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A system of functional equations and its stability using fixed point method

Mehdi Dehghanian, Yamin Sayyari, Mana Donganont* and Choonkil Park

ABSTRACT: This paper introduces and analyzes the concept of (g, h)-derivations in complex Banach algebras, extending the classical notion of g-derivations. We consider a nonlinear system of three functional equations that models approximate (g, h)-derivations and examine its Hyers-Ulam stability. Using a fixed point framework in generalized metric spaces, we derive the existence, uniqueness, and error bounds for the corresponding exact solutions. The results not only unify and extend previous stability results for derivations and homomorphisms but also offer a novel analytical method for treating operator equations with asymmetric structure.

Key Words: Hyers-Ulam stability; additive mapping, (g, h)-derivation, fixed point method, system of functional equations.

Contents

1. Introduction

Let X be a complex Banach algebra. A mapping $g : X \to X$ is called a **derivation** if it is $\mathbb{C}$-linear and satisfies the Leibniz identity

$$g(xy) = g(x)y + xg(y)$$

for all $x, y \in X$. This condition reflects the core property of differentiation within an algebraic setting and has been studied extensively in various contexts, including operator algebras and functional analysis. In contrast, a $\mathbb{C}$-linear mapping $h : X \to X$ is called a **homomorphism** if it satisfies

$$h(xy) = h(x)h(y), \quad \forall x, y \in X,$$

thus preserving the multiplicative structure of the algebra. While derivations model infinitesimal deformations, homomorphisms maintain structural invariance and are fundamental to representation theory and algebraic dynamics. To bridge these two perspectives, Mirzavaziri and Moslehian introduced the concept of a ***g*-derivation** [12], which generalizes both derivations and homomorphisms. Given a fixed $\mathbb{C}$-linear mapping $g : X \to X$, a mapping $f : X \to X$ is called a g-derivation if it satisfies

$$f(xy) = f(x)g(y) + g(x)f(y), \quad \forall x, y \in X.$$

When $g = \mathrm{id}_X$, this definition reduces to that of an ordinary derivation. On the other hand, it can be shown that every homomorphism f is a $\frac{f}{2}$-derivation, thereby placing both classical notions within a unified framework. This concept has been further explored and extended to various settings, including Jordan algebras, module homomorphisms, and C*-algebra structures (see, e.g., [6,7,13]).

* Corresponding author

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Motivated by this interplay and the desire to incorporate more flexible structural interactions, we introduce in this work the notion of a (g, h)**-derivation**. Let $g, h : X \to X$ be two given $\mathbb{C}$-linear mappings. Then a mapping $f : X \to X$ is called a (g, h)-derivation if it satisfies the functional equation

$$f(xy) = f(x)g(y) + h(x)f(y), \quad \forall x, y \in X.$$

This formulation allows the left and right multiplication behaviors to be governed independently by g and h, respectively. Such generalization opens up new avenues for examining operator identities, derivation-like structures, and stability properties under nonlinear constraints. It also provides a fertile framework for analyzing Hyers-Ulam stability of generalized derivation mappings in complex Banach algebras, as we explore in this paper.

In the context of functional equations, the notion of stability plays a fundamental role. Informally, a functional equation is said to be **stable** if any function that approximately satisfies the equation must be close to an exact solution. This concept was first introduced by S. M. Ulam in 1940 [17], in the framework of group theory, where he posed a question regarding the stability of homomorphisms under perturbations. Ulam's question was partially answered by D. H. Hyers [9], who demonstrated that every approximately additive function between Banach spaces is near a true additive function. This result laid the foundation for what is now known as **Hyers-Ulam stability**.

Since then, the theory has been considerably developed and generalized by many researchers. Various forms of stability—such as Hyers-Ulam-Rassias, generalized, and fuzzy stability—have been studied across different algebraic and analytical settings (see [1,2,8,18]). These advancements have significantly relaxed the original constraints of Hyers' theorem, extending the applicability of stability methods to broader classes of functional equations.

Two principal techniques have been employed to establish stability results: the **direct method** and the **fixed point method**. The direct method, originally introduced by Hyers [9], involves explicitly constructing an exact solution as the limit of a convergent sequence, derived from an approximate solution. This approach has been further refined in several works, including [4,16], and remains a fundamental tool for studying linear and nonlinear functional equations. On the other hand, the fixed point method has emerged as a powerful alternative, in which the desired exact solution is obtained as a fixed point of a suitably defined self-mapping on a complete metric or generalized metric space. This method not only ensures the existence and uniqueness of solutions but also often yields quantitative estimates of stability (see [3,10,14]). The use of fixed point theory has proven particularly effective in analyzing systems of functional equations under weaker regularity assumptions. We remember a fundamental result in fixed point theorem.

