.

Lemma 2.5 *([9], Theorem 1.5.2)*
Let $\{z_j\}_{j\geq 1}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then, there exists a subsequence $\{z_k\}_k \subset \{z_j\}_j$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\phi \in \mathcal{C}(\mathbb{R}^m)$ we have

$$\phi(z_k) \rightharpoonup^* \ \overline{\phi} \text{ weakly in } L^\infty(\Omega;\mathbb{R}^m)$$

where $\overline{\phi}(x) = \langle \nu_x, \phi \rangle = \displaystyle\int_{\mathbb{R}^m} \phi(\lambda) d\nu_x(\lambda)$ for a.e. $x \in \Omega$.

We call $\{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequence $\{z_k\}_{k\geq 1}$.

Young's measure theorem provides a way to analyze the behavior of bounded sequences of measurable functions. It essentially quantifies the effects of oscillations of these sequences, particularly when dealing with variational problems and non-linear PDEs.
The fundamental theorem on Young measure can be stated in the following lemma :

Lemma 2.6 *[11] Let $\Omega \subset \mathbb{R}^N$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $z_j : \Omega \longrightarrow \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $z_j \longrightarrow K$ in measure as $j \longrightarrow \infty$, i.e., given any open neighbourhood U of K in $\mathbb{R}^m$*

$$\lim_{j \longrightarrow \infty} \Big| \{x \in \Omega \ : z_j(x) \notin U\} \Big| = 0.$$

Then there exists a subsequence z_k and a family $\{\nu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$, such that

(i) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\nu_x(\lambda) \leq 1$ *for almost $x \in \Omega$.*

(ii) $\phi(z_k) \rightharpoonup^* \ \overline{\phi}$ weakly in $L^\infty(\Omega)$ *for all $\mathcal{C}_0(\mathbb{R}^m)$, where $\overline{\phi} = \langle \nu_x, \phi\rangle$.*

(iii) *If for all $R > 0$,*
$$\lim_{L \longrightarrow \infty} \sup_{k\in\mathbb{N}} \Big| \{x \in \Omega \cap B_R(0) \ : |z_k(x)| \geq L\} \Big| = 0, \tag{2.2}$$
then $\|\nu_x\| = 1$ for almost every $x \in \Omega$, and for any measurable $\Omega' \subset \Omega$, we have $\phi(z_k) \rightharpoonup \ \overline{\phi} = \langle \nu_x, \phi\rangle$ weakly in $L^1(\Omega')$ for continuous function ϕ provided the sequence $\phi(z_k)$ is weakly compact in $L^1(\Omega')$.

In this paper, we show the existence of the weak solution of problem (1.1) with the method of Young's measure. We consider the problem under the following conditions:

(A_1) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $|\Theta(x) - \Theta(y)| \leq C_\Theta |x-y|$ where C_Θ is a positive constant and satisfying
$$C_\Theta \leq \frac{1}{diam(\Omega)} \Big(\frac{1}{2}\Big)^{1/p}.$$

(A_2) $H : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a Carathéodory function, i.e., $x \mapsto H(x,u)$ is measurable for every $u \in \mathbb{R}^m$ and $u \longmapsto H(x,u)$ is continuous for a.e. $x \in \Omega$. Moreover, the Carathéodory's function H satisfies only the growth condition :
$$|H(x,u)| \leq \delta\Big(d(x) + |u|^{p-1}\Big)$$
where δ is a positive constant, $d(x)$ is a positive function in $L^{p'}(\Omega)$.

Also, we assume that $H(x,\zeta)\cdot\zeta \geq 0$ for all $\zeta \in \mathbb{R}^m$.

(A_3) $f : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is also a Carathéodory function (see the definition in (A_2)) under the following growth: there exist $c > 0$ and $0 \leq \beta < p-1$ such that
$$|f(x,\xi)| \leq a(x) + c|\xi|^\beta$$
where $0 \leq a \in L^{p'}(\Omega)$.

We can give the definition of weak solutions for problem (1.1).

Definition 2.1 *We say that $u \in W_0$ is a weak solution of the problem (1.1) if and only if*
$$\int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(v(x) - v(y)\Big)dxdy + \int_\Omega H(x,u)vdx = \langle f, v\rangle$$
for all $v \in W_0$ and where
$$\Upsilon^u_\Theta(x,y) = u(x) - u(y) - \Theta(u(x)) + \Theta(u(y)).$$

Throughout this paper, $\langle \cdot,\cdot\rangle$ denotes the duality pairing of W_0^* and W_0 for some $p \in (1,\infty)$.
The main result of this paper is the following :

Theorem 2.1 *Assume that the assumptions $(A_1) - (A_3)$ are satisfied. Then, the problem (1.1) has a weak solution in the sense of Definition (2.1).*

