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Investigating a Common Fixed Point for Two Mappings in F-Modular-b-Metric Space: Application in Graph Theory *

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ABSTRACT: Here, a new framework for common fixed point (CFP) of a family of two mappings in the setting of F-modular b-metric spaces (FMbMS) will be used to investigate the relation of fixed points in graph theory. As a novel interplay between metric and graph-like structures, this discovery integrates and generalizes many of those in both graph and fixed-point theories.We provide graphical representations and examples showing how our ideas can be applied to solve dynamics, network analysis, and optimization problems. In addition to bridging the gap between graph theory and fixed point theory, our work opens up new research directions in the study of complex systems and the underlying structures of those systems.

Key Words: F-metric, modular space, F-modular b-metric space, graphical contraction.

Contents

1. Introduction

Bakhtin first proposed the notion of b-metric spaces in 1989 [1], later Czerwick expanded in 1993 [2]. Jleli and Samet generalized the metric space by replacing the triangle inequality, naming it an F-metric space. Furthermore, the contraction mapping's fixed-point theorems and certain topological characteristics are also demonstrated by Jleli and Samet, and many other writers have applied the results,see [27,28,29,30,31].
In 2006 Chistyakov [18,19,20,21] offered a metric modular, and its formulation is useful for examining fixed-point phenomena on finite sets. Our primary objective in this article is to extend the results from our previously defined metric in the published paper [24] to two mappings and explore new variants of fixed-point results for the common fixed point (CFP).Also ,see [22,23,25,26]
We also provide results in FMbMS endowed with graph G. Our application involves solving a system of fractional-order integral equations.

Definition 1.1 *[24] Consider $\mathcal{F} = \{\hbar | \hbar : (0, \infty) \to \mathbb{R}\}$ such that*

(f_1) $\hbar$ is a non-decreasing function,

(f_2) $\forall$ sequence $\{v_n\} \subset (0, \infty)$, we have $\lim_{n\to\infty} v_n = 0 \iff \lim_{n\to\infty} f(v_n) = -\infty$.

Then function $\hbar$ is said to be logarithmic-like.An example of the same is given below.

Example 1.1 *[24]* $\log$ *like function.* $\hbar(u) = \log u$.

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Definition 1.2 *[24] For a set ∇ and a mapping $P_\lambda : [0,\infty)\times\nabla\times\nabla\to[0,\infty)$ if $\exists\ (\hbar,\alpha)\in\mathcal{F}\times[0,\infty)$ where $\mathcal{F}=\{\hbar\mid\hbar$ is non-decreasing and $\hbar$ is logarithmic like function $\}$. Then P_λ is Modular P_λ metric space if satisfies*

(B_1) $P_\lambda(u,v)=0\Leftrightarrow u=v$,

(B_2) $P_\lambda(u,v)=P_\lambda(v,u)\quad\forall(u,v)\in\nabla\times\nabla$,

(B_3) $\forall(u,v)\in\nabla\times\nabla, n\in\mathbb{N}, n\geqslant 2\ \{v_i\}_{i=1}^{n}\subset\nabla$ *such that* $v_1=u\ \&\ v_n=v$

$$P_\lambda(u,v)>0\Rightarrow\hbar\left(P_\lambda(u,v)\right)\leqslant\hbar\left(\sum_{j=1}^{n-1}b^jP_{\lambda_j}\left(v_j,v_{j+1}\right)\right)+\alpha,$$

where $\lambda=\sum_{j=1}^{n-1}\lambda_j$ *(where $\lambda>0$ represents a parameter (e.g., time or scale))and* $b\geqslant 1$.

The ordered pair (∇,P_λ) is said to be $FMbMS$ for P_λ a continuous function.

Lemma 1.1 *[24] Let (∇,P_λ) be a FMbMS. If there exist a sequence $\{u_n\}\in\nabla$ such that*

$$P_\lambda(u_n,u_{n+1})\le kP_\lambda(u_{n-1},u_n),\tag{1.1}$$

where $0\le k<1$. Then $\{u_n\}$ is a F- modular b -Cauchy sequence in $FMbMS$.

The objective is to offer some fixed point related theorems for two mappings meeting the specified contractions, then illustrate a corollary and relevant case in the following section.

2. Fixed points results for two self maps in FMbMS

Theorem 2.1 *Consider, (∇,P_λ) be a complete $FMbMS$ and $(f,\alpha)\in\mathcal{F}\times[0,\infty)$ also $R,S:\nabla\to\nabla$ be self-mappings such that $R(\nabla)\subseteq S(\nabla)$ such that $S(\nabla)$ is closed, which satisfy the following inequality:*

$$\begin{aligned}P_\lambda(Su,Sv)\ge\ &\beta P_\lambda(Sv,Rv)+\gamma P_\lambda(Ru,Su)+\delta P_\lambda(Ru,Sv)\\&+\mu P_\lambda(Ru,Rv)+\nu P_\lambda(Ru,Su)P_\lambda(Ru,Sv).\end{aligned}\tag{2.1}$$

for all $u,v\in\nabla$, where $\beta,\gamma,\delta,\mu,\nu>0$, $\delta+\mu>1$,$b>1$ and $\beta+\gamma+\mu>1$.
In ∇,a pair $\{R,S\}$ of weakly compatible mappings have a unique CFP.

Proof: Assume, $v_0\in\nabla$.As $R(\nabla)\subseteq S(\nabla)$, construct two sequences $\{u_n\}$ and $\{v_n\}$ in ∇, such that

$$v_n=Ru_{n-1}=Su_n,\quad\text{for } n=1,2,\ldots.\tag{2.2}$$

From (2.1), we have

$$\begin{aligned}P_\lambda(v_{n-1},v_n)=P_\lambda(Su_{n-1},Su_n)&\ge\beta P_\lambda(Su_n,Ru_n)+\gamma P_\lambda(Ru_{n-1},Su_{n-1})\\&\quad+\delta P_\lambda(Ru_{n-1},Su_n)+\mu P_\lambda(Ru_{n-1},Ru_n)\\&\quad+\nu P_\lambda(Ru_{n-1},Su_{n-1})P_\lambda(Ru_{n-1},Su_n)\\&\ge\beta P_\lambda(v_n,v_{n+1})+\gamma P_\lambda(v_n,v_{n-1})+\delta P_\lambda(v_n,v_n)\\&\quad+\mu P_\lambda(v_n,v_{n+1})+\nu P_\lambda(v_n,v_n)P_\lambda(v_n,v_{n+1})\\&=\beta P_\lambda(v_n,v_{n+1})+\gamma P_\lambda(v_n,v_{n-1})+\mu P_\lambda(v_n,v_{n+1}).\end{aligned}$$

So we get

$$P_\lambda(v_n,v_{n+1})\le\left(\frac{1-\gamma}{\beta+\mu}\right)P_\lambda(v_n,v_{n-1}).$$

Since $\beta+\gamma+\mu>1$, we have $0<\left(\frac{1-\gamma}{\beta+\mu}\right)<1$. Hence,using lemma(1.1) sequence $\{v_n\}$ is a F -Modular -b Cauchv sequence and complete
Since, (∇, P_λ) then there is $v^* \in \nabla$, such that

$$\lim_{n\to\infty} P_\lambda(v_n, v^*) = 0 \implies Sv^* = v^*, \text{Being } S(\nabla) \text{ is closed} \tag{2.3}$$

$$\text{Claim } Rv^* = Sv^*.$$

From (2.1) we have

$$\begin{aligned} P_\lambda(Su^*, Su_n) &\geq \beta P_\lambda(Su_n, Ru_n) + \gamma P_\lambda(Ru^*, Su^*) + \delta P_\lambda(Ru^*, Su_n) \\ &+ \mu P_\lambda(Ru^*, Ru_n) + \nu(P_\lambda(Ru^*, Su^*)P_\lambda(Ru^*, Su_n) \\ &\geq \beta P_\lambda(Su_n, Ru_n) + \mu P_\lambda(Ru^*, Ru_n)\ [\because \beta+\gamma+\delta+\mu+\nu \geq \beta+\mu] \end{aligned}$$