Theorem 1.1 *[5] Assume that (X, d) is a complete generalized metric space and $\mathcal{J} : X \to X$ is a strictly contractive mapping, that is,*

$$d(\mathcal{J}u, \Im v) \leq Ld(u, v)$$

for all $u, v \in X$ and a Lipschitz constant $L < 1$. Then for each given element $u \in X$, either

$$d(\mathcal{J}^n u, \mathcal{J}^{n+1} u) = +\infty, \qquad \forall n \geq 0,$$

or

$$d(\mathcal{J}^n u, \mathcal{J}^{n+1} u) < +\infty, \qquad \forall n \geq n_0,$$

for some positive integer n_0. Furthermore, if the second alternative holds, then

(i) the sequence $(\mathcal{J}^n u)$ is convergent to a fixed point p of $\mathcal{J}$;

(ii) p is the unique fixed point of $\mathcal{J}$ in the set $V := \{v \in X, d(\mathcal{J}^{n_0} u, v) < +\infty\}$;

(iii) $d(v, p) \leq \frac{1}{1-L} d(v, \mathcal{J}v)$ for all $u, v \in V$.

Throughout this paper, suppose that X is a complex Banach algebra and k, m, n are fixed positive real numbers. Taking ideas from the definition of approximate g-derivation, we motivate to introduce the notion of approximate (g, h)-derivation, which is related to the following system of three functional equations

$$\begin{cases} kf(2x) + kf(2y) - nh(-z) = mg(x+y+z) \\ 2kf(x+y+z) - mg(x+z) = mg(x-z) + nh(y-x+z) \\ nh\left(\frac{x+y}{2} + z\right) + mg\left(\frac{x-y}{2}\right) = nh(x) + 2kf(z) \end{cases} \tag{1.1}$$

for all $x, y, z \in X$.

This paper is motivated by the growing interest in the stability of functional equations in Banach algebras, particularly those involving generalized derivations beyond classical forms. While g-derivations unify derivations and homomorphisms, the more recent (g, h)-derivations further generalize this framework by introducing asymmetric operator actions on algebraic products, enriching both theoretical structure and application scope. Our main objective is to solve a specific system of three nonlinear functional equations, given in (1.1), that characterizes approximate (g, h)-derivations, and to establish their Hyers-Ulam stability via the fixed point method. This technique ensures existence, uniqueness, and error bounds for the exact solutions, providing a rigorous foundation for stability under perturbation. The key contributions lie in formulating a unified fixed point approach to study the stability of (g, h)-derivations, extending previous results on g-derivations, and addressing functional systems with dual operator interactions. The findings enhance ongoing developments in applying fixed point theory to stability problems within generalized metric spaces.

The paper is structured as follows. Section 2 analyzes the functional system and derives necessary properties of the involved mappings. Section 3 presents the main stability results using fixed point techniques, including quantitative estimates and corollaries for special cases. Section 4 concludes the study and outlines potential directions for future research.

2. Solution and stability of the system of functional equations (1.1)

We solve and investigate the system of additive functional equations (1.1) in complex Banach algebras.

Lemma 2.1 *[15] Let X be a complex Banach algebra and $\mathcal{F} : X \to X$ be an additive mapping such that $\mathcal{F}(\alpha x) = \alpha \mathcal{F}(x)$ for all $\alpha \in \mathbb{T}^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and all $x \in X$. Then $\mathcal{F}$ is $\mathbb{C}$-linear.*

Lemma 2.2 *Let $f, g, h : X \to X$ be mappings satisfying (1.1) for all $x, y, z \in X$. Then the mappings $f, g, h : X \to X$ are additive.*

Proof: Letting $x = y = z = 0$ in (1.1), we obtain

$$\begin{cases} 2kf(0) - nh(0) = mg(0) \\ 2kf(0) - nh(0) = 2mg(0) \\ mg(0) = 2kf(0) \end{cases}$$

for all $x, y, z \in X$. Thus

$$f(0) = g(0) = h(0) = 0.$$

Putting $x = y = 0$ in (1.1), we have

$$\begin{cases} -nh(-z) = mg(z) \\ 2kf(z) - mg(z) = mg(-z) + nh(z) \\ nh(z) = 2kf(z) \end{cases} \tag{2.1}$$

for all $z \in X$. It follows from (1.1) and (2.1) that

$$f(2x) + f(2y) - 2f(-z) = -2f(-x-y-z) \tag{2.2}$$

for all $x, y, z \in X$.

Taking $z = 0$ and replacing y by $-x$ in (2.2), we get $f(2x) = -f(-2x)$ for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $y = z = 0$ in (2.2), we have $f(2x) = -2f(-x) = 2f(x)$ for all $x \in X$.

