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Signed Graphs with Fuzzy Labels and their Energy

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ABSTRACT: In a mapping $\eta : E \to \{+, -\}$ a fuzzy graph $\Gamma(\sigma, \mu)$ is said to be a fuzzy signed graph if each edge is signed to $\{+$ or $-\}$. A vertex signed fuzzy graph is one in which each node has a signed value of $\{+$ or $-\}$. The difference between positive degree and negative degree of a vertex is called signed degree of that vertex. The notion of signed graphs with fuzzy labels is presented making use of signed degree. The adjacency matrix and characteristic polynomial of several class of signed graphs with fuzzy labels are obtained. Additionally, we find the spectra, energy, and some limitations of signed graphs with fuzzy labels.

Keywords: Vertex label, fuzzy signed graph, signed degree, matrix, spectra, energy.

Contents

1. Introduction

Let Γ represent finite, undirected simple graph and $A(\Gamma)$, represent the adjacency matrix of order n of Γ. In case the vertices v_i and v_j are adjacent, the (i, j) entry in the matrix is 1; otherwise, it is 0. The characteristic equation of the graph Γ is $det(xI_n - A(\Gamma))$, denoted as $\phi(\Gamma, x)$. The eigenvalue of the graph Γ are defined as the eigenvalue of its adjacency matrix $A(\Gamma)$. Here $A(\Gamma)$ is a real symmetric matrix since the roots of $\phi(\Gamma, x) = 0$are all real integers. They are identified by the letters $\psi_1, \psi_2, ..., \psi_p$ and collectively referred as the Γ spectrum [9].
Graph spectral properties have been well studied, especially those of the characteristic equation. I. Gutman [10] in 1978 defined energy of graph Γ as $\sum_{i=1}^{p} |\psi_i|$. Later in 1977, Gutman [11] gave few results pertaining to energy of trees. Also in 2001, Gutman [12] discussed some old and new results on energy of graph. F. Zhang [24,25] in 1983 proved some theorems on comparison of trees , comparison of bipartite graphs , by their energy. Various graph energies were introduced namely maximum degree energy, minimum covering energy, color energy, distance energy, etc. in [1,2,3,13] In 2015, E. Sampathkumar [21] and et al introduced the partition energy of graph. Nutan G. Nayak [19] introduced the spectra and energy of signed graphs. In 2013, Anjali [4] and et al extended the concept of energy of graphs to energy of fuzzy graphs.

2. Title Material

Definition 2.1 *A fuzzy graph [5]* $\Gamma = (V, \sigma, \mu)$ *is a graph where* σ *is function on the vertex set as* $\sigma : V \to [0, 1]$, *which satisfies the relation* $\mu(x, y) \leq min\{\sigma(x), \sigma(y)\}, \quad \forall\ x, y \in V$ *and E is the set of edges in* Γ.
The degree of Γ *is given by* $d(u) = \Sigma\mu(u, v)$, *where the vertex u is incident to the fuzzy edges under consideration.*

Definition 2.2 *Fuzzy signed graph [22] is a graph with a mapping* $\delta : E \to \{+, -\}$ *such that each edge is signed* $\{+, -\}$ *In a vertex-signed fuzzy graph, each node is signed* $\{+$ *or* $-\}$.
The positive degree of v is the sum of the membership values of all positive edges incident with v.

$$deg^+[\sigma(v)] = \Sigma_{\mu^+(v,v_i)\in E}\mu^+(v, v_i).$$

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In a comparable way, the negative degree of vertex v is the sum of membership values of all negative edges incident with v.

$$deg^{-}[\sigma(v)] = \Sigma_{\mu^{-}(v,v_i)\in E}\mu^{-}(v,v_i)].$$

The difference between positive degree and negative degree of a vertex is the signed degree of that vertex.

