



Characterization of Generalized Skew Lie Derivation on \ast -Algebras

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ABSTRACT: Let \mathcal{A} be a \ast -algebra over the complex field \mathbb{C} . For $A, B \in \mathcal{A}$, $[A, B]_\ast = AB - BA^\ast$ denotes the skew Lie product of A, B . In this article, it is shown that under mild assumptions a map $\xi : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) satisfies $\xi([A, B]_\ast) = [\xi(A), B]_\ast + [A, \xi(B)]_\ast$ for all $A, B \in \mathcal{A}$ if and only if ξ is an additive \ast -derivation. Further, we introduce the notion of a nonlinear generalized skew Lie derivation on \ast -algebra and proved that every nonlinear generalized skew Lie derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(T) = ZT + \xi(T)$ for all $T \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is an additive \ast -derivation. As applications, we apply our main results to some special classes of unital \ast -algebras such as prime \ast -algebra, factor von Neumann algebra and von Neumann algebra with no central summands of type I_1 .

Keywords: Additive \ast -derivation, skew Lie derivation, multiplicative generalized skew Lie derivation, \ast -algebras, Factor Von Neumann algebra.

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1. Introduction

Let \mathcal{A} be an associative \ast -algebra over the complex field \mathbb{C} . For $A, B \in \mathcal{A}$, define the skew Lie product of A and B by $[A, B]_\ast = AB - BA^\ast$. Similar type of product based on the involution (\ast) , such as the skew Jordan product ($A \circ B = AB + BA^\ast$), the bi-skew Lie product ($[A, B]_\bullet = AB^\ast - BA^\ast$) and the bi-skew Jordan product ($A \bullet B = AB^\ast + BA^\ast$), etc., have been extensively studied on various \ast -algebra. In the last decade, many mathematicians devoted themselves to the study of mappings involving new products on various kind of algebras. These kind of new products are found playing a more and more important role in some research topics, and their study has attracted many author's attention (see [8,2,12,17,19,6,5,9,10,15,14,16,18,23,24,13]).

Recall that a mapping $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is called a derivation if $\xi(A + B) = \xi(A) + \xi(B)$ and $\xi(AB) = \xi(A)B + A\xi(B)$ holds for all $A, B \in \mathcal{A}$. If algebra \mathcal{A} admits an involution ' \ast ' then an additive mapping $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is called an additive \ast -derivation, if $\xi(AB) = \xi(A)B + A\xi(B)$ and $\xi(A^\ast) = \xi(A)^\ast$ holds for all $A, B \in \mathcal{A}$. A mapping $\xi : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) is called a nonlinear skew Lie derivation if

$$\xi([A, B]_\ast) = [\xi(A), B]_\ast + [A, \xi(B)]_\ast \tag{1.1}$$

holds for all $A, B \in \mathcal{A}$. A mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) is called a nonlinear generalized skew Lie derivation if there exists a nonlinear skew Lie derivation ξ of \mathcal{A} such that

$$D([A, B]_\ast) = [D(A), B]_\ast + [A, \xi(B)]_\ast \tag{1.2}$$

holds for all $A, B \in \mathcal{A}$. Moreover, any nonlinear skew Lie derivation is an example of a nonlinear generalized skew Lie derivation (take $D = \xi$).

In 2016, Zhang [21] studied nonlinear skew Jordan derivations on factor von Neumann algebras and proved that every non linear skew Jordan derivation on a factor von Neumann algebra is an additive

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*-derivation. Later, this result has been extended to skew Jordan-type derivation on *-algebras in [11]. Khan et al. [7] proved that every multiplicative bi-skew Jordan triple derivation on *-algebra is an additive *-derivation. This result has been generalized in [3] by Ashraf et al., where they proved that every non linear bi-skew Jordan-type derivation on a factor von Neumann algebra is an additive *-derivation. Yu et al. in [20] proved that if \mathcal{A} is factor von Neumann algebra acting on a complex Hilbert space H with $\dim(H) \geq 2$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is a skew Lie derivation, then ξ is an additive *-derivation. Zhao et al. in [22], proved that if \mathcal{A} is von Neumann algebra with no central summands of type I_1 , then a mapping $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is a skew Jordan triple derivation if and only if ξ is an additive *-derivation. Benkovič [4] described the form of generalized Lie n -derivations of a unital algebra \mathcal{A} with a nontrivial idempotent and proved that under certain assumptions every generalized Lie n -derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is a Lie n -derivation. In [1], Ashraf and Ansari extended the above results to the case of nonlinear generalized Lie n -derivations.

Motivated by the above cited work, in this article, it is shown that under certain mild assumptions every nonlinear skew Lie derivation is an additive *-derivation (Theorem 2.1), and further it is proved that every nonlinear generalized skew Lie derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is a skew Lie derivation (Theorem 3.2).