Setting $z = 0$ in (2.2), we obtain

$$f(2x) + f(2y) = f(2x + 2y)$$

for all $x, y \in X$. Therefore the mapping $f : X \to X$ is additive and so by (2.1) the mappings $g, h : X \to X$ are additive □

Using the fixed point method, we prove the Hyers-Ulam stability of the system of functional equations (1.1) in complex Banach algebras.

Theorem 2.3 *Suppose that $\psi_i : X^3 \to [0, \infty)$ are functions such that there exists an $L < 1$ with*

$$\psi_i\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{2}\psi_i(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$ and all $i = 1, 2, 3$. Let $f, g, h : X \to X$ be mappings satisfying

$$\begin{cases} \|kf(2x) + kf(2y) - nh(-z) - mg(x + y + z)\| \le \psi_1(x, y, z) \\ \|2kf(x + y + z) - mg(x + z) - mg(x - z) - nh(y - x + z)\| \le \psi_2(x, y, z) \\ \left\|nh\left(\frac{x+y}{2} + z\right) + mg\left(\frac{x-y}{2}\right) - nh(x) - 2kf(z)\right\| \le \psi_3(x, y, z) \end{cases} \tag{2.4}$$

for all $x, y, z \in X$. Then there exist unique additive mappings $F, G, H : X \to X$ such that

$$\|F(x) - f(x)\| \le \frac{L}{2k(1 - L)}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{1}{2}\psi_2(x, x, 0)\right], \tag{2.5}$$

$$\|G(x) - g(x)\| \le \frac{L}{2m(1 - L)}[\psi_1(x, x, 0) + \psi_2(x, x, 0)], \tag{2.6}$$

$$\|H(x) - h(x)\| \le \frac{L}{2n(1 - L)}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, -x\right) + \psi_3(x, x, x)\right], \tag{2.7}$$

for all $x \in X$.

Proof: Taking $x = y = z = 0$ in (2.4), we get

$$\begin{cases} 2kf(0) - nh(0) - mg(0) = 0 \\ 2kf(0) - 2mg(0) - nh(0) = 0 \\ mg(0) - 2kf(0) = 0 \end{cases}$$

since $\psi_1(0, 0, 0) = \psi_2(0, 0, 0) = \psi_3(0, 0, 0) = 0$. So $f(0) = g(0) = h(0) = 0$.

Let us take $y = x$ and $z = 0$ in the first and second equations of (2.4). Then we have

$$\|2kf(2x) - mg(2x)\| \le \psi_1(x, x, 0), \tag{2.8}$$

$$\|2kf(2x) - 2mg(x)\| \le \psi_2(x, x, 0) \tag{2.9}$$

for all $x \in X$. From (2.8) and (2.9), we obtain

$$\|g(2x) - 2g(x)\| \le \frac{1}{m}[\psi_1(x, x, 0) + \psi_2(x, x, 0)] \tag{2.10}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (2.8), we get

$$\|2kf(x) - mg(x)\| \le \psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) \tag{2.11}$$

for all $x \in X$. It follows from (2.9) and (2.11) that

$$\|f(2x) - 2f(x)\| \leq \frac{1}{k}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{1}{2}\psi_2(x, x, 0)\right] \tag{2.12}$$

for all $x \in X$. Letting $y = z = x$ in (2.4), we have

$$\|nh(2x) - nh(x) - 2kf(x)\| \leq \psi_3(x, x, x) \tag{2.13}$$

for all $x \in X$. Setting $z = -x$ and replacing x and y by $\frac{x}{2}$ and $\frac{x}{2}$ in (2.4), we have

$$\|2kf(x) - nh(x)\| \leq \psi_1\left(\frac{x}{2}, \frac{x}{2}, -x\right) \tag{2.14}$$

for all $x \in X$. From (2.13) and (2.14), we get

$$\|h(2x) - 2h(x)\| \leq \frac{1}{n}\left[\psi_3(x, x, x) + \psi_1\left(\frac{x}{2}, \frac{x}{2}, -x\right)\right] \tag{2.15}$$

for all $x \in X$.