$$sdeg(v) = |deg^{+}[\sigma(v)] - deg^{-}[\sigma(v)]|.$$

Definition 2.3 *The assignment of values to the edges and vertices of a graph is called graph labeling. For a given graph* Γ*, its edge labeling is a bijection from* $E(\Gamma)$ *to the set* $\{1, 2, \cdots, |E(\Gamma)|\}$*, and its vertex labeling is an injective function* $f : V(\Gamma) \to N$*. A graph with bijective functions* $\sigma : V \to [0, 1]$ *and* $\mu : V \times V \to$ *so that the membership values of vertices and edges are different and* $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ *for all* $u, v \in V$ *is referred to as a fuzzy labeled graph [23].*

3. Mathematics

Matrix and energy of signed graphs with Fuzzy labels

Let $\Gamma = (V, E)$ be a simple graph with $|V| = p$ vertices and $|E| = q$ edges. For each $i = 1, 2, ..., p$, let d_i be the degree of vertex v_i . The fuzzy labeled signed graph of Γ is then constructed from [16] by labeling the odd-degree vertex by value less than 0.5 and the even-degree vertex by value larger than 0.5.

Let γ-cut be the relationship between the edges in Γ, and γ-value be the average of weights labeled to the edges. The signed graph with fuzzy label of the graph Γ is obtained by assigning a positive sign to edge labels greater than γ and a negative sign to edge labels less than γ.

The absolute magnitude of the difference between the weights of positive and negative signed edges occurring with a vertex v_i, represented by $Sdegv_i$, is the signed degree of the vertex. The matrix of signed graph with fuzzy labels is given by

$$A[FLS(\Gamma)] = [a_{ij}] = \begin{cases} sdegv_i & \text{if } v_i \text{ is equal to } v_j \\ l(v_i v_j) & \text{if } v_i \text{ is not equal to } v_j\text{, but } v_i,\ v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

The matrix of fuzzy labeled signed graph is symmetric and its eigenvalue are $\psi_1 \geq \psi_2 \geq \psi_3 \geq .... \geq \psi_p$ The characteristic equation of the matrix is given by

$$det|\psi I - A[FLS(\Gamma)]|.$$

The energy of signed graph with fuzzy label is the summation of absolute values of ψ_i for i = 1, 2,......, p.

$$E[FLS(\Gamma)] = \sum_{i=1}^{p} |\psi_i|$$

Properties related to energy of signed graphs with fuzzy labels

Theorem 3.1 *If eigenvalue of* $FLS(\Gamma)$ *are* $\psi_1 > \psi_2 > \cdots > \psi_p$*, then*

1. $\displaystyle\sum_{i=1}^{p} \psi_i = \sum_{i=1}^{p}(Sdegv_i)$
2. $\displaystyle\sum_{i=1}^{p} \psi_i^2 = \sum_{i=1}^{p}(Sdegv_i)^2 + 2\sum_{i=1}^{p} l(v_i v_j)^2 = \xi$

where $\displaystyle\xi = \sum_{i=1}^{p}(Sdegv_i)^2 + 2\sum_{i=1}^{p} l(v_i v_j)^2$

Proof: (1). Since the diagonal elements in the matrix correspond to the signed degree of the vertex, the sum of the signed degrees is equal to the total of the first diagonal entries in $FLS(\Gamma)$.

Hence $\sum_{i=1}^{p} \psi_i = \sum_{i=1}^{p} (Sdegv_i)$

(2). The sum of squares of roots of characteristic equation of $FLS(\Gamma)$ is the sum of roots of characteristic equation of $[FLS(\Gamma)]^2$

$$\sum_{i=1}^{n} \psi_i^2 = \sum_{i=1}^{p}\sum_{j=1}^{n} u_{ij}\, u_{ji}$$

$$= \sum_{i} (u_{ii})^2 + 2\sum_{i<j} (u_{ij})^2$$

$$= \sum_{i=1}^{p} (Sdegv_i)^2 + 2\sum_{i=1}^{p} l(v_i v_j)^2$$

$$= \xi$$

□

Theorem 3.2 *If $c_0, c_1,$ and c_2 are the first three coefficients of the characteristic polynomial of the $FLS(\Gamma)$ matrix, then*

1. $c_0 = 1$
2. $c_1 = -\sum_{i=1}^{p} (Sdegv_i)$
3. $c_2 = \sum_{i<j} (\psi_i \psi_j)$