2. Nonlinear Skew Lie derivations

As preliminaries, we first introduce some notation that will play an important role in the proof of our main results. Throughout the paper, we assume that \mathcal{A} is a *-algebra with identity I and nontrivial projection P_1 (that is, $P_1 \neq 0, I$ and $P_1^2 = P_1 = P_1^*$), and write $P_2 = I - P_1$. Assume $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ for $i, j = 1, 2$. Then by the Peirce decomposition of \mathcal{A} , we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Clearly any $A \in \mathcal{A}$ can be written as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. Furthermore, assume that \mathcal{A} satisfies the following property

$$X \mathcal{A} P_k = 0 \implies X = 0 \quad (k = 1, 2). \quad (\spadesuit)$$

Some specific examples of unital *-algebras satisfying the property (\spadesuit) are prime *-algebras, factor von Neumann algebras, von Neumann algebras with no central summands of type I_1 .

Now we state the first main result of this article.

Theorem 2.1 *Let \mathcal{A} be a unital *-algebra with a nontrivial projection P_1 satisfying (\spadesuit) . Then a map $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is a nonlinear skew Lie derivation if and only if ξ is an additive *-derivation.*

To complete the proof of the above theorem, we need the following several Lemmas:

Lemma 2.1 $\xi(0) = 0$.

Proof: $\xi(0) = \xi([0, 0]_*) = [\xi(0), 0]_* + [0, \xi(0)]_* = 0$. □

Lemma 2.2 *For any $H_{12} \in \mathcal{A}_{12}$ and $H_{21} \in \mathcal{A}_{21}$, we have*

$$\xi(H_{12} + H_{21}) = \xi(H_{12}) + \xi(H_{21}).$$

Proof: Let $T = \xi(H_{12} + H_{21}) - \xi(H_{12}) - \xi(H_{21})$. We show that $T = 0$. Using the fact that $[i(P_2 - P_1), H_{12}]_* = [i(P_2 - P_1), H_{21}]_* = 0$ and Lemma 2.1, we have

$$\begin{aligned} & \xi([i(P_2 - P_1), H_{12} + H_{21}]_*) \\ &= \xi([i(P_2 - P_1), H_{12}]_* + [i(P_2 - P_1), H_{21}]_*) \\ &= \xi([i(P_2 - P_1), H_{12}]_*) + \xi([i(P_2 - P_1), H_{21}]_*) \\ &= [\xi(i(P_2 - P_1)), H_{12}]_* + [i(P_2 - P_1), \xi(H_{12})]_* + [\xi(i(P_2 - P_1)), H_{21}]_* + [i(P_2 - P_1), \xi(H_{21})]_* \\ &= [\xi(i(P_2 - P_1)), H_{12} + H_{21}]_* + [i(P_2 - P_1), \xi(H_{12}) + \xi(H_{21})]_* \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} & \xi([i(P_2 - P_1), H_{12} + H_{21}]_*) \\ &= [\xi(i(P_2 - P_1)), H_{12} + H_{21}]_* + [i(P_2 - P_1), \xi(H_{12} + H_{21})]_* \end{aligned}$$

Comparing the above two expressions for $\xi([i(P_2 - P_1), H_{12} + H_{21}]_*)$, we find that $[i(P_2 - P_1), T]_* = 0$, which in turn implies that $T_{11} = T_{22} = 0$.

Evoking the fact $[H_{21}, P_2] = 0$ and using Lemma 2.1, we have

$$\begin{aligned} & \xi([H_{12} + H_{21}, P_2]_*) \\ &= \xi([H_{12}, P_2]_*) + \xi([H_{21}, P_2]_*) \\ &= [\xi(H_{12}), P_2]_* + [H_{12}, \xi(P_2)]_* + [\xi(H_{21}), P_2]_* + [H_{21}, \xi(P_2)]_* \\ &= [\xi(H_{12}) + \xi(H_{21}), P_2]_* + [H_{12} + H_{21}, \xi(P_2)]_* \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} & \xi([H_{12} + H_{21}, P_2]_*) \\ &= [\xi(H_{12} + H_{21}), P_2]_* + [H_{12} + H_{21}, \xi(P_2)]_* \end{aligned}$$

Comparing the above two expressions for $\xi([H_{12} + H_{21}, P_2]_*)$, we find that $[T, P_2]_* = 0$, which in turn implies that $T_{12} = 0$. Similarly, we obtain $T_{21} = 0$. Thus $T = 0$, that is,

$$\xi(H_{12} + H_{21}) = \xi(H_{12}) + \xi(H_{21}).$$

□

Lemma 2.3 For any $H_{11} \in \mathcal{A}_{11}, H_{12} \in \mathcal{A}_{12}, H_{21} \in \mathcal{A}_{21}$, and $H_{22} \in \mathcal{A}_{22}$, we have

$$\xi(H_{11} + H_{12} + H_{21}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21})$$