Let $\Delta = \{\delta : X \to X : \delta(0) = 0\}$. We define generalized metrics on Δ as follows: $d_1, d_2, d_3 : \Delta \times \Delta \longrightarrow [0, \infty]$ by

$$\begin{aligned}
d_1(\sigma, \delta) &= \inf\left\{\alpha \in \mathbb{R}_+ : \|\sigma(x) - \delta(x)\| \leq \alpha\frac{1}{k}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{1}{2}\psi_2(x, x, 0)\right], \forall x \in X\right\}, \\
d_2(\sigma, \delta) &= \inf\left\{\beta \in \mathbb{R}_+ : \|\sigma(x) - \delta(x)\| \leq \beta\frac{1}{m}[\psi_1(x, x, 0) + \psi_2(x, x, 0)], \forall x \in X\right\}, \\
d_3(\sigma, \delta) &= \inf\left\{\gamma \in \mathbb{R}_+ : \|\sigma(x) - \delta(x)\| \leq \gamma\frac{1}{n}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, -x\right) + \psi_3(x, x, x)\right], \forall x \in X\right\}
\end{aligned}$$

and we consider $\inf \emptyset = +\infty$. Then it is easy to show that d_1, d_2 and d_3 are complete generalized metrics on Δ (see [11]).

Now, we define the mappings $\mathcal{J}_1 : (\Delta, d_1) \to (\Delta, d_1), \mathcal{J}_2 : (\Delta, d_2) \to (\Delta, d_2)$ and $\mathcal{J}_3 : (\Delta, d_3) \to (\Delta, d_3)$ such that

$$\mathcal{J}_1\delta_1(x) := 2\delta_1\left(\frac{x}{2}\right), \qquad \mathcal{J}_2\delta_2(x) := 2\delta_2\left(\frac{x}{2}\right), \qquad \mathcal{J}_3\delta_3(x) := 2\delta_3\left(\frac{x}{2}\right)$$

for all $x \in X$.

Actually, let $\delta_1, \sigma_1 \in (\Delta, d_1), \delta_2, \sigma_2 \in (\Delta, d_2)$ and $\delta_3, \sigma_3 \in (\Delta, d_3)$ be given such that $d_1(\delta_1, \sigma_1) = \alpha, d_2(\delta_2, \sigma_2) = \beta$ and $d_3(\delta_3, \sigma_3) = \gamma$. Then

$$\begin{aligned}
\|\delta_1(x) - \sigma_1(x)\| &\leq \alpha\frac{1}{k}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{1}{2}\psi_2(x, x, 0)\right], \\
\|\delta_2(x) - \sigma_2(x)\| &\leq \beta\frac{1}{m}[\psi_1(x, x, 0) + \psi_2(x, x, 0)], \\
\|\delta_3(x) - \sigma_3(x)\| &\leq \gamma\frac{1}{n}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, -x\right) + \psi_3(x, x, x)\right]
\end{aligned}$$

for all $x \in X$. Hence

$$\begin{aligned}
\|\mathcal{J}_1\delta_1(x) - \mathcal{J}_1\sigma_1(x)\| &= \left\|2\delta_1\left(\frac{x}{2}\right) - 2\sigma_1\left(\frac{x}{2}\right)\right\| \\
&\leq 2\alpha\frac{1}{k}\left[\psi_1\left(\frac{x}{4}, \frac{x}{4}, 0\right) + \frac{1}{2}\psi_2\left(\frac{x}{2}, \frac{x}{2}, 0\right)\right] \\
&\leq L\alpha\frac{1}{k}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{1}{2}\psi_2(x, x, 0)\right],
\end{aligned}$$

$$\begin{aligned}\|\mathcal{J}_2\delta_2(x)-\mathcal{J}_2\sigma_2(x)\| &= \left\|2\delta_2\left(\frac{x}{2}\right)-2\sigma_2\left(\frac{x}{2}\right)\right\|\\ &\le 2\beta\frac{1}{m}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},0\right)+\psi_2\left(\frac{x}{2},\frac{x}{2},0\right)\right]\\ &\le L\beta\frac{1}{m}[\psi_1(x,x,0)+\psi_2(x,x,0)],\end{aligned}$$

$$\begin{aligned}\|\mathcal{J}_3\delta_3(x)-\mathcal{J}_3\sigma_3(x)\| &= \left\|2\delta_3\left(\frac{x}{2}\right)-2\sigma_3\left(\frac{x}{2}\right)\right\|\\ &\le 2\gamma\frac{1}{n}\left[\psi_1\left(\frac{x}{4},\frac{x}{4},-\frac{x}{2}\right)+\psi_3\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)\right]\\ &\le L\gamma\frac{1}{n}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},-x\right)+\psi_3(x,x,x)\right]\end{aligned}$$

for all $x\in X$. Since $d_1(\delta_1,\sigma_1)=\alpha, d_2(\delta_2,\sigma_2)=\beta$ and $d_3(\delta_3,\sigma_3)=\gamma$, it follows that $d_1(\mathcal{J}_1\delta_1(x),\mathcal{J}_1\sigma_1(x))\le L\alpha, d_2(\mathcal{J}_2\delta_2(x),\mathcal{J}_2\sigma_2(x))\le L\beta$ and $d_3(\mathcal{J}_3\delta_3(x),\mathcal{J}_3\sigma_3(x))\le L\gamma$ for all $x\in X$, respectively.