Proof: (i)We have $\phi(FLS(\Gamma), \psi) = det|\psi I - A\,(FLS(\Gamma))\,|$

Therefore $c_0 = 1$

(ii) $c_1 = (-1)^1 \times trace(\Gamma) = -1 \times (Sdegv_i) = 0$

(iii) By definition $c_2 = \sum_{1\le i<j\le p} \begin{vmatrix} u_{ii} & u_{ij} \\ u_{ji} & u_{jj} \end{vmatrix} = \sum_{1\le i<j\le p} (u_{ii}u_{jj} - u_{ij}u_{ji})$

$= \sum_{1\le i<j\le p} u_{ii}u_{jj} - \sum_{1\le i<j\le p} u_{ij}^2 = \sum_{i<j} (\psi_i \psi_j)$ □

McClelland's inequalities are used to find the boundaries for $FLS(\Gamma)$ [18].

Theorem 3.3 *If p vertices are present in the graph Γ, then the upper bound of $E\,[FLS(\Gamma)]$ is*

$$E\,[FLS(\Gamma)] \le \sqrt{p\psi}$$

Proof: Let the eigenvalue of $FLS(\Gamma)$ be $\psi_1 \ge \psi_2 \ge \cdots \ge \psi_p$ then by applying Cauchy-Schwarz inequality we get

$$\left[\sum_{i=1}^{p} u_i v_i\right]^2 \le \left[\sum_{i=1}^{p} u_i^2\right]\left[\sum_{i=1}^{p} v_i^2\right].$$

Choose $u_i = 1, v_i = |\psi_i|$ and by Theorem 3.1

$$\left[\sum_{i=1}^{p} |\psi_i|\right]^2 \le \left[\sum_{i=1}^{p} 1\right]\left[\sum_{i=1}^{p} |\psi_i|^2\right] = p\sum_{i=1}^{p} \psi_i^2$$

$$[E[FLS(\Gamma)]]^2 \leq p\psi.$$

Hence

$$E[FLS(\Gamma)] \leq \sqrt{p\psi}$$

□

Theorem 3.4 *Let the graph* Γ *has* p *vertices, then lower bound of* $E[FLS(\Gamma)]$ *where* $\tau = |FLS(\Gamma)|$ *of* Γ *is*

$$E[FLS(\Gamma)] \geq \sqrt{\psi + p(p-1)\tau^{\frac{2}{p}}}.$$

Proof: By definition we have,

$$[E[FLS(\Gamma)]]^2 = \left[\sum_{i=1}^{p} |\psi_i|\right]^2 = \left[\sum_{i=1}^{p} |\psi_i|\right]\left[\sum_{j=1}^{p} |\psi_j|\right]$$

$$= \sum_{i=1}^{p} |\psi_i|^2 + \sum_{i \neq j} |\psi_i||\psi_j|.$$

From the inequality of arthimetic and geometric means

$$\frac{1}{p(p-1)} \sum_{i \neq j} |\psi_i||\psi_j| \geq \left[\prod_{i \neq j} |\psi_i||\psi_j|\right]^{\frac{1}{p(p-1)}}.$$

Therefore

$$[E[FLS(\Gamma)]]^2 \geq \sum_{i=1}^{p} |\psi_i|^2 + p(p-1)\left[\prod_{i \neq j} |\psi_i||\psi_j|\right]^{\frac{1}{p(p-1)}}$$

$$\geq \sum_{i=1}^{p} |\psi_i|^2 + p(p-1)\left[\prod_{i=1}^{p} |\psi_i|^{2(p-1)}\right]^{\frac{1}{p(p-1)}}$$

$$= \sum_{i=1}^{p} |\psi_i|^2 + p(p-1)\left|\prod_{i=1}^{p} \psi_i\right|^{\frac{2}{p}}$$