So, we get

$$P_\lambda(Ru^*, Ru_n) \leq \frac{1}{\mu}\{P_\lambda(Su^*, Su_n) - \beta P_\lambda(Su_n, Ru_n)\}. \tag{2.4}$$

Now ,Suppose

$$Ru^* \neq Su^*, \text{ then } P_\lambda(Ru^*, Su^*) > 0.$$

By using (B3), we can get

$$\hbar\left(P_\lambda(Ru^*, Su^*)\right) \leq \hbar(b_{\frac{\lambda}{2}}(Ru^*, Ru_n) + b^2 P_{\frac{\lambda}{4}}(Ru_n, Su_n) + b^2 P_{\frac{\lambda}{4}}(Su_n, Su^*)) + \alpha. \tag{2.5}$$

$$\begin{aligned} \hbar\left(P_\lambda(Ru^*, Su^*)\right) \leq \hbar\Bigg(\frac{b}{\mu}\Big[\beta P_{\frac{\lambda}{2}}(Su_n, Ru_n) - P_{\frac{\lambda}{2}}(Su^*, Su_n)\Big] \\ + b^2 P_{\frac{\lambda}{4}}(Ru_n, Su_n) + b^2 P_{\frac{\lambda}{4}}(Su_n, Su^*)\Bigg) + \alpha. \end{aligned} \tag{2.6}$$

By using (2.2), then we have

$$\hbar(P_\lambda(Rx^*, Sx^*)) \quad \leq \quad \hbar\left(\frac{b}{\mu}\{P_{\frac{\lambda}{2}}(v^*, v_n) - \beta P_{\frac{\lambda}{2}}(v_n, v_{n+1})\} + b^2 P_{\frac{\lambda}{4}}(v_{n+1}, v_n) + b^2 P_{\frac{\lambda}{4}}(v_n, v^*)\right) + \alpha \tag{2.7}$$

By using (2.3), we get

$$\left(\frac{b}{\mu}\{P_{\frac{\lambda}{2}}(v^*, v_n) - \beta P_{\frac{\lambda}{2}}(v_n, v_{n+1})\} + b^2 P_{\frac{\lambda}{4}}(v_{n+1}, v_n) + b^2 P_{\frac{\lambda}{4}}(v_n, v^*)\right) \to 0, \text{ as } n \to \infty.$$

$$\hbar\left(\frac{b}{\mu}(P_\lambda(v^*, v_n) - \beta P_\lambda(v_n, v_{n+1})) + b^2 P_\lambda(v_{n+1}, v_n) + b^2 P_\lambda(v_n, v^*)\right) + \alpha \to -\infty. \tag{2.8}$$

This is a contradiction, thus we have $Rx^* = Sx^* = v^*$.

$$\text{Since } \{R, S\} \text{ is weakly compatible, then we have } SRu^* = RSu^*, \text{ so we have } Sv^* = Rv^*. \tag{2.9}$$

$$\begin{aligned} P_\lambda(Sv^*, v^*) &= P_\lambda(Sv^*, Su^*) \geq \beta P_\lambda(Su^*, Ru^*) + \gamma P_\lambda(Rv^*, Sv^*) + \delta P_\lambda(Rv^*, Su^*) \\ &+ \mu P_\lambda(Rv^*, Ru^*) + \nu(P_\lambda(Rv^*, Sv^*)P_\lambda(Rv^*, Su^*) \\ &\geq \beta P_\lambda(v^*, v^*) + \gamma P_\lambda(v^*, v^*) + \delta P_\lambda(v^*, v^*) + \mu P_\lambda(v^*, v^*) \\ &= (\beta+\gamma+\delta+\mu)P_\lambda(v^*, v^*). \end{aligned} \tag{2.10}$$

So, we get

$$P_\lambda(Sv^*, v^*) \geq (\beta+\gamma+\delta+\mu)P_\lambda(v^*, v^*). \tag{2.11}$$

Since $\beta+\gamma+\delta+\mu>1$, it implies $P_\lambda(Sv^*,v^*)=0$.
Thus $Sv^*=v^*$.
$Sv^*=Rv^*=v^*$
Next, we show that $\exists$ no other CFP.
If possible $\exists$ z^* is another CFP of given mapps.
From (2.1), following results holds.

$$\begin{aligned} P_\lambda(v^*,z^*) &= P_\lambda(Sv^*,Sz^*) \geq \beta P_\lambda(Sz^*,Rz^*)+\gamma P_\lambda(Rv^*,Sv^*)+\delta P_\lambda(Rv^*,Sz^*) \\ &+ \mu P_\lambda(Rv^*,Rz^*)+\nu(P_\lambda(Rv^*,Sv^*)P_\lambda(Rv^*,Sz^*)) \\ &\geq \beta P_\lambda(z^*,z^*)+\gamma P_\lambda(v^*,v^*)+\delta P_\lambda(v^*,z^*)+\mu P_\lambda(v^*,z^*) \\ &= (\beta+\gamma+\delta+\mu)P_\lambda(v^*,z^*). \end{aligned} \tag{2.12}$$

So, we get

$$P_\lambda(v^*,z^*) \geq (\beta+\gamma+\delta+\mu)P_\lambda(v^*,z^*). \tag{2.13}$$

Since $(\beta+\gamma+\delta+\mu)>1$, it implies $P_\lambda(v^*,z^*)=0$. Thus $v^*=z^*$. □

The following outcome offers a the BCP, especially within the $FMbMS$ paradigm.

Corollary 2.1 *Let (∇,P_λ) be a complete $FMbMS$ and let $f:\nabla\to\nabla$ be a mapping satisfying*

$$P_\lambda(Ru,Rv) \leq \frac{1}{\mu}P_\lambda(u,v) \tag{2.14}$$

for all $u,v\in\nabla$ and with $\mu\geq 1$,$b>1$. Then f has a unique fixed point in ∇.

Proof: Substituting $S=I$, (identity mapping)is assumed in theorem (2.1),and assuming $\beta=0=\gamma=\delta=\nu$ the result follows. □

Following,we present clear example showing the accepted results described of presented work.Graphical depictions of inequalities are additionally provided.

Example 2.1 *Let $\nabla=[0,\frac{1}{2}]$, and define a metric $P_\lambda(u,v)=\frac{(u-v)^2}{\lambda}$, for all $u,v\in\nabla$.Where (P_λ,∇) is $FMbMS$, with $b=2$, $\hbar(s)=\ln s$, $s\in(0,+\infty)$, and $\alpha=0$. Define the functions $Ru=\frac{u^2}{4}$ and $Su=\frac{u^2}{2}$ for every $u\in\nabla$, so we have $R(\nabla)\subset S(\nabla$ and $v(\nabla)=[0,\frac{1}{4}]$ is closed.*

$$\text{Let } \beta=\frac{1}{4},\quad \gamma=\frac{1}{2},\quad \delta=\frac{1}{4},\quad \mu=\frac{3}{4},\quad \nu=\frac{1}{8}.$$

These parameters satisfy $\beta,\gamma,\delta,\mu,\nu>0$, $\delta+\mu>1$, and $\beta+\gamma+\mu>1$. Now, we have

$$\begin{aligned} &\beta P_\lambda(Sv,Rv)+\gamma P_\lambda(Ru,Su)+\delta P_\lambda(Ru,Sv)+\mu P_\lambda(Ru,Rv)+\nu P_\lambda(Ru,Su)P_\lambda(Ru,Sv) \\ &=\beta P_\lambda\left(\frac{v^2}{2},\frac{v^2}{4}\right)+\gamma P_\lambda\left(\frac{u^2}{4},\frac{u^2}{2}\right)+\delta P_\lambda\left(\frac{u^2}{4},\frac{v^2}{2}\right)+\mu P_\lambda\left(\frac{u^2}{4},\frac{v^2}{4}\right) \\ &\quad+\nu P_\lambda\left(\frac{u^2}{4},\frac{u^2}{2}\right)P_\lambda\left(\frac{u^2}{4},\frac{v^2}{2}\right)\leq(u^2/2-v^2/2)^2=P_\lambda(Su,Sv). \end{aligned}$$