So

$$\begin{cases} d_1(\mathcal{J}_1\delta_1(x),\mathcal{J}_1\sigma_1(x))\le Ld_1(\delta_1,\sigma_1)\\ d_2(\mathcal{J}_2\delta_2(x),\mathcal{J}_2\sigma_2(x))\le Ld_2(\delta_2,\sigma_2)\\ d_3(\mathcal{J}_3\delta_3(x),\mathcal{J}_3\sigma_3(x))\le Ld_3(\delta_3,\sigma_3)\end{cases}$$

for all $x\in X$ and all $\delta_i,\sigma_i\in\Delta$, where $i=1,2,3$.

Using (2.12), (2.10) and (2.15), we obtain that

$$\begin{cases} \left\|f(x)-2f\left(\frac{x}{2}\right)\right\|\le\frac{1}{k}\left[\psi_1\left(\frac{x}{4},\frac{x}{4},0\right)+\frac{1}{2}\psi_2\left(\frac{x}{2},\frac{x}{2},0\right)\right]\le\frac{L}{2}\frac{1}{k}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},0\right)+\frac{1}{2}\psi_2(x,x,0)\right]\\ \left\|g(x)-2g\left(\frac{x}{2}\right)\right\|\le\frac{1}{m}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},0\right)+\psi_2\left(\frac{x}{2},\frac{x}{2},0\right)\right]\le\frac{L}{2}\frac{1}{m}[\psi_1(x,x,0)+\psi_2(x,x,0)]\\ \left\|h(x)-2h\left(\frac{x}{2}\right)\right\|\le\frac{1}{n}\left[\psi_1\left(\frac{x}{4},\frac{x}{4},-\frac{x}{2}\right)+\psi_3\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)\right]\le\frac{L}{2}\frac{1}{n}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},-x\right)+\psi_3(x,x,x)\right]\end{cases}$$

for all $x\in X$, which implies that $d_1(f,\mathcal{J}_1f)\le\frac{L}{2}, d_2(g,\mathcal{J}_2g)\le\frac{L}{2}$ and $d_3(h,\mathcal{J}_3h)\le\frac{L}{2}$.

Using the fixed point alternative, we deduce the existence of a unique fixed point of $\mathcal{J}_1$, a unique fixed point of $\mathcal{J}_2$ and a unique fixed point of $\mathcal{J}_3$, that are, the existence of mappings $F,G,H:X\to X$, respectively, such that

$$F(x)=2F\left(\frac{x}{2}\right),\qquad G(x)=2G\left(\frac{x}{2}\right),\qquad H(x)=2H\left(\frac{x}{2}\right)$$

with the following property: there exist $\alpha,\beta,\gamma\in(0,\infty)$ satisfying

$$\begin{cases} \|f(x)-F(x)\|\le\frac{\alpha}{k}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},0\right)+\frac{1}{2}\psi_2(x,x,0)\right]\\ \|g(x)-G(x)\|\le\frac{\beta}{m}[\psi_1(x,x,0)+\psi_2(x,x,0)]\\ \|h(x)-H(x)\|\le\frac{\gamma}{n}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},-x\right)+\psi_3(x,x,x)\right]\end{cases}$$

for all $x\in X$.

Since $\lim_{j\to\infty}d_1(\mathcal{J}_1^jf,F)=0$, $\lim_{j\to\infty}d_2(\mathcal{J}_2^jg,G)=0$ and $\lim_{j\to\infty}d_3(\mathcal{J}_2^jh,H)=0$,

$$\lim_{j\to\infty}2^jf\left(\frac{x}{2^j}\right)=F(x),\quad \lim_{j\to\infty}2^jg\left(\frac{x}{2^j}\right)=G(x),\quad \lim_{j\to\infty}2^jh\left(\frac{x}{2^j}\right)=H(x)$$

for all $x\in X$.

Next, $d_1(f,F)\le\frac{1}{1-L}d_1(f,\mathcal{J}_1f)$, $d_2(g,G)\le\frac{1}{1-L}d_2(g,\mathcal{J}_2g)$ and $d_3(h,H)\le\frac{1}{1-L}d_3(h,\mathcal{J}_2h)$ which implies

$$\begin{cases} \|f(x)-F(x)\|\le\frac{L}{2k(1-L)}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},0\right)+\frac{1}{2}\psi_2(x,x,0)\right]\\ \|g(x)-G(x)\|\le\frac{L}{2m(1-L)}[\psi_1(x,x,0)+\psi_2(x,x,0)]\\ \|h(x)-H(x)\|\le\frac{L}{2n(1-L)}\left[\psi_1\left(\frac{x}{2},\frac{x}{2},-x\right)+\psi_3(x,x,x)\right]\end{cases}$$

for all $x \in X$.