$$= \psi + p(p-1)\tau^{\frac{2}{p}}.$$

Hence

$$E[FLS(\Gamma)] \geq \sqrt{\psi + p(p-1)\tau^{\frac{2}{p}}}$$

□

Standard signed graphs with fuzzy labels and their energy

Theorem 3.5 *Let* $FLS(K_p)$ *be a signed complete graph with fuzzy labels to* p *vertices, where* p *is odd and* $p \geq 3$, *then* $FLS(K_p)$ *has an energy*

$$E[FLS(K_p)] = {}^{p}C_2.$$

Proof: The matrix of signed complete graph with fuzzy labels to p-vertices, where p is odd and $p \geq 3$ is

$$A\left[FLS(K_p)\right] = \begin{bmatrix} \frac{p-1}{2} & 0.5 & 0.5 & \dots & 0.5 \\ 0.5 & \frac{p-1}{2} & 0.5 & \dots & 0.5 \\ 0.5 & 0.5 & \frac{p-1}{2} & \dots & 0.5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0.5 & \dots & 0.5 & \frac{p-1}{2} \end{bmatrix}.$$

Its characteristic equation is

$\left[\psi - (\frac{p-2}{2})\right]^{(p-1)} \left[\psi - (p-1)\right] = 0$

$$Spectra\left[FLS(K_p)\right] = \begin{pmatrix} \frac{p-2}{2} & p-1 \\ p-1 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} E\left[FLS(K_p)\right] &= |\frac{p-2}{2}|(p-1) + |(p-1)|(1) \\ &= {}^{p}C_2 \end{aligned}$$

□

Theorem 3.6 *Let $FLS(K_p)$ be a signed complete graph with fuzzy labels to p vertices, where p is even and $p \geq 4$, then $FLS(K_p)$ has an energy*

$$E\left[FLS(K_p)\right] = 0.6\,{}^{p}C_2$$

Proof: The matrix of signed complete graph with fuzzy labels to p-vertices, where p is even and $p \geq 4$.

$$A\left[FLS(K_p)\right] = \begin{bmatrix} 0.3(p-1) & 0.3 & 0.3 & \dots & 0.3 \\ 0.3 & 0.3(p-1) & 0.3 & \dots & 0.3 \\ 0.3 & 0.3 & 0.3(p-1) & \dots & 0.3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.3 & 0.3 & \dots & 0.3 & 0.3(p-1) \end{bmatrix}.$$

Its characteristic polynomial is

$\left[\psi - (0.3p - 0.6)\right]^{(p-1)} \left[\psi - 0.6(p-1)\right] = 0$

$$Spectra\left[FLS(K_p)\right] = \begin{pmatrix} 0.3(p-2) & 0.6(p-1) \\ (p-1) & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} E\left[FLS(K_p)\right] &= |0.3(p-2)|(p-1) + |0.6(p-1)|(1) \\ &= 0.6\,{}^{p}C_2 \end{aligned}$$

□

Theorem 3.7 *If $FLS(K_1, p-1)$ is a signed star graph with fuzzy labels to p vertices, then its energy is given by*

$$E\left[FLS(K_1, p-1)\right] = 0.6p - 0.6.$$

Proof: The matrix of signed star graph with fuzzy labels to p vertices is

$$A\left[FLS(K_1,p-1)\right]=\begin{bmatrix}0.3(p-1) & 0.3 & 0.3 & \dots & 0.3\\ 0.3 & 0.3 & 0.3 & \dots & 0.3\\ 0.3 & 0.3 & 0.3 & \dots & 0.3\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0.3 & 0.3 & \dots & 0.3 & 0.3\end{bmatrix}.$$

Its characteristic polynomial is $[\psi-0.3]^{(p-2)}[\psi-0.3p]=0$

$$Spectra\left[FLS(K_1,p-1)\right]=\begin{pmatrix}0.3 & 0.3p\\ (p-2) & 1\end{pmatrix}.$$

Therefore

$$\begin{aligned}E\left[FLS(K_1,p-1)\right]&=|0.3|(p-2)+|0.3n)|(1)\\&=0.6p-0.6\end{aligned}$$