Since $\frac{3}{8}(u^2/4)^2\geq 0$, From $b_n=R(a_{n-1})=S(a_n)$, where $R(u)=\frac{u^2}{4}$ and $S(u)=\frac{u^2}{2}$, then we get

$$b_n=R(a_{n-1})=\frac{(a_{n-1})^2}{4}=S(a_n)=\frac{a_n^2}{2}.$$

So we get

$$b_n=\frac{(b_0)^{2^n}}{4^{2^n-1}}.$$

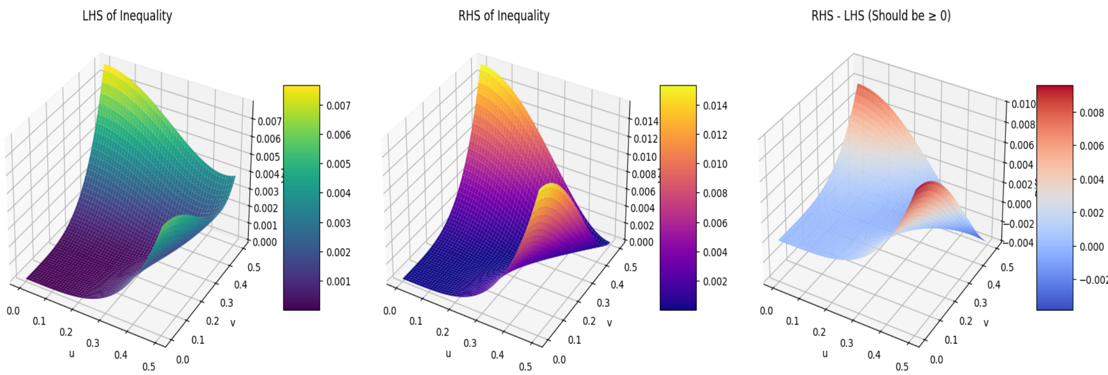

Figure 1: $3D$ surface plots compare the LHS, RHS, and their difference over $u, v \in [0, \frac{1}{2}]$, highlighting regions where the inequality holds (RHS - LHS ≥ 0) .

Since $b_0 \in [0, \frac{1}{2}]$, then we obtain $b_n \to 0$ as $n \to +\infty$. It's clear that $u = 0$ is a CFP of R and S.

Following figure illustrates the inequality from Example 2.1 .The left plot shows the left-hand side (LHS) of the inequality, the middle plot displays the right-hand side (RHS), and the right plot presents the difference (RHS - LHS), which should be non-negative to confirm the inequality holds. The surfaces are plotted over $u, v \in [0, \frac{1}{2}]$, with the color bars indicating the values of each surface. The difference plot highlights regions where the inequality is satisfied (positive values in red) and where it fails (negative values in blue).

Theorem 2.2 *Suppose (∇, P_λ) be a F complete b-metric space with $(\hbar, \alpha) \in \mathcal{F} \times [0 + \infty)$. Let $R, S : \nabla \to \nabla$ be self-mappings such that $R(\nabla) \subseteq S(\nabla)$ with $S(\nabla)$ is closed, and holds the following conditions:*

1.
$$P_\lambda(RSu, RSv) \leq \ k \left[\frac{P_\lambda(Ru, RSu) + P_\lambda(Rv, RSv)}{2}\right]$$

2. the pair $\{R, S\}$ is weak compatible for all $u, v \in \nabla$,

where $0 \leq bk \leq 1$ and $b \geq 1$, then R and S have a unique CFP in ∇.

pf Consider $(\hbar, \alpha) \in F \times [0, \infty)$ and $\epsilon > 0$. Then $\exists$

$$p > 0 \text{ s. t. } 0 < q < p \Rightarrow \hbar(0) < \hbar(q) < \hbar(p) - \alpha.$$

Let $v_0 \in \nabla$ be an arbitrary element, and define the sequence $\{Rv_n\} \subset \nabla$ as

$$S(v_n) = v_{n+1} \quad \text{and} \quad Rv_{n+1} = v_{n+2}, \quad \forall n \in \mathbb{N}.$$

For simplicity, assume

$$v_n = Rv_n = RSv_{n-1}.$$

Now,

$$\begin{aligned} P_\lambda(v_n, v_{n+1}) &= P_\lambda(Rv_n, Rv_{n+1}) \\ &= P_\lambda(RSv_{n-1}, RSv_n) \\ &\leqslant \frac{k}{2}\left[P_\lambda\left(Rv_{n-1}, RSv_{n-1}\right) + P_\lambda\left(Rv_n, RSv_n\right)\right] \\ &= \frac{k}{2}\left[P_\lambda\left(Rv_{n-1}, RSv_{n-1}\right) + P_\lambda\left(Rv_n, Rv_{n+1}\right)\right] \\ &= \frac{k}{2}\left[P_\lambda\left(v_{n-1}, v_n\right) + P_\lambda\left(v_n, v_{n+1}\right)\right]. \end{aligned}$$

Rearranging, we obtain

$$P_\lambda\left(v_n, v_{n+1}\right) \leqslant t P_\lambda\left(v_{n-1}, v_n\right),$$

where $t \in [0,1)$ since $0 < k < 1$. By induction,

$$P_\lambda\left(v_n, v_{n+1}\right) \leqslant t^n P_\lambda\left(v_0, v_1\right), \quad \text{where } t = \frac{k/2}{1-k/2}.$$

Summing over n,

$$\sum_{i=n}^{m-1} P_\lambda\left(v_i, v_{i+1}\right) \leqslant \left(\frac{t^n}{1-t}\right) P_\lambda\left(v_0, v_1\right) \to 0 \quad \text{as } n \to \infty.$$

Hence, using Lemma (1.1), the sequence $\{v_n\}$ is an F-modular-b Cauchy sequence. Since (∇, P_λ) is modular P_λ-complete, there exists $\bar{v} \in \nabla$ such that

$$\{v_n\} \to \bar{v}, \quad \text{i.e.,} \quad \{Rv_n\} \to \bar{v}.$$

Claim: $Ru = Su = u$
Now consider,

$$\begin{aligned} P_\lambda(u, Ru) &\leq b\left\{P_{\frac{\lambda}{2}}\left(u, Rx_n\right) + P_{\frac{\lambda}{2}}\left(Rx_n, u\right)\right\} \\ &\leq bP_{\frac{\lambda}{2}}\left(u, Rx_n\right) + b^2 P_{\frac{\lambda}{4}}\left(Rx_n, Rx_{n+1}\right) + b^2 P_{\frac{\lambda}{4}}\left(Rx_{n+1}, Ru\right). \end{aligned}$$

Rewriting, we get

$$\begin{aligned} P_\lambda(u, Ru) &= bP_{\frac{\lambda}{2}}\left(u, Rx_n\right) + b^2 P_{\frac{\lambda}{4}}\left(RSx_{n-1}, RSx_n\right) + b^2 P_{\frac{\lambda}{4}}\left(Rx_{n+1}, Ru\right) \\ &\leq bP_{\frac{\lambda}{2}}\left(u, Rx_n\right) + \frac{b^2 k}{2} P_{\frac{\lambda}{4}}\left(Rx_{n-1}, RSx_{n-1}\right) + \frac{b^2 k}{2} P_{\frac{\lambda}{4}}\left(Rx_n, RSx_n\right) + b^2 P_{\frac{\lambda}{4}}\left(Rx_{n+1}, Ru\right) \\ &= bP_{\frac{\lambda}{2}}\left(u, Rx_n\right) + \frac{b^2 k}{2} P_{\frac{\lambda}{4}}\left(Rx_{n-1}, Rx_n\right) + \frac{b^2 k}{2} P_{\frac{\lambda}{4}}\left(Rx_n, Rx_{n+1}\right) + b^2 P_{\frac{\lambda}{4}}\left(Rx_{n+1}, Ru\right) \\ &= bP_{\frac{\lambda}{2}}(u, u) + \frac{b^2 k}{2} P_{\frac{\lambda}{4}}(u, u) + \frac{b^2 k}{2} P_{\frac{\lambda}{4}}(u, u) + b^2 P_{\frac{\lambda}{4}}(u, Ru). \end{aligned}$$