and

$$\xi(H_{12} + H_{21} + H_{22}) = \xi(H_{12}) + \xi(H_{21}) + \xi(H_{22}).$$

Proof: Let $T = \xi(H_{11} + H_{12} + H_{21}) - \xi(H_{11}) - \xi(H_{12}) - \xi(H_{21})$. We show that $T = 0$. Using the fact that $[iP_2, H_{11}]_* = 0$ and Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} & \xi([iP_2, H_{11} + H_{12} + H_{21}]_*) \\ &= \xi([iP_2, H_{11}]_*) + \xi([iP_2, H_{12} + H_{21}]_*) \\ &= [\xi(iP_2), H_{11}]_* + [iP_2, \xi(H_{11})]_* + [\xi(iP_2), H_{12} + H_{21}]_* + [iP_2, \xi(H_{12}) + \xi(H_{21})]_* \\ &= [\xi(iP_2), H_{11} + H_{12} + H_{21}]_* + [iP_2, \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21})]_* \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} & \xi([iP_2, H_{11} + H_{12} + H_{21}]_*) \\ &= [\xi(iP_2), H_{11} + H_{12} + H_{21}]_* + [iP_2, \xi(H_{11} + H_{12} + H_{21})]_* \end{aligned}$$

Comparing the above two expressions for $\xi([iP_2, H_{11} + H_{12} + H_{21}]_*)$, we find that $[iP_2, T]_* = 0$, which in turn implies that $T_{12} = T_{21} = T_{22} = 0$.

Evoking the fact $[i(P_2 - P_1), H_{12}]_* = [i(P_2 - P_1), H_{21}]_* = 0$ and using Lemma 2.1, we have

$$\begin{aligned} & \xi([i(P_2 - P_1), H_{11} + H_{12} + H_{21}]_*) \\ &= \xi([i(P_2 - P_1), H_{11}]_* + \xi([i(P_2 - P_1), H_{12}]_* + [i(P_2 - P_1), H_{21}]_*)) \\ &= [\xi(i(P_2 - P_1)), H_{11}]_* + [i(P_2 - P_1), \xi(H_{11})]_* + [\xi(i(P_2 - P_1)), H_{12}]_* \\ &\quad + [i(P_2 - P_1), \xi(H_{12})]_* + [\xi(i(P_2 - P_1)), H_{21}]_* + [i(P_2 - P_1), \xi(H_{21})]_* \\ &= [\xi(i(P_2 - P_1)), H_{11} + H_{12} + H_{21}]_* + [i(P_2 - P_1), \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21})]_* \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} & \xi([i(P_2 - P_1), H_{11} + H_{12} + H_{21}]_*) \\ &= [\xi(i(P_2 - P_1)), H_{11} + H_{12} + H_{21}]_* + [i(P_2 - P_1), \xi(H_{11} + H_{12} + H_{21})]_* . \end{aligned}$$

Comparing the above two expressions for $\xi([i(P_2 - P_1), H_{11} + H_{12} + H_{21}]_*)$, we find that $[i(P_2 - P_1), T]_* = 0$, which in turn implies that $T_{11} = 0$. Thus $T = 0$, that is,

$$\xi(H_{11} + H_{12} + H_{21}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21}).$$

Similarly, we can show that

$$\xi(H_{12} + H_{21} + H_{22}) = \xi(H_{12}) + \xi(H_{21}) + \xi(H_{22}).$$

□

Lemma 2.4 For any $H_{11} \in \mathcal{A}_{11}, H_{12} \in \mathcal{A}_{12}, H_{21} \in \mathcal{A}_{21}$, and $H_{22} \in \mathcal{A}_{22}$, we have

$$\xi(H_{11} + H_{12} + H_{21} + H_{22}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21}) + \xi(H_{22}).$$

Proof: Let $T = \xi(H_{11} + H_{12} + H_{21} + H_{22}) - \xi(H_{11}) - \xi(H_{12}) - \xi(H_{21}) - \xi(H_{22})$. We show that $T = 0$. Using the fact that $[iP_1, H_{22}]_* = 0$ and Lemmas 2.1 and 2.3, we find that

$$\begin{aligned} & \xi([iP_1, H_{11} + H_{12} + H_{21} + H_{22}]_*) \\ &= \xi([iP_1, H_{11} + H_{12} + H_{21}]_*) + \xi([iP_1, H_{22}]_*) \\ &= \xi([iP_1, H_{11}]_*) + \xi([iP_1, H_{12}]_*) + \xi([iP_1, H_{21}]_*) + \xi([iP_1, H_{22}]_*) \\ &= [\xi(iP_1), H_{11}]_* + [iP_1, \xi(H_{11})]_* + [\xi(iP_1), H_{12}]_* + [iP_1, \xi(H_{12})]_* \\ &\quad + [\xi(iP_1), H_{21}]_* + [iP_1, \xi(H_{21})]_* + [\xi(iP_1), H_{22}]_* + [iP_1, \xi(H_{22})]_* \\ &= [\xi(iP_1), H_{11} + H_{12} + H_{21} + H_{22}]_* + [iP_1, \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21}) + \xi(H_{22})]_* . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \xi([iP_1, H_{11} + H_{12} + H_{21} + H_{22}]_*) \\ &= [\xi(iP_1), H_{11} + H_{12} + H_{21} + H_{22}]_* + [iP_1, \xi(H_{11} + H_{12} + H_{21} + H_{22})]_* . \end{aligned}$$

Comparing the above two expressions for $\xi([iP_1, H_{11} + H_{12} + H_{21} + H_{22}]_*)$, we obtain that $[iP_1, T]_* = 0$ which further implies that $T_{11} = T_{12} = T_{21} = 0$. Similarly we can show that $T_{22} = 0$. Thus $T = 0$, that is,

$$\xi(H_{11} + H_{12} + H_{21} + H_{22}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{21}) + \xi(H_{22}).$$

□

Lemma 2.5 $\xi(I) \in \mathcal{Z}(\mathcal{A})$.