□

Theorem 3.8 *Let $FLS(S_p^0), p\geq 3$ be a signed crown graph with fuzzy labels to $2p$ vertices where p is odd, then the energy of $FLS(S_p^0)$ is*

$$E\left[FLS(S_p^0)\right]=\frac{p^2+3p-5}{2}.$$

Proof: With $2p$-vertices and p being odd, the matrix of signed crown graph with fuzzy labels is

$$A\left[FLS(S_p^0)\right]=\begin{bmatrix}\frac{p-1}{2} & 0 & 0 & \dots & 0 & 0 & 0.5 & \dots & 0.5 & 0.5\\ 0 & \frac{p-1}{2} & 0 & \dots & 0 & 0.5 & 0 & \dots & 0.5 & 0.5\\ 0 & 0 & \frac{p-1}{2} & \dots & 0 & 0.5 & 0.5 & \dots & 0.5 & 0.5\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \frac{p-1}{2} & 0.5 & 0.5 & \dots & & 0\\ 0 & 0.5 & 0.5 & \dots & 0.5 & \frac{p-1}{2} & 0 & \dots & 0 & 0\\ 0.5 & 0 & 0.5 & \dots & 0.5 & 0 & \frac{p-1}{2} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0.5 & 0.5 & 0 & \dots & 0.5 & 0 & 0 & \dots & \frac{p-1}{2} & 0\\ 0.5 & 0.5 & 0.5 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{p-1}{2}\end{bmatrix}.$$

Its characteristic polynomial is $\left[\psi-\frac{p-2}{2}\right]^{\frac{p+3}{2}}\left[\psi-\frac{p}{2}\right]^{\frac{p+3}{2}}[\psi-(p-1)]=0$

$$Spectra\left[FLS(S_p^0)\right]=\begin{pmatrix}\frac{p-2}{2} & \frac{p}{2} & p-1\\ \frac{p+3}{2} & \frac{p+3}{2} & 1\end{pmatrix}.$$

Therefore

$$E\left[FLS(S_p^0)\right]=\left|\frac{p-2}{2}\right|\left(\frac{p+3}{2}\right)+\left|\frac{p}{2}\right|\left(\frac{(p+3)}{2}\right)+\left|p-1\right|(1)$$

$$=\frac{p^2+3p-5}{2}$$

□

Theorem 3.9 *If $FLS(S_p^0), p \geq 4$ is signed crown graph with fuzzy labels to $2p$ vertices where p is even, then the energy of $FLS(S_p^0)$ is*

$$E\left[FLS(S_p^0)\right] = 1.2\,{}^pC_2.$$

Proof: With $2p$-vertices and p being even, the matrix of signed crown graph with fuzzy labels is

$$A\left[FLS(S_p^0)\right] = \begin{bmatrix} \frac{3(p-1)}{10} & 0 & 0 & \dots & 0 & 0 & 0.3 & \dots & 0.3 & 0.3 \\ 0 & \frac{3(p-1)}{10} & 0 & \dots & 0 & 0.3 & 0 & \dots & 0.3 & 0.3 \\ 0 & 0 & \frac{3(p-1)}{10} & \dots & 0 & 0.3 & 0.3 & \dots & 0.3 & 0.3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{3(p-1)}{10} & 0.3 & 0.3 & \dots & & 0 \\ 0 & 0.3 & 0.3 & \dots & 0.3 & \frac{3(p-1)}{10} & 0 & \dots & 0 & 0 \\ 0.3 & 0 & 0.3 & \dots & 0.3 & 0 & \frac{3(p-1)}{10} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.3 & 0.3 & 0 & \dots & 0.3 & 0 & 0 & \dots & \frac{3(p-1)}{10} & 0 \\ 0.3 & 0.3 & 0.3 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{3(p-1)}{10} \end{bmatrix}.$$