Using (2.3) and (2.4), we conclude that

$$\begin{aligned}
&\|kF(2x) + kF(2y) - nH(-z) - mG(x+y+z)\| \\
&= \lim_{j\to\infty} 2^j \left\| kf\left(\frac{2x}{2^j}\right) + kf\left(\frac{2y}{2^j}\right) - nh\left(\frac{-z}{2^j}\right) - mg\left(\frac{x+y+z}{2^j}\right)\right\| \\
&\le \lim_{j\to\infty} 2^j \psi_1\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \le \lim_{j\to\infty} L^j \psi_1(x,y,z),
\end{aligned}$$

$$\begin{aligned}
&\|2kF(x+y+z) - mG(x+z) - mG(x-z) - nH(y-x+z)\| \\
&= \lim_{j\to\infty} 2^j \left\| 2kf\left(\frac{x+y+z}{2^j}\right) - mg\left(\frac{x+z}{2^j}\right) - mg\left(\frac{x-z}{2^j}\right) - nh\left(\frac{y-x+z}{2^j}\right)\right\| \\
&\le \lim_{j\to\infty} 2^j \psi_2\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \le \lim_{j\to\infty} L^j \psi_2(x,y,z)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| nH\left(\frac{x+y}{2} + z\right) + mG\left(\frac{x-y}{2}\right) - nH(x) - 2kF(z)\right\| \\
&= \lim_{j\to\infty} 2^j \left\| nh\left(\frac{x+y}{2^{j+1}} + \frac{z}{2^j}\right) + mg\left(\frac{x-y}{2^{j+1}}\right) - nh\left(\frac{x}{2^j}\right) - 2kf\left(\frac{z}{2^j}\right)\right\| \\
&\le \lim_{j\to\infty} 2^j \psi_3\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \le \lim_{j\to\infty} L^j \psi_3(x,y,z)
\end{aligned}$$

for all $x, y, z \in X$. Hence

$$\begin{cases}
kF(2x) + kF(2y) - nH(-z) = mG(x+y+z) \\
2kF(x+y+z) - mG(x+z) = mG(x-z) + nH(y-x+z) \\
nH\left(\frac{x+y}{2} + z\right) + mG\left(\frac{x-y}{2}\right) = nH(x) + 2kF(z)
\end{cases}$$

for all $x, y, z \in X$, since $L < 1$. Therefore by Lemma 2.2, the mappings $F, G, H : X \to X$ are additive. □

3. Hyers-Ulam stability of (g, h)-derivations in Banach algebras

In this section, by using the fixed point technique, we prove the Hyers-Ulam stability of (g, h)-derivations in complex Banach algebras.

Theorem 3.1 *Suppose that $\psi_i : X^3 \to [0, \infty)$ are functions such that there exists an $L < 1$ with*

$$\psi_i\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{4}\psi_i(x,y,z) \tag{3.1}$$

for all $x, y, z \in X$ and all $i = 1, 2, 3$. Let $f, g, h : X \to X$ be mappings satisfying

$$\begin{cases}
\|kf(2\lambda x) + kf(2\lambda y) - n\lambda h(-z) - mg(\lambda(x+y+z))\| \le \psi_1(x,y,z) \\
\|2kf(\lambda(x+y+z)) - m\lambda g(x+z) - m\lambda g(x-z) - n\lambda h(y-x+z)\| \le \psi_2(x,y,z) \\
\left\| nh\left(\frac{\lambda(x+y)}{2} + \lambda z\right) + mg\left(\frac{\lambda(x-y)}{2}\right) - n\lambda h(x) - 2\lambda kf(z)\right\| \le \psi_3(x,y,z)
\end{cases} \tag{3.2}$$

and

$$\|f(xy) - f(x)g(y) - h(x)f(y)\| \le \sum_{i=1}^{3} \psi_i(x,y,z) \tag{3.3}$$

for all $x, y, z \in X$ and all $\lambda \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $F, G, H : X \to X$ such that F is a (G, H)-derivation and

$$\|F(x) - f(x)\| \leq \frac{L}{2k(1-L)}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{1}{2}\psi_2(x, x, 0)\right], \tag{3.4}$$

$$\|G(x) - g(x)\| \leq \frac{L}{2m(1-L)}[\psi_1(x, x, 0) + \psi_2(x, x, 0)], \tag{3.5}$$

$$\|H(x) - h(x)\| \leq \frac{L}{2n(1-L)}\left[\psi_1\left(\frac{x}{2}, \frac{x}{2}, -x\right) + \psi_3(x, x, x)\right], \tag{3.6}$$

for all $x \in X$.