Proof: We have $[I, I]_* = 0$

$$\begin{aligned} 0 &= \xi([I, I]_*) \\ &= [\xi(I), I]_* + [I, \xi(I)]_* \\ &= \xi(I) - \xi(I)^* . \end{aligned}$$

We obtain

$$\xi(I) = \xi(I)^*$$

For any $B \in \mathcal{A}$, we have $[I, B]_* = 0$

$$\begin{aligned} 0 &= \xi([I, B]_*) \\ &= [\xi(I), B]_* + [I, \xi(B)]_* \\ &= \xi(I)B - B\xi(I)^*. \end{aligned}$$

Using the above condition, we get

$$\xi(I)B = B\xi(I).$$

Hence, the result follows. \square

Lemma 2.6 For any $A \in \mathcal{A}$, if $A^* = A$ then $\xi(A^*) = \xi(A) = \xi(A)^*$

Proof: Invoking the fact that $[A, I]_* = 0$ and using Lemma 2.5, we obtain

$$\begin{aligned} 0 &= \xi([A, I]_*) \\ &= [\xi(A), I]_* \\ &= \xi(A) - \xi(A)^*. \end{aligned}$$

We obtain that

$$\xi(A)^* = \xi(A).$$

\square

Lemma 2.7 For any $A \in \mathcal{A}$, $\xi(iA) = i\xi(A)$

Proof: Invoking the fact that $-\frac{I}{2} = [\frac{iI}{2}, \frac{iI}{2}]_*$, we have

$$\begin{aligned} \xi\left(-\frac{I}{2}\right) &= \xi\left(\left[\frac{iI}{2}, \frac{iI}{2}\right]_*\right) \\ &= \left[\xi\left(\frac{iI}{2}\right), \frac{iI}{2}\right]_* + \left[\frac{iI}{2}, \xi\left(\frac{iI}{2}\right)\right]_* \\ &= i\xi\left(\frac{iI}{2}\right) + \frac{iI}{2}\left(\xi\left(\frac{iI}{2}\right) - \xi\left(\frac{iI}{2}\right)^*\right). \end{aligned}$$

Observe that $\xi\left(-\frac{I}{2}\right)$ and $\frac{iI}{2}\left(\xi\left(\frac{iI}{2}\right) - \xi\left(\frac{iI}{2}\right)^*\right)$ are self-adjoint by Lemma 2.6. Therefore, $i\xi\left(\frac{iI}{2}\right)$ is also self-adjoint; consequently, we get

$$-\xi\left(\frac{iI}{2}\right) = \xi\left(\frac{iI}{2}\right)^*. \quad (2.1)$$

Invoking (2.1) in the above relation, we obtain

$$\xi\left(-\frac{I}{2}\right) = 2i\xi\left(\frac{iI}{2}\right). \quad (2.2)$$

Evoking the fact $-\frac{I}{2} = \left[-\frac{iI}{2}, -\frac{iI}{2}\right]_*$, we have

$$\begin{aligned} \xi\left(-\frac{I}{2}\right) &= \xi\left(\left[-\frac{iI}{2}, -\frac{iI}{2}\right]_*\right) \\ &= \left[\xi\left(-\frac{iI}{2}\right), -\frac{iI}{2}\right]_* + \left[-\frac{iI}{2}, \xi\left(-\frac{iI}{2}\right)\right]_* \\ &= -i\xi\left(-\frac{iI}{2}\right) + \frac{iI}{2}\left(\xi\left(-\frac{iI}{2}\right)^* - \xi\left(-\frac{iI}{2}\right)\right). \end{aligned}$$

Observe that $-i\xi\left(-\frac{iI}{2}\right)$ is self-adjoint by Lemma 2.6, we get

$$-\xi\left(-\frac{iI}{2}\right) = \xi\left(-\frac{iI}{2}\right)^*. \quad (2.3)$$

Invoking (2.3) in the above relation, we obtain

$$\xi\left(-\frac{I}{2}\right) = 2i\xi\left(\frac{iI}{2}\right). \quad (2.4)$$

Equations (2.2) and (2.4) implies that

$$-2i\xi\left(-\frac{iI}{2}\right) = 2i\xi\left(\frac{iI}{2}\right). \quad (2.5)$$