3. Existence of weak solutions

3.1. Galerkin approximations and a priori estimates

To construct the approximating solutions, we will use the Galerkin method. For this, we define the operator $T\ :\ W_0 \longrightarrow W_0^*$ in the following way :

$$\begin{aligned}\langle T(u),\phi\rangle = \int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)dxdy\\ +\int_\Omega H(x,u)\phi(x)dx - \int_\Omega f(x,u)\phi(x)dx\end{aligned}$$

for all $\phi \in W_0$.
Therefore, our problem (1.1) is then equivalent to finding $u \in W_0$ such that $\langle T(u),\phi\rangle = 0$ for all $\phi \in W_0$.

Assertion 1 : We claim that $T(u)$ is linear, well defined and bounded.
For arbitrary $u \in W_0$, $T(u)$ is linear. For all $\phi \in W_0$, we have

$$\begin{aligned}|\langle T(u),\phi\rangle| &= \Big|\int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)dxdy\\ &\quad +\int_\Omega H(x,u)\phi(x)dx - \int_\Omega f(x,u)\phi(x)dx\Big|\\ &\leq \Big|\int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)\Big|dxdy\\ &\quad +\int_\Omega |H(x,u)||\phi|dx\ +\int_\Omega |f(x,u)||\phi|dx\\ &:= J_1+J_2+J_3\end{aligned}$$

To establish that the operator T is bounded, we use the Hölder inequality, assumptions $(A_1)-(A_3)$ and Lemma 2.4. We obtain, firstly, that :

$$\begin{aligned}J_1 &:= \int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-1}}{|x-y|^{N+ps}}\Big|\phi(x)-\phi(y)\Big|dxdy\\ &\leq 2^{p-2}\int\int_Q\Big(\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}+\frac{|\Theta(u(x))-\Theta(u(y))|^{p-1}}{|x-y|^{N+ps}}\Big)\Big|\phi(x)-\phi(y)\Big|dxdy\\ &\leq 2^{p-2}(C_\Theta^{p-1}+1)\int\int_Q\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}|\phi(x)-\phi(y)|dxdy\\ &\leq C_0\Big(\int\int_Q\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+ps}}dxdy\Big)^{\frac{p-1}{p}}\times\Big(\int\int_Q\frac{|\phi(x)-\phi(y)|^{p}}{|x-y|^{N+ps}}dxdy\Big)^{\frac{1}{p}}\\ &\leq C_0\|u\|^{p-1}_{W_0}\|\phi\|_{W_0},\end{aligned}$$

where $C_0 = 2^{p-2}(C_\Theta^{p-1}+1)$.
Therefore, for the second term, we consider the growth condition in (A_2). We obtain :

$$\begin{aligned}J_2 &:= \int_\Omega |H(x,u)||\phi|dx\\ &\leq \delta\Big(\|d\|_{p'}\|\phi\|_p+\|u\|_p^{p-1}\|\varphi\|_p\Big)\\ &= \delta\Big(\|d\|_{p'}+\|u\|_p^{p-1}\Big)\|\phi\|_p\\ &\leq C_1\|\phi\|_{W_0}.\end{aligned}$$

where $C_1 = \delta(\|d\|_{p'} + \|u\|_p^{p-1})$.
For the third term, we proceed as previously. The growth condition allows us to estimate (by application of the Hölder inequality)

$$
\begin{aligned}
J_3 &:= \int_\Omega |f(x,u)||\phi|dx \\
&\leq \|a\|_{p'}\|\phi\|_p + c\|u\|_p^{p-1}\|\phi\|_p \\
&= \Big(\|a\|_{p'} + c\|u\|_p^{p-1}\Big)\|\phi\|_p \\
&\leq C_2\|\phi\|_{W_0}.
\end{aligned}
$$

where $C_2 = (\|d\|_{p'} + c\|u\|_p^{p-1})$.
By virtue of Poincaré inequality, the J_i for $i = 1, \ldots, 3$ are finite, then $T(u)$ is well defined. Moreover, for all $\phi \in W_0$, we have

$$
\Big|\langle T(u), \phi\rangle\Big| \leq \sum_{i=1}^{3} J_i \leq C_3\|\phi\|_{W_0},
$$

and this implies that $T(u)$ is bounded.