Its characteristic polynomial is $\left[\psi - \frac{3p-6}{10}\right]^{p-1}\left[\psi - \frac{3p}{10}\right]^{p-1}\left[\psi - \frac{6(n-1)}{10}\right] = 0$

$$Spectra\left[FLS(S_p^0)\right] = \begin{pmatrix} 0 & 0.3p-0.6 & 0.3p & 0.6p-0.6 \\ 1 & p-1 & p-1 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} E\left[FLS(S_p^0)\right] &= \left| 0.3p-0.6 \right|(p-1) + \left| 0.3p \right|(p-1) + \left| 0.6p-0.6 \right|(1) \\ &= 1.2\,{}^pC_2 \end{aligned}$$

□

Theorem 3.10 *If $FLS(K_{p\times 2})$ is a signed cocktail party graph with fuzzy labels to $2p$ vertices, then its energy is given by*

$$E\left[FLS(K_{p\times 2})\right] = 4\,{}^pC_2.$$

Proof: The matrix of signed cocktail party graph with fuzzy labels to $2p$-vertices is

$$A\left[FLS(K_{p\times 2})\right] = \begin{bmatrix} p-1 & 0.5 & 0.5 & \dots & 0.5 & 0 & 0.5 & \dots & 0.5 & 0.5 \\ 0.5 & p-1 & 0.5 & \dots & 0.5 & 0.5 & 0 & \dots & 0.5 & 0.5 \\ 0.5 & 0.5 & p-1 & \dots & 0.5 & 0.5 & 0.5 & \dots & 0.5 & 0.5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.5 & 0.5 & 0.5 & \dots & p-1 & 0.5 & 0.5 & \dots & 0.5 & 0 \\ 0 & 0.5 & 0.5 & \dots & 0.5 & p-1 & 0.5 & \dots & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 & \dots & 0.5 & 0.5 & p-1 & \dots & 0.5 & 0.5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.5 & 0.5 & 0 & \dots & 0.5 & 0.5 & 0.5 & \dots & p-1 & 0.5 \\ 0.5 & 0.5 & 0.5 & \dots & 0 & 0.5 & 0.5 & \dots & 0.5 & p-1 \end{bmatrix}.$$

Its characteristic polynomial is
$[\psi - (p-2)]^{p-1}\,[\psi - (p-1)]^p\,[\psi - (2p-2)] = 0$

$$Spectra\,[FLS(K_{p\times 2})] = \begin{pmatrix} p-2 & p-1 & 2p-2 \\ p-1 & p & 1 \end{pmatrix}.$$

Therefore

$$E\,[FLS(K_{p\times 2})] = \left| p-2 \right| (p-1) + \left| p-1 \right| (p) + \left| 2p-2 \right| (1)$$
$$= 4\,{}^{p}C_2$$

□

Theorem 3.11 *If $FLS(K_{p,p})$ is signed double star graph with fuzzy labels to p vertices, where p is odd and $p \geq 3$, then its energy is given by*

$$E\,[FLS(K_{p,p})] = 1.2p - 0.6.$$

Proof: The matrix of signed double star graph with fuzzy labels to p vertices, p being odd and $p \geq 3$ is

$$A\,[FLS(K_{p,p})] = \begin{bmatrix} 0.3p & 0.3 & 0.3 & \dots & 0.3 & 0.3 & 0 & \dots & 0 & 0 \\ 0.3 & 0.3 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0.3 & 0 & 0.3 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.3 & 0 & 0 & \dots & 0.3 & 0 & 0 & \dots & 0 & 0 \\ 0.3 & 0 & 0 & \dots & 0 & 0.3p & 0.3 & \dots & 0.3 & 0.3 \\ 0 & 0 & 0 & \dots & 0 & 0.3 & 0.3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0.3 & 0 & \dots & 0 & 0.3 \end{bmatrix}.$$

Its characteristic polynomial is
$[\psi - 0.3]^{2p-4}\,[\psi - 0.3p]\,\left[\psi^2 - 0.3(p+2)\psi + 0.18\right] = 0$

$$Spectra\,[FLS(K_{p,p})] = \begin{pmatrix} 0.3 & 0.3p & \frac{0.3(p+2)\pm 0.3\sqrt{p^2+4p-4}}{2} \\ 2p-4 & 1 & 1 \end{pmatrix}.$$