Proceeding with the fact that $\frac{iI}{2} = \left[-\frac{iI}{2}, -\frac{I}{2}\right]_*$ and using equation (2.5), we have

$$\begin{aligned} \xi\left(\frac{iI}{2}\right) &= \xi\left(\left[-\frac{iI}{2}, -\frac{I}{2}\right]_*\right) \\ &= \left[\xi\left(-\frac{iI}{2}\right), -\frac{I}{2}\right]_* + \left[-\frac{iI}{2}, \xi\left(-\frac{I}{2}\right)\right]_* \\ &= -\xi\left(-\frac{iI}{2}\right) - i\xi\left(-\frac{I}{2}\right) \\ &= \xi\left(\frac{iI}{2}\right) - i\xi\left(-\frac{I}{2}\right). \end{aligned}$$

This implies that $\xi\left(-\frac{I}{2}\right) = 0$. So by the equation (2.2), we obtain

$$\xi\left(\frac{iI}{2}\right) = 0. \quad (2.6)$$

For any $A \in \mathcal{A}$ and using Equation (2.6), we obtain

$$\xi(iA) = \xi\left(\left[\frac{iI}{2}, A\right]\right) = \left[\frac{iI}{2}, \xi(A)\right] = i\xi(A).$$

□

Lemma 2.8 For any $H_{12}, H'_{12} \in \mathcal{A}_{12}$ and $H_{21}, H'_{21} \in \mathcal{A}_{21}$ we have

$$\xi(H_{12} + H'_{12}) = \xi(H_{12}) + \xi(H'_{12}) \quad \text{and} \quad \xi(H_{21} + H'_{21}) = \xi(H_{21}) + \xi(H'_{21}).$$

Proof: Making use of the fact $[iP_1 + iH_{12}, P_2 + H'_{12}]_* = iH_{12} + iH'_{12} + iH_{12}^* + iH'_{12}H_{12}^*$ and applying Lemmas 2.4 and 2.7, we have

$$\begin{aligned} &\xi(iH_{12} + iH'_{12}) + \xi(iH_{12}^*) + \xi(iH'_{12}H_{12}^*) \\ &= \xi(iH_{12} + iH'_{12} + iH_{12}^* + iH'_{12}H_{12}^*) \\ &= \xi([iP_1 + iH_{12}, P_2 + H'_{12}]_*) \\ &= [\xi(iP_1 + iH_{12}), P_2 + H'_{12}]_* + [iP_1 + iH_{12}, \xi(P_2 + H'_{12})]_* \\ &= [\xi(iP_1) + \xi(iH_{12}), P_2 + H'_{12}]_* + [iP_1 + iH_{12}, \xi(P_2) + \xi(H'_{12})]_* \\ &= \xi([iP_1, P_2]_*) + \xi([iP_1, H'_{12}]_*) + \xi([iH_{12}, P_2]_*) + \xi([iH_{12}, H'_{12}]_*) \\ &= \xi(iH_{12}) + \xi(iH'_{12}) + \xi(iH_{12}^*) + \xi(iH'_{12}H_{12}^*). \end{aligned}$$

Hence $\xi(H_{12} + H'_{12}) = \xi(H_{12}) + \xi(H'_{12})$ for any $H_{12}, H'_{12} \in \mathcal{A}_{12}$. Similarly, we can prove other part. □

Lemma 2.9 For any $H_{ii}, H'_{ii} \in \mathcal{A}_{ii}$ for $(i = 1, 2)$, we have

$$\xi(H_{11} + H'_{11}) = \xi(H_{11}) + \xi(H'_{11}) \quad \text{and} \quad \xi(H_{22} + H'_{22}) = \xi(H_{22}) + \xi(H'_{22}).$$

Proof: Let $T = \xi(H_{11} + H'_{11}) - \xi(H_{11}) - \xi(H'_{11})$, we show that $T = 0$. Making use of the fact that $[iP_2, H_{11}]_* = [iP_2, H'_{11}]_* = 0$ with Lemma 2.1, we obtain

$$\begin{aligned} \xi([iP_2, H_{11} + H'_{11}]_*) &= \xi([iP_2, H_{11}]_*) + \xi([iP_2, H'_{11}]_*) \\ &= [\xi(iP_2), H_{11}]_* + [iP_2, \xi(H_{11})]_* + [\xi(iP_2), H'_{11}]_* + [iP_2, \xi(H'_{11})]_* \\ &= [\xi(iP_2), H_{11} + H'_{11}]_* + [iP_2, \xi(H_{11}) + \xi(H'_{11})]_* \end{aligned}$$

On the other hand, we have

$$\xi([iP_2, H_{11} + H'_{11}]_*) = [\xi(iP_2), H_{11} + H'_{11}]_* + [iP_2, \xi(H_{11} + H'_{11})]_*.$$