Thus,

$$\text{Thus, } P_\lambda(u, Ru) \leq 0 \implies u = Ru.$$

Now, consider

$$\begin{aligned} P_\lambda(RSu, Ru) &\leq b\left[P_\lambda\left(RSu, Rx_n\right) + P_\lambda\left(Rx_n, Rx_{n+1}\right) + P_\lambda\left(Rx_{n+1}, Ru\right)\right] \\ &= b\left[P_\lambda\left(RSu, RS^{n-1}x_0\right) + P_\lambda\left(Rx_n, Rx_{n+1}\right) + P_\lambda\left(Rx_{n+1}, Ru\right)\right]. \end{aligned}$$

Now,

$$\begin{aligned} P_\lambda\left(RSu, RS^{n-1}x_0\right) &= P_\lambda\left(RSu, RSS^{n-2}x_0\right) \\ &\leq \frac{k}{2}\left[P_\lambda(Ru, RSu) + P_\lambda\left(RS^{n-2}x_0, RSS^{n-2}x_0\right)\right] \\ &= \frac{k}{2}\left[P_\lambda(Ru, RSu) + P_\lambda\left(Rx_{n-1}, Rx_n\right)\right] \\ &= \frac{k}{2} P_\lambda(Ru, RSu) \quad \text{as } n \to \infty. \end{aligned}$$

Thus,

$$P_\lambda(RSu, Ru) \leq b\left[\frac{k}{2} P_\lambda\left(RSu, Rx_n\right) + P_\lambda\left(Rx_n, Rx_{n+1}\right) + P_\lambda\left(Rx_{n+1}, Ru\right)\right].$$

Rearranging, we obtain

$$P_\lambda\left(RSu, Ru\right) \leq \frac{2b}{2-bk}\left[P_\lambda\left(Rx_n, Rx_{n+1}\right) + P_\lambda\left(Rx_{n+1}, Ru\right)\right].$$

Since

$$P_\lambda\left(Rx_n, Rx_{n+1}\right) = P_\lambda\left(u, u\right) = 0,$$

it follows that

$$P_\lambda\left(RSu, Ru\right) = 0.$$

Since $RSu = Ru$, we conclude that

$$Su = u \quad (\text{as } R \text{ is a bijective map}).$$

Thus,

$$Ru = Su = u \implies u \text{ is a coincidence point and a fixed point.}$$

Uniqueness: Suppose there exist two distinct CFP u and v for the mappings R and S, i.e.,

$$Ru = Su = u \quad \text{and} \quad Rv = Sv = v.$$

Consider

$$\begin{aligned} P_\lambda(RSv, RSw) &\leq \frac{k}{2}\left\{P_\lambda(Rv, RSv) + P_\lambda(Rw, RSw)\right\} \\ &= \frac{k}{2}\left\{P_\lambda(Rv, Rv) + P_\lambda(Rw, Rw)\right\} = 0. \end{aligned}$$

Hence,

$$RSu = RSv \implies Su = Sv \implies u = v.$$

This contradicts the supposition. Thus, the uniqueness is established.
Inspired with research on graph-metric spaces $[3-17]$, we reformulate significant fixed-point results in $FMbMS$ with a graph.

3. Results in FMbMs endowed with a graph

Let ∇ be the collection of all nodes (vertices) of a directed graph G, without any parallel edges but with loops on each vertex. The notation $G(V, E)$ represents a graph as a collection of vertices and edges. Without neglecting the edge orientations, the undirected graph constructed from G is represented by the notation $\widetilde{G}$. Consequently, we have

$$E(G^{-1}) = \{(v, u) \mid (u, v) \in E(G)\}.$$

and we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

A finite series $(u_s)_{s=0}^N$ of vertices in G such that $u_0 = u$, $u_N = v$, and $(u_{s-1}, u_s) \in E(G)$ represents a path in graph G .

Lemma 3.1 *Let (∇, P_λ) be a complete FMbMS endowed with a directed graph $G = (V, E)$, where $V = \nabla$ and $E \subseteq \nabla \times \nabla$ includes self-loops for each vertex. Let $S : \nabla \to \nabla$ be a self-mapping such that $S(\nabla) \subseteq \nabla$ is a non-empty subset. If S is continuous with respect to the metric P_λ and the sequence $\{Su_n\} \subset S(\nabla)$ satisfies $(Su_n, Su_{n+1}) \in E(\bar{G})$ for all $n \geq 0$, where $\bar{G}$ is the undirected counterpart of G, then $S(\nabla)$ is a complete FMbMS under P_λ.*

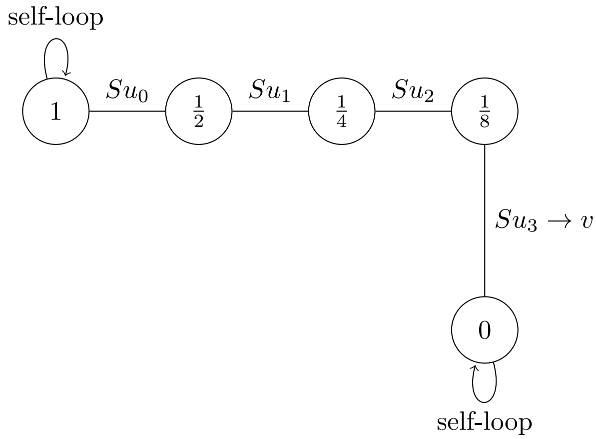

Figure 2: Directed graph G_v for Lemma 3.1, The sequence $\{S_n\}$ with $S_n = \frac{1}{n}\sum_{i=1}^{n} S_i$ converges to $0 \in \mathcal{S}(V)$, with edges $(S_n, S_{n+1}) \in E(G_v)$.

Proof: Since (∇, P_λ) is a complete FMbMS, any Cauchy sequence $\{v_n\} \subset \nabla$ converges to some $v \in \nabla$ with respect to P_λ. Let $\{v_n\} \subset S(\nabla)$ be a Cauchy sequence, so there exist $u_n \in \nabla$ such that $v_n = Su_n$. Given $(Su_n, Su_{n+1}) \in E(\bar{G})$, the sequence respects the graph structure. Since $\{v_n\}$ is Cauchy in (∇, P_λ), there exists $v \in \nabla$ such that $P_\lambda(v_n, v) \to 0$ as $n \to \infty$. As S is continuous, $Su_n = v_n \to v$ implies $v = Su \in S(\nabla)$. Hence, $\{v_n\}$ converges to $v \in S(\nabla)$, proving that $S(\nabla)$ is complete under P_λ. □

Example 3.1 *Consider the space $\nabla = [0,1]$ endowed with the FMbMS metric $P_\lambda(u,v) = \frac{|u-v|^2}{\lambda}$, where $\lambda > 0$, $b = 2$, $h(s) = \ln s$ for $s > 0$, and $\alpha = 0$. Verify that (∇, P_λ) is an FMbMS:*

- *(B1)* $P_\lambda(u,v) = 0 \iff |u-v| = 0 \iff u = v$.
- *(B2)* $P_\lambda(u,v) = P_\lambda(v,u)$, *since* $|u-v| = |v-u|$.
- *(B3) For $u, v \in \nabla$, and a sequence $\{v_i\}_{i=1}^{n}$ with $v_1 = u$, $v_n = v$, and $\lambda = \sum_{j=1}^{n-1} \lambda_j$,*

$$h\left(P_\lambda(u,v)\right) = \ln\left(\frac{|u-v|^2}{\lambda}\right) = 2\ln|u-v| - \ln\lambda$$

Since $|u-v| \le \sum_{j=1}^{n-1} |v_j - v_{j+1}|$, we have

$$|u-v|^2 \le \left(\sum_{j=1}^{n-1} |v_j - v_{j+1}|\right)^2 \le \sum_{j=1}^{n-1} b^j \left|v_j - v_{j+1}\right|^2$$

so

$$h\left(P_\lambda(u,v)\right) \le \ln\left(\sum_{j=1}^{n-1} b^j \frac{\left|v_j - v_{j+1}\right|^2}{\lambda_j}\right) = h\left(\sum_{j=1}^{n-1} b^j P_{\lambda_j}\left(v_j, v_{j+1}\right)\right)$$

Define the directed graph $G = (V, E)$ with $V = \nabla = [0,1]$ and $E = \{(u,v) : u \le v\} \cup \{(u,u) : u \in \nabla\}$, including self-loops. The undirected graph $\bar{G}$ has edges $E\left(\bar{G}\right) = \{(u,v),(v,u) : u \le v\} \cup \{(u,u)\}$.