Assertion 2 : We show that the restriction of T to a finite linear subspace of W_0 is continuous.

$$
\begin{aligned}
&\Big|\langle T(u_k), \phi\rangle - \langle T(u), \phi\rangle\Big| \\
&= \Big|\int\int_Q \frac{|\Upsilon_\Theta^{u_k}(x,y)|^{p-2}\Upsilon_\Theta^{u_k}(x,y) - |\Upsilon_\Theta^{u}(x,y)|^{p-2}\Upsilon_\Theta^{u}(x,y)}{|x-y|^{N+ps}}(\phi(x)-\phi(y))dxdy \\
&\qquad + \int_\Omega \Big(H(x,u_k) - H(x,u)\Big)\phi(x)dx - \int_\Omega \Big(f(x,u_k) - f(x,u)\Big)\phi(x)dx\Big| \\
&\leq \Big(\int\int_Q \Big|\Upsilon_\Theta^{u_k}(x,y)|^{p-2}\Upsilon_\Theta^{u_k}(x,y) - |\Upsilon_\Theta^{u}(x,y)|^{p-2}\Upsilon_\Theta^{u}(x,y)\Big|^{\frac{p}{p-1}} \Big/ |x-y|^{(N+ps)\frac{p}{p-1}}dxdy\Big)^{\frac{p-1}{p}} \\
&\qquad + \Big(\int_\Omega \Big|H(x,u_k) - H(x,u)\Big|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}} + \Big(\int_\Omega \Big|f(x,u_k) - f(x,u)\Big|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}.
\end{aligned}
$$

In the one part, since

$$
\Big||a|^{p-2}a - |b|^{p-2}b\Big| \leq 2^{p-2}(p-1)|a-b|\Big(|a|+|b|\Big)^{p-2},
$$

we obtain the following estimate by considering the first term of the right-hand side

$$
\begin{aligned}
&\Big|\int\int_Q \frac{|\Upsilon_\Theta^{u_k}(x,y)|^{p-2}\Upsilon_\Theta^{u_k}(x,y) - |\Upsilon_\Theta^{u}(x,y)|^{p-2}\Upsilon_\Theta^{u}(x,y)}{|x-y|^{N+ps}}(\phi(x)-\phi(y))dxdy\Big| \\
&= \Big(\int\int_Q \Big|\Upsilon_\Theta^{u_k}(x,y)|^{p-2}\Upsilon_\Theta^{u_k}(x,y) - |\Upsilon_\Theta^{u}(x,y)|^{p-2}\Upsilon_\Theta^{u}(x,y)\Big|^{\frac{p}{p-1}} \Big/ |x-y|^{(N+ps)\frac{p}{p-1}}dxdy\Big)^{\frac{p-1}{p}} \\
&\leq C_p\Big[\int\int_Q (1-C_\Theta)\Big|u_k(x) - u_k(y) - (u(x)-u(y))\Big|^{\frac{p}{p-1}} \\
&\qquad \times \Big((1-C_\Theta)(|u_k(x)-u_k(y)| + |u(x)-u(y)|)\Big)^{\frac{p(p-2)}{p-1}} \Big/ |x-y|^{(N+ps)\frac{p}{p-1}}dxdy\Big]^{\frac{p-1}{p}} \\
&\qquad\qquad \leq C_4\|u_k - u\|_{W_0}\Big(\|u_k\|_{W_0}^{p-2} + \|u\|_{W_0}^{p-2}\Big).
\end{aligned}
$$

Then, we get

$$\begin{aligned}\Big|\langle T(u_k),\varphi\rangle-\langle T(u),\varphi\rangle\Big| &\le C_4\|u_k-u\|_{W_0}\Big(\|u_k\|_{W_0}^{p-2}+\|u\|_{W_0}^{p-2}\Big)\\ &+\Big(\int_\Omega\Big|H(x,u_k)-H(x,u)\Big|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}+\Big(\int_\Omega\Big|f(x,u_k)-f(x,u)\Big|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}.\end{aligned} \tag{3.1}$$

On the other hand, we consider the second and the third terms. As

$$|a-b|^p\le 2^{p-1}\Big(|a|^p+|b|^p\Big),\ 1<p$$

and since $1<\dfrac{p}{p-1}$, it follows that

$$\int_\Omega\Big|H(x,u_k)\Big|^{\frac{p}{p-1}}dx\le 2^{\frac{1}{p-1}}\cdot\delta\int_\Omega\Big(|d(x)|^{p'}+|u_k|^p\Big)dx\le C_5 \tag{3.2}$$

and

$$\int_\Omega\Big|f(x,u_k)\Big|^{\frac{p}{p-1}}dx\le 2^{\frac{1}{p-1}}\int_\Omega\Big(|a(x)|^{p'}+|u_k|^p\Big)dx\le C_6 \tag{3.3}$$

by the boundedness of u_k in $L^p(\Omega,\mathbb{R}^m)$. It follows from (3.2) and (3.3) that the sequences $\Big\{\Big|H(x,u_k)-H(x,u)\Big|^{p'}\Big\}$ and $\Big\{\Big|f(x,u_k)-f(x,u)\Big|^{p'}\Big\}$ are uniformly bounded and equi-integrable in $L^1(\Omega)$. The Vitali convergence theorem (see [18]) implies respectively

$$\int_\Omega\Big|H(x,u_k)-H(x,u)\Big|^{p'}dx=0$$

and

$$\int_\Omega\Big|f(x,u_k)-f(x,u)\Big|^{p'}dx=0.$$

By (3.1) and the definition of u_k, we deduce that

$$\Big|\langle T(u_k),\varphi\rangle-\langle T(u),\varphi\rangle\Big|\longrightarrow 0 \text{ as as } k\longrightarrow\infty.$$