Comparing the above two expressions for $\xi([iP_2, H_{11} + H'_{11}]_*)$, we find that $[iP_2, T] = 0$, which in turn gives $T_{12} = T_{21} = T_{22} = 0$. Next, we show that $T_{11} = 0$. Let $X_{12} \in \mathcal{A}_{12}$ and it is easy to observe that $[H_{11}, X_{12}]_*, [H'_{11}, X_{12}]_* \in \mathcal{A}_{12}$. Thus, using Lemma 2.8, we find that

$$\begin{aligned} \xi([H_{11} + H'_{11}, X_{12}]_*) &= \xi([H_{11}, X_{12}]_*) + \xi([H'_{11}, X_{12}]_*) \\ &= [\xi(H_{11}), X_{12}]_* + [H_{11}, \xi(X_{12})]_* + [\xi(H'_{11}), X_{12}]_* + [H'_{11}, \xi(X_{12})]_* \\ &= [\xi(H_{11}) + \xi(H'_{11}), X_{12}]_* + [H_{11} + H'_{11}, \xi(X_{12})]_* \end{aligned}$$

On the other hand, we have

$$\xi([H_{11} + H'_{11}, X_{12}]_*) = [\xi(H_{11} + H'_{11}), X_{12}]_* + [H_{11} + H'_{11}, \xi(X_{12})]_*.$$

From the last two expressions for $\xi([H_{11} + H'_{11}, X_{12}]_*)$, we get $[T, X_{12}]_* = 0$, which implies that $TX_{12} - X_{12}T^* = 0$. Multiplying both sides by P_1 from left and right and using the above condition, we obtain $P_1TP_1XP_2 = 0$ for all $X \in \mathcal{A}$. The condition (\spadesuit) yields $T_{11} = 0$. Hence $T = 0$, that is, $\xi(H_{11} + H'_{11}) = \xi(H_{11}) + \xi(H'_{11})$. Symmetrically, one can prove that $\xi(H_{22} + H'_{22}) = \xi(H_{22}) + \xi(H'_{22})$. \square

Lemma 2.10 ξ is additive on \mathcal{A} .

Proof: For any $H, K \in \mathcal{A}$, we have $H = H_{11} + H_{12} + H_{21} + H_{22}$ and $K = K_{11} + K_{12} + K_{21} + K_{22}$. With the help of Lemmas 2.4, 2.8 and 2.9, we obtain

$$\begin{aligned} \xi(H + K) &= \xi(H_{11} + H_{12} + H_{21} + H_{22} + K_{11} + K_{12} + K_{21} + K_{22}) \\ &= \xi(H_{11} + K_{11}) + \xi(H_{12} + K_{12}) + \xi(H_{21} + K_{21}) + \xi(H_{22} + K_{22}) \\ &= \xi(H_{11}) + \xi(K_{11}) + \xi(H_{12}) + \xi(K_{12}) + \xi(H_{21}) + \xi(K_{21}) + \xi(H_{22}) + \xi(K_{22}) \\ &= \xi(H_{11} + H_{12} + H_{21} + H_{22}) + \xi(K_{11} + K_{12} + K_{21} + K_{22}) \\ &= \xi(H) + \xi(K). \end{aligned}$$

\square

Lemma 2.11 For any $H \in \mathcal{A}$, we have $\xi(H^*) = \xi(H)^*$.

Proof: Observe that for any $H \in \mathcal{A}$, we have $H = H_1 + iH_2$ such that $H_1^* = H_1$ and $H_2^* = H_2$. Making use of Lemmas 2.6, 2.7 and 2.10, we find that

$$\begin{aligned}\xi(H^*) &= \xi(H_1 - iH_2) \\ &= \xi(H_1) - i\xi(H_2) \\ &= \xi(H_1)^* + (i\xi(H))^* \\ &= \xi(H_1)^* + \xi(iH)^* \\ &= \xi(H_1 + iH)^*\end{aligned}$$

which implies that

$$\xi(H^*) = \xi(H)^*.$$

□

Lemma 2.12 $\xi(AB) = \xi(A)B + A\xi(B)$, for all $A, B \in \mathcal{A}$.

Proof: Utilizing the fact that ξ is additive, Equation (2.6) implies that $\xi(iI) = 0$. Invoking the fact that $[iI + A, B]_* = 2iB + AB - BA^*$ and making use of Lemmas 2.7 and 2.11, we obtain

$$\begin{aligned}\xi(2iB) + \xi(AB) - \xi(BA^*) &= \xi([iI + A, B]_*) \\ &= [\xi(iI + A), B]_* + [iI + A, \xi(B)]_* \\ &= [\xi(iI) + \xi(A), B]_* + [iI + A, \xi(B)]_* \\ &= 2i\xi(B) + \xi(A)B + A\xi(B) - \xi(B)A^* - B\xi(A)^*\end{aligned}$$

which further implies from Lemma 2.7 that

$$\xi(AB) - \xi(BA^*) = \xi(A)B + A\xi(B) - \xi(B)A^* - B\xi(A)^*.$$

Converting A by iA in the above relation, we obtain

$$\xi(AB) + \xi(BA^*) = \xi(A)B + A\xi(B) + \xi(B)A^* + B\xi(A)^*.$$

Hence

$$\xi(AB) = \xi(A)B + A\xi(B).$$

□

Proof of Theorem 2.1: In view of Lemmas 2.10, 2.11, and 2.12, ξ is an additive $*$ -derivation. This completes the proof of Theorem 2.1.

Now we apply Theorem 2.1 to certain special classes of $*$ -algebra, prime $*$ -algebras, standard operator algebras, factor von Neumann algebras and von Neumann algebras with no central summands of type I_1 .