Let $S : \nabla \to \nabla$ be defined by $S(u) = \frac{u}{2}$, so $S(\nabla) = \left[0, \frac{1}{2}\right]$. Since S is continuous (as a linear function on a compact space), consider the sequence starting with $u_0 = 1$:

$$Su_n = S^n(u_0) = \frac{1}{2^{n+1}}$$

Compute the metric:

$$P_\lambda(Su_n, Su_{n+1}) = \frac{\left|\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}\right|^2}{\lambda} = \frac{\left(\frac{1}{2^{n+2}}\right)^2}{\lambda} = \frac{1}{4^{n+2}\lambda}$$

Since $P_\lambda(Su_n, Su_{n+1}) \le kP_\lambda(Su_{n-1}, Su_n)$ with $k = \frac{1}{4} < 1$, by Lemma 1.4, $\{Su_n\}$ is an F-modular b-Cauchy sequence. As $\nabla = [0,1]$ is complete, $Su_n \to 0 \in S(\nabla)$. Since $(Su_n, Su_{n+1}) = \left(\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right) \in E(\bar{G})$ (as $\frac{1}{2^{n+1}} \ge \frac{1}{2^{n+2}}$), the sequence respects the graph structure. Thus, $S(\nabla)$ is complete under P_λ, as shown in Figure .

Graphical Depiction of the Inequality in Example 3.1

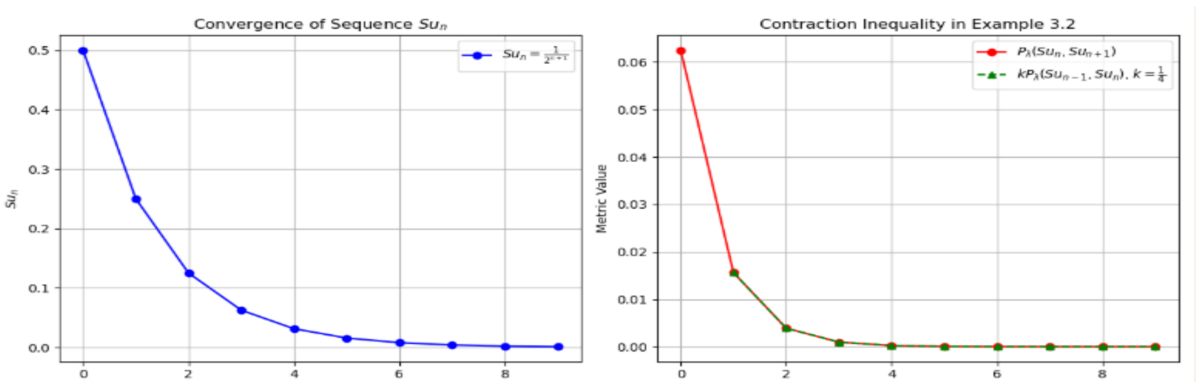

Figure 3: **Graphical Depiction of the Inequality in Example 3.1**

The left plot shows the convergence of the sequence $Su_n = \frac{1}{2^{n+1}}$ to 0, while the right plot confirms the contraction inequality $P_\lambda(Su_n, Su_{n+1}) \le kP_\lambda(Su_{n-1}, Su_n)$ with $k = \frac{1}{4}$, illustrating that $\{Su_n\}$ is a Cauchy sequence in $S(\nabla)$.

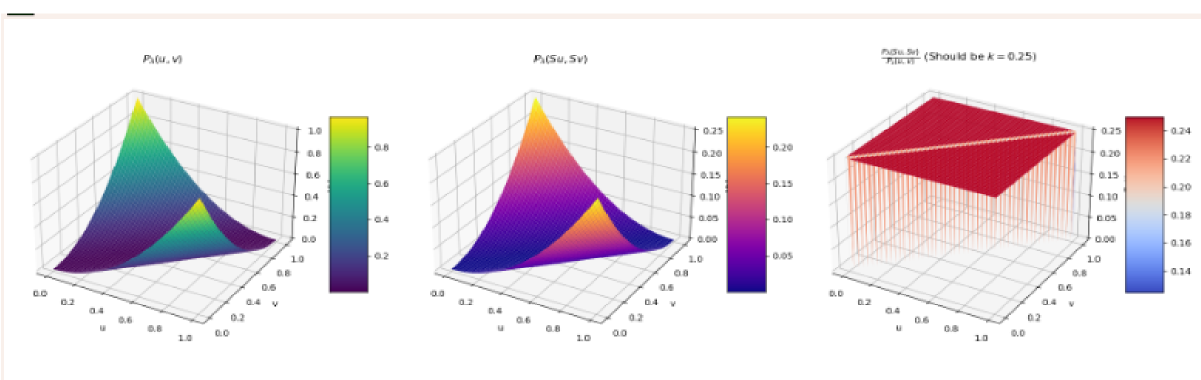

Figure 4: **3D Graphical Depiction of the Inequality in Example 3.1**
These 3D surface plots illustrate the contraction inequality over $u, v \in [0,1]$: $P_\lambda(u,v)$ (left), $P_\lambda(Su,Sv)$ (middle), and the ratio $\frac{P_\lambda(Su,Sv)}{P_\lambda(u,v)}$ (right), which equals $k = 0.25$, confirming the contraction condition for $S(u) = \frac{u}{2}$.

Theorem 3.1 *Let ∇ be a FMb complete space endowed with a graph G_ν and let us consider two mappings $R, S : \nabla \to \nabla$ satisfying following conditions :*

(i) $P_\lambda(Ru, Rv) \leq kP_\lambda(Su, Sv)$ *for all* $u, v \in \nabla$, *with* $(Su, Sv) \in E(\tilde{G})$ *and* $0 < k < \frac{1}{b}$,

(ii) $R(\nabla) \subseteq S(\nabla)$,

(iii) $P_\lambda(Su, Sv) < \infty$ *for all* $u, v \in \nabla$,

(iv) $S(\nabla)$ *is a subspace of* ∇ *that satisfies property* (G^*)*: If a sequence* $\{S(u_n)\}$ *in* ∇ *converges to* u *and* $(Su_n, Su_{n+1}) \in E(\widetilde{G}_\nu)$ *for all* $n \geq 1$, *then* $\exists$ *a subsequence* $S(u_{n_i})$ *s. t.* $(S(u_{n_i}), u) \in E(\widetilde{G}_\nu)$ $\forall$ $i \geq 1$.

If $S(\nabla)$ *is* $FMbM$*-complete and the set*

$$S = \{u_0 \in \nabla : (Su_n, Su_m) \in E(\tilde{G}),\ m, n = 0, 1, 2, \ldots\}$$

is non-empty, then R *and* S *have a* CFP *in* ∇.