Assertion 3 : The operator T is coercive. For any $u\in W_0$, we have

$$\begin{aligned}\Big\langle T(u),u\Big\rangle &= \int\int_Q\frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(u(x)-u(y)\Big)dxdy\\ &\quad+\int_\Omega H(x,u)udx\ -\int_\Omega f(x,u)udx\\ &\ge \int\int_Q\frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(u(x)-u(y)\Big)dxdy\ -\int_\Omega f(x,u)udx.\end{aligned}$$

Moreover, by Hölder inequality and Lemma 2.2, there exists a positive constant

$$\begin{aligned}\int_\Omega|f(x,u)u|dx &\le \|d\|_{p'}\|u\|_p+c\|u\|_p^{\beta+1}\\ &\le C_\eta\|d\|_{p'}\|u\|_{W_0}+cC_\eta\|u\|_{W_0}^{\beta+1}\end{aligned}$$

where C_η is the constant of the embedding $W_0\hookrightarrow L^p(\Omega;\mathbb{R}^m)$.
This implies that

$$\langle T(u),u\rangle\ge\langle Y(u),u\rangle-C_\eta\|d\|_{p'}\|u\|_{W_0}-cC_\eta\|u\|_{W_0}^{\beta+1}, \tag{3.4}$$

where $\langle Y(u),u\rangle := \displaystyle\int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(u(x)-u(y)\Big)dxdy.$
On the other hand, we can write that

$$\begin{aligned}\langle Y(u),u\rangle &= \int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big(u(x)-u(y)\Big)dxdy\\ &= \int\int_Q \frac{|\Upsilon^u_\Theta(x,y)|^{p-2}\Upsilon^u_\Theta(x,y)}{|x-y|^{N+ps}}\Big[u(x)-u(y)-\Big(\Theta(u(x))-\Theta(u(y))\Big)\\ &\qquad\qquad +\Big(\Theta(u(x))-\Theta(u(y))\Big)\Big]dxdy.\end{aligned}$$

Thanks to Lemma 2.4, we deduce that

$$\begin{aligned}&\frac{1}{2^{p-1}}|u(x)-u(y)|^p\\ =\;&\frac{1}{2^{p-1}}\Big|u(x)-u(y)-\Big(\Theta(u(x))-\Theta(u(y))\Big)+\Big(\Theta(u(x))-\Theta(u(y))\Big)\Big|^p\\ \leq\;&\Big|u(x)-u(y)-\Big(\Theta(u(x))-\Theta(u(y))\Big)\Big|^p+\Big|\Big(\Theta(u(x))-\Theta(u(y))\Big)\Big|^p.\end{aligned}$$

Then,

$$\frac{1}{2^{p-1}}\Big|u(x)-u(y)\Big|^p-\Big|\Theta(u(x))-\Theta(u(y))\Big)\Big|^p\leq\Big|u(x)-u(y)-\Big(\Theta(u(x))-\Theta(u(y))\Big)\Big|^p.$$

Consequently,

$$\begin{aligned}\langle Y(u),u\rangle &\geq \int\int_Q \frac{1}{p}\Big[\frac{1}{2^{p-1}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}-\frac{2|\Theta(u(x))-\Theta(u(y))|^p}{|x-y|^{N+ps}}\Big]dxdy\\ &\geq \int\int_Q\Big(\frac{1}{p}\frac{1}{2^{p-1}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy-\frac{2C^p_\Theta}{p}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\Big)dxdy\\ &\geq \frac{1}{p}\Big(\frac{1}{2^{p-1}}-2C^p_\Theta\Big)\|u\|^p_{W_0}.\end{aligned}$$

So, the choice of constant C_Θ in (A_1) gives the existence of a positive constant C_7 such that

$$\langle Y(u),u\rangle\geq C_7\|u\|^p_{W_0}.$$

Then, inequality (3.4) becomes

$$\langle T(u),u\rangle\geq C_7\|u\|^p_{W_0}-C_\eta\|d\|_{p'}\|u\|_{W_0}-cC_\eta\|u\|^{\beta+1}_{W_0}.$$

Therefore, we have

$$\frac{\langle T(u),u\rangle}{\|u\|_{W_0}}\longrightarrow+\infty \text{ as } \|u\|_{W_0}\longrightarrow+\infty.$$

This allows us to conclude that the operator T is coercive.