Recall that an algebra \mathcal{A} is prime if for any $A, B \in \mathcal{A}$, $AAB = \{0\}$ implies that either $A = 0$ or $B = 0$. It is easy to verify that every prime $*$ -algebra satisfies (\spadesuit). Therefore, as a direct consequence of Theorem 2.1, we have the following result:

Corollary 2.1 *Let \mathcal{A} be a unital prime $*$ -algebra containing a non trivial projection. Then, every non-linear skew Lie derivation of \mathcal{A} is an additive $*$ -derivation.*

A von Neumann algebra \mathcal{A} is a weakly closed, self adjoint algebra of operators on a Hilbert space \mathcal{H} containing the identity operator I . A von Neumann algebra \mathcal{A} is a factor von Neumann algebra if its centre contains only the scalar operators. It is well known that a factor von Neumann algebra is prime, thus, it always satisfies (\spadesuit). Hence, as an immediate consequence of Corollary 2.1, we get the following result:

Corollary 2.2 *Let \mathcal{A} be a factor von Neumann algebra with $\dim(\mathcal{A}) \geq 2$. Then, every nonlinear skew Lie derivation of \mathcal{A} is an additive *-derivation.*

Further, it is well-known that every von Neumann algebra with no central summands of type I_1 satisfies (\spadesuit) (see [5,9] for details). Therefore, applying Theorem 2.1, we have the following result:

Corollary 2.3 *Let \mathcal{A} be a von Neumann algebra having no central summands of type I_1 . Then, every nonlinear skew Lie derivation of \mathcal{A} is an additive *-derivation.*

3. Nonlinear Generalized Skew Lie derivations

In this section, we present the second main result of this article which gives a description of nonlinear generalized skew Lie derivations of unital *-algebras with nontrivial projections. Observe that if $D : \mathcal{A} \rightarrow \mathcal{A}$ is a nonlinear generalized skew Lie derivation associated with nonlinear skew Lie derivation $\xi : \mathcal{A} \rightarrow \mathcal{A}$, then follows from (1.1) and (1.2) that a map $\psi = D - \xi$ satisfies

$$\psi([A, B]_*) = [\psi(A), B]_* \quad (3.1)$$

for all $A, B \in \mathcal{A}$. Therefore, to obtain a description of the map D it suffices to consider a map ψ with property (3.1). The following theorem provides a description of such maps on unital *-algebra containing nontrivial projections.

Theorem 3.1 *Let \mathcal{A} be a unital *-algebra with a nontrivial projection. Then, a map $\psi : \mathcal{A} \rightarrow \mathcal{A}$ satisfying $\psi([A, B]_*) = [\psi(A), B]_*$ for all $A, B \in \mathcal{A}$ is of the form $\psi(A) = ZA$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$.*

To complete the proof of the above theorem, we need the following several Lemmas:

Lemma 3.1 $\psi(0) = 0$.

Proof: $\psi(0) = \psi([0, 0]_*) = [\psi(0), 0]_* = 0$. □

Lemma 3.2 $\psi(\alpha I) \in \mathcal{Z}(\mathcal{A})$, for any $\alpha \in \mathbb{R}$, (where \mathbb{R} is the field of real numbers).

Proof: We have $[\alpha I, I]_* = 0$

$$\begin{aligned} 0 &= \psi([\alpha I, I]_*) = [\psi(\alpha I), I]_* \\ &= \psi(\alpha I) - \psi(\alpha I)^*. \end{aligned}$$

We obtain

$$\psi(\alpha I) = \psi(\alpha I)^*$$

For any $B \in \mathcal{A}$, we have $[\alpha I, B]_* = 0$

$$\begin{aligned} 0 &= \psi([\alpha I, B]_*) = [\psi(\alpha I), B]_* \\ &= \psi(\alpha I)B - B\psi(\alpha I)^*. \end{aligned}$$

Using the above condition, we get

$$\psi(\alpha I)B = B\psi(\alpha I).$$

Hence, the result follows. □

Lemma 3.3 $\psi(-\frac{I}{2}) = i\psi(\frac{iI}{2}) = -i\psi(\frac{iI}{2})^*$.

Proof: Invoking the fact that $[\frac{iI}{2}, \frac{iI}{2}]_* = [-\frac{iI}{2}, -\frac{iI}{2}]_* = -\frac{I}{2}$

$$\begin{aligned} \psi\left(-\frac{I}{2}\right) &= \psi\left(\left[\frac{iI}{2}, \frac{iI}{2}\right]_*\right) = \left[\psi\left(\frac{iI}{2}\right), \frac{iI}{2}\right]_* \\ &= \frac{iI}{2}\left(\psi\left(\frac{iI}{2}\right) - \psi\left(\frac{iI}{2}\right)^*\right). \end{aligned} \quad (3.2)$$