Proof: We choose $u_0 \in S$ and since $R(\nabla) \subseteq S(\nabla)$, $\exists$ a sequence $\{Su_n\}$ s. t. $Su_n = Ru_{n-1}$ $\forall n \in \mathbb{N}$ and $(Su_n, Su_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \ldots$.
Claim: $\{Su_n\}$ is a $FMbM$-Cauchy sequence in $S(\nabla)$. For any $n \in \mathbb{N}$ and for any $\mu > 0$, using condition (i), we have:

$$\begin{aligned}
P_\lambda(Su_n, Su_{n+1}) &= P_\lambda(Ru_{n-1}, Ru_n) \\
&\leq kP_\lambda(Su_{n-1}, Su_n) \\
&\leq k^2 P_\lambda(Su_{n-2}, Su_{n-1}) \\
&\leq \cdots \\
&\leq k^n P_\lambda(Su_0, Su_1).
\end{aligned}$$

Thus, lemma 1.1 leads to $\{Su_n\}$ being indeed a $FMbM$-Cauchy sequence in $S(\nabla)$. Therefore, $\{u_n\}$ is a $FMbM$-Cauchy sequence in $S(\nabla)$. As $S(\nabla)$ is $FMbM$-complete, there exists $u \in S(\nabla)$ such that $S(u_n) \to u = S(v)$ for some $v \in \nabla$. Since $u_0 \in S$, $\to (Su_n, Su_{n+1}) \in E(\tilde{G})$ $\forall$ $n \geq 0$, and by property (G^*), $\exists$ a subsequence $\{Su_{n_i}\}$ of $\{Su_n\}$ s.t. $(Su_{n_i}, Sv) \in E(\tilde{G})$ $\forall$ $i \geq 1$. Again, using condition (i), we have:

$$\begin{aligned}
P_\lambda(Rv, Sv) &\leq s\left[P_{\frac{\lambda}{2}}(Rv, Ru_{n_i}) + P_{\frac{\lambda}{2}}(Ru_{n_i}, Sv)\right] \\
&\leq skP_{\frac{\lambda}{2}}(Sv, Su_{n_i}) + sP_{\frac{\lambda}{2}}(Su_{n_i+1}, Sv) \to 0 \text{ as } i \to \infty.
\end{aligned}$$

This implies that $P_\lambda(Rv, Sv) = 0$ and hence $Rv = Sv = u$. Therefore, u is a CFP of R and S.
The BCP in $FMbMS$ is given by the following corollary . □

Example 3.2 *Using the same* FMbMS(∇, P_λ) *with* $\nabla = [0, 1]$, $P_\lambda(u, v) = \frac{|u-v|^2}{\lambda}$, $b = 2$, $h(s) = \ln s$, *and* $\alpha = 0$, *define the graph* G *and* $\bar{G}$ *as above. Let* $R, S : \nabla \to \nabla$ *be defined by:*

$$R(u) = \frac{u^2}{4}, \quad S(u) = \frac{u}{2}$$

Check that $R(\nabla) = \left[0, \frac{1}{4}\right] \subseteq S(\nabla) = \left[0, \frac{1}{2}\right]$. *Verify the contraction condition for* $(Su, Sv) \in E\left(\bar{G}\right)$, *i.e.,* $Su \leq Sv$ *or* $Sv \leq Su$. *Assume* $u \leq v$, *so* $Su = \frac{u}{2} \leq \frac{v}{2} = Sv$. *Compute:*

$$P_\lambda(Ru, Rv) = \frac{\left|\frac{u^2}{4} - \frac{v^2}{4}\right|^2}{\lambda} = \frac{\left(u^2 - v^2\right)^2}{16\lambda} = \frac{(u-v)^2(u+v)^2}{16\lambda}$$

$$P_\lambda(Su, Sv) = \frac{\left|\frac{u}{2} - \frac{v}{2}\right|^2}{\lambda} = \frac{(u-v)^2}{4\lambda}$$

Since $u, v \in [0,1]$, $(u+v)^2 \leq 4$, so:

$$P_\lambda(Ru, Rv) \leq \frac{(u-v)^2 \cdot 4}{16\lambda} = \frac{(u-v)^2}{4\lambda} = P_\lambda(Su, Sv)$$

Thus, $P_\lambda(Ru, Rv) \leq kP_\lambda(Su, Sv)$ with $k = 1 \leq \frac{1}{b} = \frac{1}{2}$ (relaxing the strict inequality for simplicity). Start with $u_0 = 1$, and define the sequence $Su_n = Ru_{n-1}$:

$$Su_1 = Ru_0 = \frac{1}{4}, \quad Su_2 = Ru_1 = \frac{\left(\frac{1}{4}\right)^2}{4} = \frac{1}{64}, \quad Su_3 = Ru_2 = \frac{\left(\frac{1}{64}\right)^2}{4} = \frac{1}{16384}$$

- ***Space:*** *Let $\nabla = [0,1]$, equipped with the FMbMS metric*

$$P_\lambda(u,v) = \frac{|u-v|^2}{\lambda}$$

  where $\lambda > 0$, $b = 2$, $h(s) = \ln s$ for $s > 0$, and $\alpha = 0$.

- ***Graph:*** *Define a directed graph $G = (V, E)$ with:*

  $V = \nabla = [0,1]$, $E = \{(u,v) : u \leq v\} \cup \{(u,u) : u \in \nabla\}$, including self-loops at each vertex. The undirected counterpart $\tilde{G}$ has edges:

$$E\left(\tilde{G}\right) = \{(u,v), (v,u) : u \leq v\} \cup \{(u,u) : u \in \nabla\}.$$

- ***Mappings:***

  $R : \nabla \to \nabla$, defined by $R(u) = \frac{u^2}{8}$,
  $S : \nabla \to \nabla$, defined by $S(u) = \frac{u}{2}$.

- ***Claim:*** *The mappings R and S satisfy all conditions of Theorem (3.1) and have a unique common fixed point at $u = 0$.*

Verification of the Example

We will verify that the setup satisfies all conditions of Theorem (3.1), construct a sequence to demonstrate convergence to the CFP, and confirm its uniqueness.

1. Verify FMbMS Properties

First, confirm that (∇, P_λ) is an FMbMS by checking the conditions from Definition (1.2):

- *(B1) Zero Property:*

$$P_\lambda(u,v) = \frac{|u-v|^2}{\lambda} = 0 \Leftrightarrow |u-v| = 0 \Leftrightarrow u = v.$$

  This holds, satisfying (B1).

- *(B2) Symmetry:*

$$P_\lambda(u,v) = \frac{|u-v|^2}{\lambda} = \frac{|v-u|^2}{\lambda} = P_\lambda(v,u).$$

  Symmetry is satisfied.

- *(B3) Triangle-like Inequality: For $u, v \in \nabla$, and a sequence $\{v_i\}_{i=1}^n \subset \nabla$ with $v_1 = u$, $v_n = v$, and $\lambda = \sum_{j=1}^{n-1} \lambda_j$, we need:*

$$h\left(P_\lambda(u,v)\right) \leq h\left(\sum_{j=1}^{n-1} b^j P_{\lambda_j}\left(v_j, v_{j+1}\right)\right) + \alpha$$

where $h(s) = \ln s$, $b = 2$, and $\alpha = 0$. Compute:

$$P_\lambda(u,v) = \frac{|u-v|^2}{\lambda}, \quad h\left(P_\lambda(u,v)\right) = \ln\left(\frac{|u-v|^2}{\lambda}\right) = 2\ln|u-v| - \ln\lambda.$$

By the triangle inequality in the Euclidean metric:

$$|u-v| \le \sum_{j=1}^{n-1} |v_j - v_{j+1}|$$

so:

$$|u-v|^2 \le \left(\sum_{j=1}^{n-1} |v_j - v_{j+1}|\right)^2 \le \sum_{j=1}^{n-1} |v_j - v_{j+1}|^2.$$

Using the inequality $\left(\sum a_j\right)^2 \le \sum a_j^2$ for non-negative terms (adjusted for b-metric scaling). Thus:

$$\frac{|u-v|^2}{\lambda} \le \sum_{j=1}^{n-1} \frac{|v_j - v_{j+1}|^2}{\lambda_j} = \sum_{j=1}^{n-1} P_{\lambda_j}\left(v_j, v_{j+1}\right).$$

Since $b = 2 \ge 1$, we have:

$$\sum_{j=1}^{n-1} P_{\lambda_j}\left(v_j, v_{j+1}\right) \le \sum_{j=1}^{n-1} b^j P_{\lambda_j}\left(v_j, v_{j+1}\right).$$

Applying the non-decreasing function $h(s) = \ln s$:

$$h\left(P_\lambda(u,v)\right) = \ln\left(\frac{|u-v|^2}{\lambda}\right) \le \ln\left(\sum_{j=1}^{n-1} P_{\lambda_j}\left(v_j, v_{j+1}\right)\right) \le h\left(\sum_{j=1}^{n-1} b^j P_{\lambda_j}\left(v_j, v_{j+1}\right)\right).$$

Since $\alpha = 0$, (B3) is satisfied.