Now, we are able to construct the approximating solutions by the Galerkin method. Let $\Lambda_1\subset\Lambda_2\subset\cdots\subset W_0$ be a sequence of finite-dimensional subspaces with the property that $\bigcup_{k\geq1}\Lambda_k$ is dense in W_0. Such a sequence (Λ_k) exists since W_0 is separable.

Lemma 3.1 *(i) For all $k \in \mathbb{N}$, there exists $u_k \in \Lambda_k$ such that*

$$\langle T(u_k), \phi\rangle = 0 \text{ for all } \phi \in \Lambda_k. \tag{3.5}$$

(ii) The sequence defined in (i) is uniformly bounded in W_0, i.e. there exists a constant $R > 0$ such that

$$\|u_k\|_{W_0} \leq R \text{ for all } k \in \mathbb{N}. \tag{3.6}$$

Proof:

(i) Let fix k and assume that $dim\Lambda_k = q$. For simplicity, we can write $\sum_{1\leq i\leq q} a^i\xi_i$ where $(\xi_i)_{i=1}^q$ is a basis of Λ_k. We introduce the map :

$$\begin{array}{rcl} B:\ \mathbb{R}^q & \longrightarrow & \mathbb{R}^q \\ (a^1,\dots,a^q) & \longrightarrow & \Big(\langle T(a^i\xi_i,\xi_j)\rangle\Big)_{j=1,\dots,q} \end{array}$$

Since T restricted to Λ_k is continuous (see **Assertion 2**), then B is continuous. Let $a \in \mathbb{R}^q$ and $u = a^i\xi_i \in \Lambda_k$, then $\|a\|_{\mathbb{R}^q} \longrightarrow \infty$ is equivalent to $\|u\|_{W_0} \longrightarrow \infty$. Moreover, we have

$$B(a)\cdot a = \langle T(u), u\rangle.$$

Hence, by **Assertion 3**, we have

$$B(a)\cdot a \longrightarrow \infty \text{ as } \|a\|_{\mathbb{R}^q} \longrightarrow \infty.$$

Thus, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^q$, we have $B(a)\cdot a > 0$. According to the usual topological argument (see e.g., [24] Proposition 2.8), $B(y) = 0$ has a solution $y \in B_R(0)$. Hence, for all k, there exists $u_k \in \Lambda_k$ such that

$$\langle T(u_k), \phi\rangle = 0 \text{ for all } \phi \in \Lambda_k.$$

(ii) Since $\langle T(u), u\rangle \longrightarrow \infty$ as $\|u\|_{W_0} \longrightarrow \infty$, it follows that there exists $R > 0$ with the property that $\langle T(u), u\rangle > 1$ whenever $\|u\|_{W_0} > R$. Consequently, for the sequences of Galerkin approximation $u_k \in V_k$ which satisfy $\langle T(u_k), u_k\rangle = 0$ by (3.5), we have the uniform bound

$$\|u_k\|_{W_0} \leq R \text{ for all } k \in \mathbb{N}.$$

□

3.2. Passage to the limit

As mentioned in the introduction, we consider the framework of Young's measure theory to establish the existence of the weak solution. This section is devoted first to identifying weak limits of sequences by means of the Young measures, and then we pass to the limit in the approximating equations. Now, we collect some facts about the Young measure $\nu_{(x,y)}$ in the following lemma.

Lemma 3.2 *Let (w_k) the sequence defined in Lemma 3.1. Then there exists a Young measure $\nu_{(x,y)}$ generated by $w_k \in L^p(Q;\mathbb{R}^m)$ such that :*

1. *$\nu_{(x,y)}$ is a probability measure, i.e. $\|\nu_{(x,y)}\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for almost every $(x,y) \in Q$.*
2. *The weak L^1-limit of w_k is given by $\langle \nu_{(x,y)}, id\rangle = \int_{\mathbb{R}^m} \lambda d\nu_{(x,y)}(\lambda)$.*
3. *$\nu_{(x,y)}$ satisfies $\langle \nu_{(x,y)}, id\rangle = w(x,y)$ for almost every $(x,y) \in Q$.*

Proof:

1. Let us consider

$$w_k(x,y) = \frac{\Upsilon_\Theta^{\vartheta_k}(x,y)}{|x-y|^{\frac{N}{p}+s}} = \frac{\vartheta_k(x)-\vartheta_k(y)-\Theta(\vartheta_k(x))+\Theta(\vartheta_k(y))}{|x-y|^{\frac{N}{p}+s}} \in L^p(Q;\mathbb{R}^m)$$

for every $\vartheta_k \in W_0$.