Proof: Let $\lambda = 1$ in (3.2). By the same reasoning as in the proof of Theorem 2.3, there exist unique additive mappings $F, G, H : X \to X$ satisfying (3.4), (3.5) and (3.6), respectively, which are given by

$$F(x) = \lim_{j\to\infty} 2^j f\left(\frac{x}{2^j}\right), \quad G(x) = \lim_{j\to\infty} 2^j g\left(\frac{x}{2^j}\right), \quad H(x) = \lim_{j\to\infty} 2^j h\left(\frac{x}{2^j}\right)$$

for all $x \in X$.

Letting $y = x$ and $z = 0$ in (3.2), we get

$$\|2kf(2\lambda x) - mg(2\lambda x)\| \leq \psi_1(x, x, 0), \tag{3.7}$$

$$\|2kf(2\lambda x) - 2m\lambda g(x)\| \leq \psi_2(x, x, 0) \tag{3.8}$$

and

$$\|h(\lambda x) - \lambda h(x)\| \leq \frac{1}{n}\psi_3(x, x, 0) \tag{3.9}$$

for all $x \in X$. From (3.7) and (3.8), we obtain

$$\|g(2\lambda x) - 2\lambda g(x)\| \leq \frac{1}{m}[\psi_1(x, x, 0) + \psi_2(x, x, 0)] \tag{3.10}$$

for all $x \in X$. Setting $y = 0$ and $z = -x$ in the first inequalitie of (3.2), we have

$$\|kf(2\lambda x) - n\lambda h(x)\| \leq \psi_1(x, 0, -x) \tag{3.11}$$

for all $x \in X$. Putting $x = y = 0$ in the third inequality of (3.2), we get

$$\|nh(\lambda z) - 2k\lambda f(z)\| \leq \psi_3(0, 0, z) \tag{3.12}$$

for all $z \in X$. From (3.9), (3.11) and (3.12), we get

$$\|f(2\lambda x) - 2\lambda f(x)\| \leq \frac{1}{k}\left[\psi_1(x, 0, -x) + \psi_3(x, x, 0) + \psi_3(0, 0, x)\right] \tag{3.13}$$

for all $x \in X$. Now, from (3.1) and (3.13), we have

$$\begin{aligned}\|2F(\lambda x) - 2\lambda F(x)\| = \|F(2\lambda x) - 2\lambda F(x)\| &= \lim_{j\to\infty} 2^j \left\|f\left(\frac{2\lambda x}{2^j}\right) - 2\lambda f\left(\frac{x}{2^j}\right)\right\| \\ &\leq \lim_{j\to\infty} 2^j \frac{1}{k}\left[\psi_1\left(\frac{x}{2^j}, 0, -\frac{x}{2^j}\right) + \psi_3\left(\frac{x}{2^j}, \frac{x}{2^j}, 0\right) + \psi_3\left(0, 0, \frac{x}{2^j}\right)\right] \\ &\leq \lim_{j\to\infty} \frac{L^j}{k2^j}\left[\psi_1(x, 0, -x) + \psi_3(x, x, 0) + \psi_3(0, 0, x)\right]\end{aligned}$$

which tends to zero as $j \to \infty$ and so $F(\lambda x) = \lambda F(x)$ for all $x \in X$ and all $\lambda \in \mathbb{T}^1$. By the same reasoning as in the proof of [15, Theorem 2.1], the mapping $F : X \to X$ is $\mathbb{C}$-linear.

Similarly, by using (3.1), (3.10) and (3.9), one can show that the additive mappings $G, H : X \to X$ are $\mathbb{C}$-linear.

It follows from (3.1) and (3.3) that

$$\begin{aligned} &\|F(xy) - F(x)G(y) - H(x)F(y)\| \\ &= \lim_{j\to\infty} 4^j \left\| f\left(\frac{xy}{4^j}\right) - f\left(\frac{x}{2^j}\right) g\left(\frac{y}{2^j}\right) - h\left(\frac{x}{2^j}\right) f\left(\frac{y}{2^j}\right) \right\| \\ &\le \lim_{j\to\infty} 4^j \left[\psi_1\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \psi_2\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \psi_3\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \right] \\ &\le \lim_{j\to\infty} L^j [\psi_1(x,y,z) + \psi_2(x,y,z) + \psi_3(x,y,z)] = 0 \end{aligned}$$

for all $x, y, z \in X$. Therefore the $\mathbb{C}$-linear mapping $F : X \to X$ is a (G, H)-derivation. □