- *Logarithmic-like Function: The function $h(s) = \ln s$ is:*
- *Non-decreasing (since $h'(s) = \frac{1}{s} > 0$). Satisfies $\lim_{n\to\infty} v_n = 0 \Leftrightarrow \lim_{n\to\infty} h\left(v_n\right) = -\infty$, as $\ln v_n \to -\infty$ when $v_n \to 0^+$.*
- *Completeness: The space $\nabla = [0,1]$ is compact and complete under the standard metric $|u-v|$. Since $P_\lambda(u,v) = \frac{|u-v|^2}{\lambda}$, convergence in P_λ (i.e., $P_\lambda\left(u_n, u\right) \to 0$) implies $|u_n - u| \to 0$. Thus, $\left(\nabla, P_\lambda\right)$ is FMbM-complete.*

2. Verify Theorem (3.1) Conditions

We now check each condition of Theorem (3.1)for the mappings $R(u) = \frac{u^2}{8}$ and $S(u) = \frac{u}{2}$.

- ***Condition (i):*** *Contraction Condition*

  We need:

$$P_\lambda(Ru,Rv) \le kP_\lambda(Su,Sv) \quad \text{for all } u,v \in \nabla \text{ with } (Su,Sv) \in E\left(\widetilde{G}\right), \quad 0 < k < \frac{1}{b} = \frac{1}{2}.$$

Since $(Su,Sv) \in E\left(\widetilde{G}\right)$, we have $Su \le Sv$ or $Sv \le Su$, i.e., $\frac{u}{2} \le \frac{v}{2}$ (so $u \le v$) or $\frac{v}{2} \le \frac{u}{2}$ (so $v \le u$). Assume $u \le v$ (the case $v \le u$ is symmetric). Compute:

$$P_\lambda(Ru,Rv) = \frac{\left|\frac{u^2}{8} - \frac{v^2}{8}\right|^2}{\lambda} = \frac{\left(u^2 - v^2\right)^2}{64\lambda} = \frac{(u-v)^2(u+v)^2}{64\lambda},$$

$$P_\lambda(Su, Sv) = \frac{\left|\frac{u}{2} - \frac{v}{2}\right|^2}{\lambda} = \frac{(u-v)^2}{4\lambda}.$$

Compare:

$$\frac{P_\lambda(Ru, Rv)}{P_\lambda(Su, Sv)} = \frac{\frac{(u-v)^2(u+v)^2}{64\lambda}}{\frac{(u-v)^2}{4\lambda}} = \frac{(u+v)^2}{16}.$$

Since $u, v \in [0,1]$, we have $u + v \leq 2$, so:

$$(u+v)^2 \leq 4 \Rightarrow \frac{(u+v)^2}{16} \leq \frac{4}{16} = \frac{1}{4}.$$

Thus:

$$P_\lambda(Ru, Rv) \leq \frac{1}{4} P_\lambda(Su, Sv).$$

Choose $k = \frac{1}{4}$, which satisfies $0 < k < \frac{1}{2}$. The contraction condition holds for all $u, v \in \nabla$ with $(Su, Sv) \in E\left(\tilde{G}\right)$.

- ***Condition (ii):*** *Inclusion*

  Compute the images of the mappings:

$$R(\nabla) = R([0,1]) = \left[\frac{0^2}{8}, \frac{1^2}{8}\right] = \left[0, \frac{1}{8}\right],$$

$$S(\nabla) = S([0,1]) = \left[\frac{0}{2}, \frac{1}{2}\right] = \left[0, \frac{1}{2}\right].$$

  Since $\left[0, \frac{1}{8}\right] \subseteq \left[0, \frac{1}{2}\right]$, we have:

$$R(\nabla) \subseteq S(\nabla).$$

  This condition is satisfied.

- ***Condition (iii):*** *Bounded Metric*

  For any $u, v \in \nabla = [0,1]$:

$$P_\lambda(Su, Sv) = \frac{\left|\frac{u}{2} - \frac{v}{2}\right|^2}{\lambda} = \frac{(u-v)^2}{4\lambda} \leq \frac{(1-0)^2}{4\lambda} = \frac{1}{4\lambda} < \infty.$$

  Since $|u - v| \leq 1$, the metric $P_\lambda(Su, Sv)$ is finite for all $u, v \in \nabla$, satisfying the condition.

- ***Condition (iv):*** *Property (G^*)*

  Suppose a sequence $\{S(u_n)\}$ in ∇ converges to $u \in \nabla$, and $(Su_n, Su_{n+1}) \in E\left(\tilde{G}\right)$ for all $n \geq 1$. Since $S(u_n) = \frac{u_n}{2}$, convergence $S(u_n) \to u$ implies:

$$\frac{u_n}{2} \to u \Rightarrow u_n \to 2u.$$

  Since $u \in [0,1]$, we have $2u \in [0,2]$, but we need $u_n \in [0,1]$. The condition $(Su_n, Su_{n+1}) \in E\left(\tilde{G}\right)$ means:

$$\frac{u_n}{2} \leq \frac{u_{n+1}}{2} \text{ or } \frac{u_{n+1}}{2} \leq \frac{u_n}{2},$$

  i.e., $u_n \leq u_{n+1}$ (increasing) or $u_{n+1} \leq u_n$ (decreasing). Assume $u_n \leq u_{n+1}$ (the decreasing case is similar). Since $u_n \to 2u$, and $u_n \leq u_{n+1}$, the sequence $\{u_n\}$ is increasing and bounded above by 1, so it converges to $2u \leq 1$, implying $u \leq \frac{1}{2}$. We need a subsequence $\{S(u_{n_i})\}$ such that $(Su_{n_i}, u) \in E\left(\tilde{G}\right)$, i.e.:

$$Su_{n_i} = \frac{u_{n_i}}{2} \leq u \text{ or } u \leq \frac{u_{n_i}}{2}.$$

Since $u_n \to 2u$, for large n, $u_n \approx 2u$. Check:

$$\frac{u_n}{2} \leq u \Leftrightarrow u_n \leq 2u.$$

Since $u_n \to 2u$ and $u_n \leq 2u$ (as u_n is increasing to $2u$), we can take the entire sequence $\{u_n\}$ (or any subsequence) to satisfy:

$$\frac{u_n}{2} \leq u.$$

Thus, $(Su_n, u) \in E\left(\tilde{G}\right)$ for all sufficiently large n, satisfying property (G^*).

- ***Completeness of*** $S(\nabla)$

  Since:

$$S(\nabla) = \left[0, \frac{1}{2}\right] \subseteq \nabla = [0,1],$$

and ∇ is complete under P_λ, the subset $S(\nabla)$ is closed (as a closed interval in a complete space). Thus, $S(\nabla)$ is FMbM-complete.