For every $R>0$, we have $(\Omega\cap B_R)^2 \subset \Omega\times\Omega \subsetneq Q$, where $B_R = B(0,R)$ is the ball centered in 0 with radius R. Let $L\in\mathbb{R}$ such that

$$Q_L \equiv \Big\{(x,y)\in(\Omega\cap B_R)^2 :\ |w_k(x,y)| \geq L\Big\}.$$

We have

$$\begin{aligned} \|w_k\|_{L^p(Q;\mathbb{R}^m)} &= \Big(\int\int_Q \frac{|\Upsilon_\Theta^{\vartheta_k}(x,y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/p} \\ &\leq (1+C_\Theta)\|\vartheta_k\|_{W_0} \\ &\leq (1+C_\Theta)R \end{aligned}$$

by (3.6), which implies that $\{w_k\}$ is bounded in $L^p(Q;\mathbb{R}^m)$. Hence, there exists $c\geq 0$ such that

$$c \geq \int\int_Q \Big|w_k(x,y)\Big|^p dxdy \geq \int\int_{Q_L}\Big|w_k(x,y)\Big|^p dxdy \geq L^p meas(Q_L),$$

where $meas(Q)$ is the Lebesgue measure of Q_L. Therefore, w_k satisfies equation (2.2) in Lemma 2.6; thus there is a Young measure noted by $\nu_{(x,y)}$ associated to ϑ_k such that $\|\nu_{(x,y)}\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for almost every $(x,y)\in Q$.

2. Since $L^p(Q;\mathbb{R}^m)$ is reflexive $(p>1)$, it follows by (3.6), that there exists of a subsequence (still denoted by w_k) weakly convergent in $L^p(Q;\mathbb{R}^m)$. Moreover, weakly convergent in $L^1(Q;\mathbb{R}^m)$, since $1<p$. By Lemma 2.6 -(iii), taking ϕ as the identity mapping id, we obtain

$$w_k \rightharpoonup \langle\nu_{(x,y)}, id\rangle = \int_{\mathbb{R}^m}\lambda d\nu_{(x,y)}(\lambda) \text{ weakly in } L^1(Q;\mathbb{R}^m).$$

3. By (3.6), we have $\vartheta_k \rightharpoonup \vartheta$ in W_0 and $\vartheta_k \rightharpoonup \vartheta$ in $L^p(\Omega;\mathbb{R}^m)$ (for a subsequence). Thus $w_k \rightharpoonup w$ in $L^p(Q;\mathbb{R}^m)$ where

$$w(x,y) = \frac{\Upsilon_\Theta^{\vartheta}(x,y)}{|x-y|^{\frac{N}{p}+s}} = \frac{\vartheta(x)-\vartheta(y)-\Theta(\vartheta(x))+\Theta(\vartheta(y))}{|x-y|^{\frac{N}{p}+s}}.$$

Owing to (2), the uniqueness of limits implies that

$$\langle\nu_{(x,y)}, id\rangle = w(x,y) = \frac{\vartheta(x)-\vartheta(y)-\Theta(\vartheta(x))+\Theta(\vartheta(y))}{|x-y|^{\frac{N}{p}+s}}$$

for almost every $(x,y)\in Q$.

□

Now, we have all the ingredients to pass to the limit in the approximating equations and to prove Theorem 2.1.

Proof: [**Proof of Theorem 2.1**]
Let $\{w_k\}$ be the sequence defined in the proof of Lemma (3.2), i.e.

$$w_k(x,y)=\frac{\Upsilon_{\Theta}^{\vartheta_k}(x,y)}{|x-y|^{\frac{N}{p}+s}} \text{ for every } \vartheta_k\in W_0.$$

We have

$$\begin{aligned}\int\int_Q |w_k(x,y)|^p dxdy &= \int\int_Q \frac{|\Upsilon_{\Theta}^{\vartheta_k}(x,y)|^p}{|x-y|^{\frac{N}{p}+s}}dxdy\\ &= -\int_\Omega H(x,\vartheta_k)\vartheta_k dx+\int_\Omega f(x,\vartheta_k)\vartheta_k dx.\end{aligned}$$

By (3.6), up to a subsequence,

$$\vartheta_k \longrightarrow \vartheta \text{ strongly in } L^p(\Omega;\mathbb{R}^m) \text{ and a.e. in } \Omega.$$

Firstly, by the continuity condition in (A_2), we have

$$H(x,\vartheta_k)(\vartheta_k-\vartheta)\longrightarrow 0 \text{ a.e. in } \Omega \text{ as } k\longrightarrow\infty.$$