Corollary 3.2 *Let r, η be nonnegative real numbers with $r > 2$ and $f, g, h : X \to X$ be mappings satisfying*

$$\begin{cases} \|kf(2\lambda x) + kf(2\lambda y) - n\lambda h(-z) - mg(\lambda(x+y+z))\| \le \eta(\|x\|^r + \|y\|^r + \|z\|^r) \\ \|2kf(\lambda(x+y+z)) - m\lambda g(x+z) - m\lambda g(x-z) - n\lambda h(y-x+z)\| \le \eta(\|x\|^r + \|y\|^r + \|z\|^r) \\ \left\| nh\left(\frac{\lambda(x+y)}{2} + \lambda z\right) + mg\left(\frac{\lambda(x-y)}{2}\right) - n\lambda h(x) - 2\lambda kf(z) \right\| \le \eta(\|x\|^r + \|y\|^r + \|z\|^r) \end{cases}$$

and

$$\|f(xy) - f(x)g(y) - h(x)f(y)\| \le \eta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in X$ and all $\lambda \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $F, G, H : X \to X$ such that F is a (G, H)-derivation and

$$\|F(x) - f(x)\| \le \frac{\eta(2^{1-r} + 1)}{k(2^{r-1} - 2)} \|x\|^r$$

$$\|G(x) - g(x)\| \le \frac{4\eta}{m(2^{r-1} - 2)} \|x\|^r$$

$$\|H(x) - h(x)\| \le \frac{\eta(2^{1-r} + 2^2)}{n(2^{r-1} - 2)} \|x\|^r$$

for all $x \in X$.

Proof: The proof follows from Theorem 3.1 by taking

$$\psi_i(x, y, z) = \eta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in X$ and all $i = 1, 2, 3$ and $L = 2^{2-r}$. □

Corollary 3.3 *Let p, q, r, η be nonnegative real numbers with $p + q + r > 2$ and $f, g, h : X \to X$ be mappings satisfying*

$$\begin{cases} \|kf(2\lambda x) + kf(2\lambda y) - n\lambda h(-z) - mg(\lambda(x+y+z))\| \le \eta\|x\|^p\|y\|^q\|z\|^r \\ \|2kf(\lambda(x+y+z)) - m\lambda g(x+z) - m\lambda g(x-z) - n\lambda h(y-x+z)\| \le \eta\|x\|^p\|y\|^q\|z\|^r \\ \left\| nh\left(\frac{\lambda(x+y)}{2} + \lambda z\right) + mg\left(\frac{\lambda(x-y)}{2}\right) - n\lambda h(x) - 2\lambda kf(z) \right\| \le \eta\|x\|^p\|y\|^q\|z\|^r \end{cases}$$

and

$$\|h(xy) - h(x)g(y) - f(x)h(y)\| \le \eta\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in X$ and all $\lambda \in \mathbb{T}^1$. Then there exists a unique $\mathbb{C}$-linear mapping $H : X \to X$ such that H is an (f, g)-derivation and

$$\|H(x) - h(x)\| \le \frac{\eta(1 + 2^{-(p+q)})}{n(2^{p+q+r-1} - 2)} \|x\|^{p+q+r}$$

for all $x \in X$.

Proof: The proof follows from Theorem 3.1 by taking $L = 2^{2-p-q-r}$ and

$$\psi_1(x,y,z) = \psi_2(x,y,z) = \psi_3(x,y,z) = \eta\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in X$. □

4. Conclusion

In this paper, we investigated a nonlinear system of functional equations (1.1) characterizing approximate (g,h)-derivations in complex Banach algebras. We derived the general solution and established the Hyers-Ulam stability of the associated (g,h)-derivations using the fixed point method. Our approach ensures existence, uniqueness, and provides explicit bounds for the approximation error. The results extend the classical theory of g-derivations by introducing a robust fixed point framework for analyzing generalized derivations with asymmetric operator structure. Future work may consider extensions to stochastic settings, non-Archimedean algebras, or systems involving delays, impulses, or fractional dynamics.

Declarations

Availability of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Mehdi Dehghanian,
Department of Mathematics,
Sirjan University of Technology, Sirjan,
Iran.
E-mail address: `mdehghanian@sirjantech.ac.ir`

and

Yamin Sayyari,
Department of Mathematics,
Sirjan University of Technology, Sirjan,
Iran.
E-mail address: `y.sayyari@sirjantech.ac.ir`

and

Mana Donganont,
Department of Mathematics,
University of Phayao, Phayao,
Thailand.
E-mail address: `mana.do@up.ac.th; Orcid: 0009-0008-9782-7368`

and

Choonkil Park,
Department of Mathematics,
Research Institute for Convergence of Basic Science, Hanyang University, Seoul 04763,
Korea.
E-mail address: `baak@hanyang.ac.kr`