On the other hand

$$\begin{aligned} \psi\left(-\frac{I}{2}\right) &= \psi\left(\left[-\frac{iI}{2}, -\frac{iI}{2}\right]_*\right) = \left[\psi\left(-\frac{iI}{2}\right), -\frac{iI}{2}\right]_* \\ &= \frac{iI}{2}\left(\psi\left(-\frac{iI}{2}\right)^* - \psi\left(-\frac{iI}{2}\right)\right). \end{aligned} \quad (3.3)$$

Proceeding with the fact that $\frac{iI}{2} = \left[-\frac{iI}{2}, -\frac{I}{2}\right]_*$, we have

$$\begin{aligned} \psi\left(\frac{iI}{2}\right) &= \psi\left(\left[-\frac{iI}{2}, -\frac{I}{2}\right]_*\right) = \left[\psi\left(-\frac{iI}{2}\right), -\frac{I}{2}\right]_* \\ &= \psi\left(-\frac{iI}{2}\right)\left(-\frac{I}{2}\right) - \left(-\frac{I}{2}\right)\psi\left(-\frac{iI}{2}\right)^* \\ &= \frac{1}{2}\left(\psi\left(-\frac{iI}{2}\right)^* - \psi\left(-\frac{iI}{2}\right)\right). \end{aligned} \quad (3.4)$$

On combining (3.3) and (3.4), we obtain that

$$\psi\left(-\frac{I}{2}\right) = i\psi\left(\frac{iI}{2}\right). \quad (3.5)$$

The last equation, together with (3.2), implies that

$$\psi\left(\frac{iI}{2}\right) = -\psi\left(\frac{iI}{2}\right)^*. \quad (3.6)$$

By manipulating the last two equations, we obtain that

$$\psi\left(-\frac{I}{2}\right) = -i\psi\left(\frac{iI}{2}\right)^*.$$

□

Lemma 3.4 $\psi(iA) = i\psi(I)A$ and $\psi(A) = \psi(I)A$ for any $A \in \mathcal{A}$.

Proof: Invoking the fact that $[\frac{iI}{2}, A]_* = iA$, and making use of lemmas 3.2 and 3.3, we obtain that

$$\begin{aligned} \psi(iA) &= \psi\left(\left[\frac{iI}{2}, A\right]_*\right) = \left[\psi\left(\frac{iI}{2}\right), A\right]_* \\ &= \psi\left(\frac{iI}{2}\right)A - A\psi\left(\frac{iI}{2}\right)^* \\ &= \psi\left(\frac{iI}{2}\right)A + A\psi\left(\frac{iI}{2}\right) \\ &= -i\psi\left(-\frac{I}{2}\right)A + A\left(-i\psi\left(-\frac{I}{2}\right)\right) \\ &= -i\psi\left(-\frac{I}{2}\right)A - \left(-i\psi\left(\frac{I}{2}\right)\right)A \end{aligned}$$

Therefore, we obtain

$$\psi(iA) = -2i\psi\left(-\frac{I}{2}\right)A. \quad (3.7)$$

On replacing A by $-iI$ in the above equation, we get

$$\psi(I) = -2\psi\left(-\frac{I}{2}\right). \quad (3.8)$$

By manipulating the last two equations, we obtain that

$$\psi(iA) = i\psi(I)A.$$

Substituting A by iA in the above equation, we obtain

$$\psi(A) = \psi(I)A. \quad (3.9)$$

□

Proof of Theorem 3.1: With the help of Lemma 3.2 and Equation (3.9), we get $\psi(A) = ZA$, where $Z = \psi(I)$.

Theorem 3.2 *Let \mathcal{A} be a unital *-algebra with a nontrivial projection. Then, every nonlinear generalized skew Lie derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ with associated nonlinear skew Lie derivation $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$.*

Proof: Let $D : \mathcal{A} \rightarrow \mathcal{A}$ be a nonlinear generalized skew Lie derivation with associated nonlinear skew Lie derivation $\xi : \mathcal{A} \rightarrow \mathcal{A}$. Then the map $\psi = D - \xi$ satisfies

$$\psi([A, B]_*) = [\psi(A), B]_*$$

for all $A, B \in \mathcal{A}$. Hence, it follows from Theorem 3.1 that $\psi(A) = ZA$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$. Consequently, $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$. □

As we have discussed earlier that unital prime *-algebras with nontrivial projections, factor von Neumann algebras and von Neumann algebras with no central summands of type I_1 satisfy (\spadesuit) . Therefore, as a direct consequence of Theorem 3.2, we have the following results.

Corollary 3.1 *Let \mathcal{A} be a unital prime *-algebra containing a non trivial projection. Then, every nonlinear generalized skew Lie derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is an additive *-derivation.*

Corollary 3.2 *Let \mathcal{A} be a factor von Neumann algebra with $\dim(\mathcal{A}) \geq 2$. Then, every nonlinear generalized skew Lie derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is an additive *-derivation.*

Corollary 3.3 *Let \mathcal{A} be a von Neumann algebra having no central summands of type I_1 . Then, every nonlinear generalized skew Lie derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $D(A) = ZA + \xi(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}(\mathcal{A})$ and $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is an additive *-derivation.*

4. Declaration

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