Therefore

$$E\,[FLS(K_{p,p})] = \left| 0.3 \right| (2p-4) + \left| 0.3p \right| (1) + \left| \frac{0.3(p+2)+0.3\sqrt{p^2+4p-4}}{2} \right| (1)$$

$$+ \left| \frac{0.3(p+2)-0.3\sqrt{p^2+4p-4}}{2} \right| (1)$$

$$= 1.2p - 0.6$$

□

Theorem 3.12 *If $FLS(K_{p,p})$ is signed double star graph with fuzzy labels to p vertices, where p is even and $p \geq 4$, then its energy is given by*

$$E\,[FLS(K_{p,p})] = 0.9p - 1.2 + \sqrt{0.09p^2 - 0.6p + 2.2}$$

Proof: The matrix of signed double star graph with fuzzy labels to p vertices, p being even and $p \geq 4$ is

$$A\left[FLS(K_{p,p})\right] = \begin{bmatrix} 0.3p-0.8 & 0.3 & 0.3 & \ldots & 0.3 & 0.5 & 0 & \ldots & 0 & 0 \\ 0.3 & 0.3 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0.3 & 0 & 0.3 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.3 & 0 & 0 & \ldots & 0.3 & 0 & 0 & \ldots & 0 & 0 \\ 0.5 & 0 & 0 & \ldots & 0 & 0.3p-0.8 & 0.3 & \ldots & 0.3 & 0.3 \\ 0 & 0 & 0 & \ldots & 0 & 0.3 & 0.3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0.3 & 0 & \ldots & 0 & 0.3 \end{bmatrix}.$$

Its characteristic polynomial is
$[\psi - 0.3]^{2p-4}\left[\psi - 0.3p\right]\left[\psi^2 - (0.3p-1)\psi - 0.3\right] = 0$

$$Spectra\left[FLS(K_{p,p})\right] = \begin{pmatrix} 0.3 & 0.3p & \frac{(0.3p-1)\pm\sqrt{0.09p^2-0.6p+2.2}}{2} \\ 2p-4 & 1 & 1 \end{pmatrix}.$$

Therefore

$$E\left[FLS(K_{p,p})\right] = \left| 0.3 \right| (2p-4) + \left| 0.3p \right| (1) + \left| \frac{(0.3p-1)+\sqrt{0.09p^2-0.6p+2.2}}{2} \right| (1)$$

$$+ \left| \frac{(0.3p-1)-\sqrt{0.09p^2-0.6p+2.2}}{2} \right| (1)$$

$$= 0.9p - 1.2 + \sqrt{0.09p^2-0.6p+2.2}$$

□

Theorem 3.13 *If F_p is signed friendship graph with fuzzy labels to $2p+1$ vertices, then its energy is given by*

$$E\left[FLS\left(F_p\right)\right] = 3p.$$

Proof: The matrix of signed friendship graph with fuzzy labels to $2p+1$ vertices is

$$A\left[FLS\left(F_p\right)\right] = \begin{bmatrix} p & 0.5 & 0.5 & \ldots & 0.5 \\ 0.5 & 1 & 0.5 & \ldots & 0 \\ 0.5 & 0.5 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.5 & 0 & \ldots & 0.5 & 1 \end{bmatrix}.$$

Its characteristic polynomial is
$[\psi - 0.5]^{p}\left[\psi - 1.5\right]^{p-1}\left[\psi^2 - (p+1.5)\psi + p\right] = 0$

$$Spectra\left[FLS(F_p)\right] = \begin{pmatrix} 0.5 & 1.5 & \frac{(p+1.5)\pm\sqrt{p^2-p+2.25}}{2} \\ p & p-1 & 1 \end{pmatrix}.$$

Therefore

$$E\left[FLS(F_p)\right] = \left| 0.5 \right| (p) + \left| 1.5 \right| (p-1) + \left| \frac{(p+1.5)+\sqrt{p^2-p+2.25}}{2} \right| (1)$$

$$+ \left| \frac{(p+1.5)-\sqrt{p^2-p+2.25}}{2} \right| (1)$$

$$= 3p$$

□

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