In addition, by the growth condition in (A_2), $\{H(x,\vartheta_k)(\vartheta_k-\vartheta)\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. Hence, the Vitali convergence theorem implies that

$$\lim_{k\longrightarrow\infty}\int_\Omega H(x,\vartheta_k)(\vartheta_k-\vartheta)dx=0.$$

Secondly, by the continuity condition in (A_3), we have $f(x,\vartheta_k)(\vartheta_k-\vartheta)\longrightarrow 0$ a.e. in Ω as $k\longrightarrow\infty$. Moreover, by the growth condition in (A_3), $\{f(x,\vartheta_k)(\vartheta_k-\vartheta)\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. Hence, the Vitali convergence theorem implies that

$$\lim_{k\longrightarrow\infty}\int_\Omega f(x,\vartheta_k)(\vartheta_k-\vartheta)dx=0.$$

Then, since $\vartheta_k\longrightarrow\vartheta$ in measure for $k\longrightarrow\infty$, we may infer that, after extraction of a suitable subsequence, if necessary,

$$\vartheta_k\longrightarrow\vartheta \text{ almost everywhere for } k\longrightarrow\infty.$$

Hence, for arbitrary $\phi\in W_0$, it follows from the continuity conditions in (A_3) that

$$f(x,\vartheta_k)\phi\longrightarrow f(x,\vartheta)\phi \text{ almost everywhere.}$$

As in **Assertion 2**, we have $f(x,\vartheta_k)\phi$ is equiintegrable, thus

$$f(x,\vartheta_k)\phi\longrightarrow f(x,\vartheta)\phi \text{ in } L^1(\Omega)$$

by the Vitali Convergence theorem. Consequently

$$\lim_{k\longrightarrow\infty}\int_\Omega f(x,\vartheta_k)\phi dx=\int_\Omega f(x,\vartheta)\phi dx.$$

Therefore, let's take $\vartheta_k(x)=\vartheta_k^x$ and $\vartheta_k(y)=\vartheta_k^y$.
According to a weak limit defined in Lemma 3.2 and the continuity of Θ, we can write :

$$\begin{aligned}\vartheta_k^x-\Theta(\vartheta_k^x)-\Big(\vartheta_k^y-\Theta(\vartheta_k^y)\Big) &\rightharpoonup \int_{\mathbb{R}^m}\Big[(\lambda-\Theta(\vartheta^x))-(\lambda-\Theta(\vartheta^y))\Big]d\nu_{x,y}(\lambda)\\ &= \int_{\mathbb{R}^m}\Big(\lambda-\Theta(\vartheta^x)\Big)d\nu_{x,y}(\lambda)-\Big(\lambda-\Theta(\vartheta^y)\Big)\int_{\mathbb{R}^m}d\nu_{x,y}(\lambda)\\ &= \Big[\int_{\mathbb{R}^m}\lambda d\nu_{(x,y)}(\lambda)-\Theta(\vartheta^x)\int_{\mathbb{R}^m}d\nu_{(x,y)}(\lambda)\Big]\\ &\quad -\Big[\int_{\mathbb{R}^m}\lambda d\nu_{(x,y)}(\lambda)-\Theta(\vartheta^y)\int_{\mathbb{R}^m}d\nu_{(x,y)}(\lambda)\Big]\\ &= \vartheta^x-\Theta(\vartheta^x)-\Big(\vartheta^y-\Theta(\vartheta^y)\Big)\end{aligned}$$

weakly in $L^1(Q;\mathbb{R}^m)$, since $\vartheta_k(x)-\Theta(\vartheta_k(x))$ and $\vartheta_k(y)-\Theta(\vartheta_k(y))$ are equiintegrable by the condition (A_1).
Therefore

$$\Big|\Upsilon_\Theta^{\vartheta_k}(x,y)\Big|^{p-2}\Upsilon_\Theta^{\vartheta_k}(x,y)\rightharpoonup\Big|\Upsilon_\Theta^{\vartheta}(x,y)\Big|^{p-2}\Upsilon_\Theta^{\vartheta}(x,y) \text{ weakly in } L^1(Q,\mathbb{R}^m).$$

Since $L^p(Q;\mathbb{R}^m)$ is reflexive $(p>1)$ and $\Big\{\psi_\Theta^{\vartheta_k}(x,y)\Big\}$ is bounded (see **Assertion 2**), the sequence $\Big\{\Upsilon_\Theta^{\vartheta_k}(x,y)\Big\}$ converges in $L^{p'}(Q;\mathbb{R}^m)$. Hence, its weak $L^{p'}$-limits it also $\Upsilon_\Theta^{\vartheta}(x,y)$.
We may infer that

$$\begin{aligned}&\lim_{k\longrightarrow\infty}\int\int_Q\frac{\Big|\Upsilon_\Theta^{\vartheta_k}(x,y)\Big|^{p-2}\Upsilon_\Theta^{\vartheta_k}(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)dxdy\\&=\int\int_Q\frac{\Big|\Upsilon_\Theta^{\vartheta}(x,y)\Big|^{p-2}\Upsilon_\Theta^{\vartheta}(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)dxdy,\ \forall\phi\in\bigcup_{k\geq1}\Lambda_k.\end{aligned}$$

For any $\phi\in W_0$, since $\bigcup_{k\geq1}\Lambda_k$ is dense in W_0, there exists a sequence $\{\phi_k\}\subset\bigcup_{k\geq1}\Lambda_k$ such that $\phi_k\longrightarrow\phi$ in W_0 as $k\longrightarrow\infty$. Since

$$\int\int_Q\Big[\frac{|\Upsilon_\Theta^{\vartheta_k}(x,y)|^{p-2}\Upsilon_\Theta^{\vartheta_k}(x,y)}{|x-y|^{N+ps}}\Big(\phi_k(x)-\phi_k(y)\Big)-\frac{|\Upsilon_\Theta^{\vartheta}(x,y)|^{p-2}\Upsilon_\Theta^{\vartheta}(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)\Big]dxdy$$

tends to 0 as $k\longrightarrow\infty$, we have

$$\begin{aligned}&\langle T(\vartheta_k),\phi_k\rangle-\langle T(\vartheta),\phi\rangle\\&=\int\int_Q\Big[\frac{|\Upsilon_\Theta^{\vartheta_k}(x,y)|^{p-2}\Upsilon_\Theta^{\vartheta_k}(x,y)}{|x-y|^{N+ps}}\Big(\phi_k(x)-\phi_k(y)\Big)-\frac{|\Upsilon_\Theta^{\vartheta}(x,y)|^{p-2}\Upsilon_\Theta^{\vartheta}(x,y)}{|x-y|^{N+ps}}\Big(\phi(x)-\phi(y)\Big)\Big]dxdy\\&\qquad+\int_\Omega\Big[H(x,\vartheta_k)\phi_k-H(x,\vartheta)\phi\Big]dx-\int_\Omega\Big[f(x,\vartheta_k)\phi_k-f(x,\vartheta)\phi\Big]dx.\end{aligned}$$

For simplify, we consider the following notations :

$$\left\{\begin{array}{ll}\dfrac{|\Upsilon_\Theta^{\vartheta_k}(x,y)|^{p-2}\Upsilon_\Theta^{\vartheta_k}(x,y)}{|x-y|^{N+ps}}=A_\Theta^{\vartheta_k}(x,y), & \phi_k(x)-\phi_k(y)=z_k\\ \qquad\text{and} & \\ \dfrac{|\psi_\Theta^{\vartheta}(x,y)|^{p-2}\psi_\Theta^{\vartheta}(x,y)}{|x-y|^{N+ps}}=A_\Theta^{\vartheta}(x,y), & \phi(x)-\phi(y)=z.\end{array}\right.$$

We obtain

$$\begin{aligned}&\langle T(\vartheta_k),\phi_k\rangle-\langle T(\vartheta),\phi\rangle\\&=\int\int_Q A_\Theta^{\vartheta_k}z_kdxdy-\int\int_Q A_\Theta^{\vartheta}zdxdy\ +\int_\Omega H(x,\vartheta_k)(\phi_k-\phi)dx\ +\int_\Omega\Big(H(x,\vartheta_k)-H(x,\vartheta)\Big)\phi dx\\&\qquad-\int_\Omega f(x,\vartheta_k)(\phi_k-\phi)dx-\int_\Omega\Big(f(x,\vartheta_k)-f(x,\vartheta)\Big)\phi dx\\&=\int\int_Q\Big[A_\Theta^{\vartheta_k}(x,y)(z_k-z)+(A_\Theta^{\vartheta_k}-A_\Theta^{\vartheta})z\Big]dxdy+\int_\Omega H(x,\vartheta_k)(\phi_k-\phi)dx\\&\qquad+\int_\Omega\Big(H(x,\vartheta_k)-H(x,\vartheta)\Big)\phi dx-\int_\Omega f(x,\vartheta_k)(\phi_k-\phi)dx-\int_\Omega\Big(f(x,\vartheta_k)-f(x,\vartheta)\Big)\phi dx.\end{aligned}$$

The right-hand side of the above equation tends to zero as k tends to infinity by the previous results. By virtue of Lemma 3.1, it follows that $\langle T(\vartheta), \phi\rangle = 0$ for all $\phi \in W_0$ as desired.
This establishes the existence of a weak solution to problem (1.1). $\square$

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