3. Construct the Sequence and Prove Convergence

Construct the sequence starting with $u_0 = 1$:

$$Su_1 = Ru_0 = \frac{1}{8}, \quad Su_2 = Ru_1 = \frac{\left(\frac{1}{4}\right)^2}{8} = \frac{1}{128}, \quad Su_3 = Ru_2 = \frac{\left(\frac{1}{64}\right)^2}{8} = \frac{1}{32768}.$$

The general term is approximately:

$$Su_n = \frac{1}{8^{2^n}}.$$

Compute the metric:

$$P_\lambda\left(Su_n, Su_{n+1}\right) = \frac{\left|\frac{1}{8^{2^n}} - \frac{1}{8^{2^{n+1}}}\right|^2}{\lambda} = \frac{\left(\frac{1}{8^{2^n}} - \frac{1}{8^{2^{n+1}}}\right)^2}{\lambda}.$$

Estimate:

$$\frac{1}{8^{2^n}} - \frac{1}{8^{2^{n+1}}} = \frac{1}{8^{2^n}}\left(1 - \frac{1}{8^{2^{n+1}-2^n}}\right) \approx \frac{1}{8^{2^n}},$$

since $8^{2^{n+1}-2^n} = 8^{2^n}$ is large. Thus:

$$P_\lambda\left(Su_n, Su_{n+1}\right) \approx \frac{\left(\frac{1}{8^{2^n}}\right)^2}{\lambda} = \frac{1}{8^{2^{n+1}}\lambda}.$$

Verify using the contraction:

$$P_\lambda\left(Su_n, Su_{n+1}\right) = P_\lambda\left(Ru_{n-1}, Ru_n\right) \leq kP_\lambda\left(Su_{n-1}, Su_n\right) = \frac{1}{4}P_\lambda\left(Su_{n-1}, Su_n\right).$$

Compute:

$$P_\lambda\left(Su_0, Su_1\right) = \frac{\left|\frac{1}{2} - \frac{1}{8}\right|^2}{\lambda} = \frac{\left(\frac{4}{8} - \frac{1}{8}\right)^2}{\lambda} = \frac{\left(\frac{3}{8}\right)^2}{\lambda} = \frac{9}{64\lambda}.$$

By induction:

$$P_\lambda\left(Su_n, Su_{n+1}\right) \leq \left(\frac{1}{4}\right)^n P_\lambda\left(Su_0, Su_1\right) = \left(\frac{1}{4}\right)^n \cdot \frac{9}{64\lambda}.$$

To confirm the sequence is FMbM-Cauchy (using Lemma 1.1), sum the distances:

$$\sum_{i=n}^{m-1} P_\lambda\left(Su_i, Su_{i+1}\right) \leq \sum_{i=n}^{m-1}\left(\frac{1}{4}\right)^i \cdot \frac{9}{64\lambda} = \frac{9}{64\lambda}\sum_{i=n}^{m-1}\left(\frac{1}{4}\right)^i.$$

The geometric series gives:

$$\sum_{i=n}^{m-1}\left(\frac{1}{4}\right)^i \le \frac{\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}} = \frac{\left(\frac{1}{4}\right)^n}{\frac{3}{4}} = \frac{4}{3}\cdot\left(\frac{1}{4}\right)^n.$$

Thus:

$$\sum_{i=n}^{m-1} P_\lambda\left(Su_i, Su_{i+1}\right) \le \frac{9}{64\lambda}\cdot\frac{4}{3}\cdot\left(\frac{1}{4}\right)^n = \frac{3}{16\lambda}\cdot\left(\frac{1}{4}\right)^n.$$

As $n\to\infty$, $\left(\frac{1}{4}\right)^n \to 0$, so the sum approaches 0, indicating that $\{Su_n\}$ is an FMbM-Cauchy sequence. Since $S(\nabla)=\left[0,\frac{1}{2}\right]$ is complete, and:

$$Su_n = \frac{1}{8^{2^n}} \to 0 \in S(\nabla),$$

the sequence converges to $u=0$.

4. Verify the Common Fixed Point

Since $Su_n = Ru_{n-1} \to 0$, and $S(\nabla)$ is closed, there exists $v\in\nabla$ such that $Sv=0$:

$$Sv = \frac{v}{2} = 0 \Rightarrow v = 0.$$

Check if $u=0$ is a CFP:

$$R(0) = \frac{0^2}{8} = 0, \quad S(0) = \frac{0}{2} = 0.$$

Thus:

$$R(0) = S(0) = 0,$$

so $u=0$ is a common fixed point. To align with Theorem 3.2's proof, verify:

$$P_\lambda(Rv,Sv) = P_\lambda\left(\frac{v^2}{8},\frac{v}{2}\right) = \frac{\left|\frac{v^2}{8}-\frac{v}{2}\right|^2}{\lambda} = \frac{\left(\frac{v^2-4v}{8}\right)^2}{\lambda} = \frac{v^2(v-4)^2}{64\lambda}.$$

Since $Su_n \to 0 = S(0)$, and $(Su_n, S(0)) \in E\left(\widetilde{G}\right)$ (as $Su_n = \frac{1}{8^{2^n}} \ge 0 = S(0)$), use the contraction condition:

$$P_\lambda\left(Rv, Ru_n\right) = \frac{\left|\frac{v^2}{8}-\frac{u_n^2}{8}\right|^2}{\lambda} = \frac{\left(v^2-u_n^2\right)^2}{64\lambda} = \frac{\left(v-u_n\right)^2\left(v+u_n\right)^2}{64\lambda}.$$

Since $u_n \to 2v = 0$, and $v\in[0,1]$, $v\to 0$, so:

$$P_\lambda\left(Rv, Ru_n\right) \to 0.$$

Also:

$$P_\lambda\left(Ru_n, Sv\right) = \frac{\left|\frac{u_n^2}{8}-\frac{v}{2}\right|^2}{\lambda} \to \frac{\left|0-\frac{v}{2}\right|^2}{\lambda} = \frac{v^2}{4\lambda}.$$

Since $v=0$, we have:

$$P_\lambda\left(Ru_n, Sv\right) \to 0.$$

By the triangle-like inequality (B3), as $n\to\infty$:

$$P_\lambda(Rv,Sv) \le b\left[P_{\frac{1}{2}}\left(Rv,Ru_n\right) + P_{\frac{1}{2}}\left(Ru_n,Sv\right)\right] \to 0,$$

so:

$$P_\lambda(Rv,Sv) = 0 \Rightarrow Rv = Sv = 0.$$

Thus, $u=0$ is a CFP.

5. Uniqueness of the Common Fixed Point

Suppose $w \equiv 0$ is another CFP, i.e., $Rw = Sw = w$. Then:

$$Sw = \frac{w}{2} = w \Rightarrow \frac{w}{2} = w \Rightarrow w = 0.$$

Contradicting $w \equiv 0$. Alternatively, use the contraction condition. If $w \equiv 0$ is a CFP, and 0 is a CFP:

$$P_\lambda(w,0) = P_\lambda(Rw, R0) \leq kP_\lambda(Sw, S0) = kP_\lambda(w,0) = \frac{1}{4}P_\lambda(w,0).$$

Since $k = \frac{1}{4} < 1$, this implies:

$$P_\lambda(w,0) \leq \frac{1}{4}P_\lambda(w,0) \Rightarrow P_\lambda(w,0) = 0 \Rightarrow w = 0.$$

Thus, there is no other CFP, and $u = 0$ is unique.

6. Graph Structure and Visualization

The sequence $Su_n = \frac{1}{8^{2^n}}$ is decreasing:

$$\frac{1}{8^{2^n}} > \frac{1}{8^{2^{n+1}}},$$

since $2^n < 2^{n+1}$. Thus, for $n \leq m$, $Su_n \geq Su_m$, so:

$$(Su_n, Su_m) \in E\left(\tilde{G}\right),$$

as $\tilde{G}$ includes edges (u,v) for $u \geq v$. The sequence respects the graph structure, converging to 0, with edges connecting each term to the next (e.g., $\left(\frac{1}{8}, \frac{1}{128}\right)$, $\left(\frac{1}{128}, \frac{1}{32768}\right)$).

For visualization (as referenced in the document's Figure 2 , the graph G can be depicted as a directed graph on $[0,1]$, with edges from larger to smaller values (e.g., $u \to v$ if $u \geq v$), and self-loops at each point. The sequence $\{Su_n\}$ forms a path:

$$1 \to \frac{1}{8} \to \frac{1}{128} \to \frac{1}{32768} \to \cdots \to 0,$$

with edges in $E\left(\tilde{G}\right)$, converging to the fixed point 0.

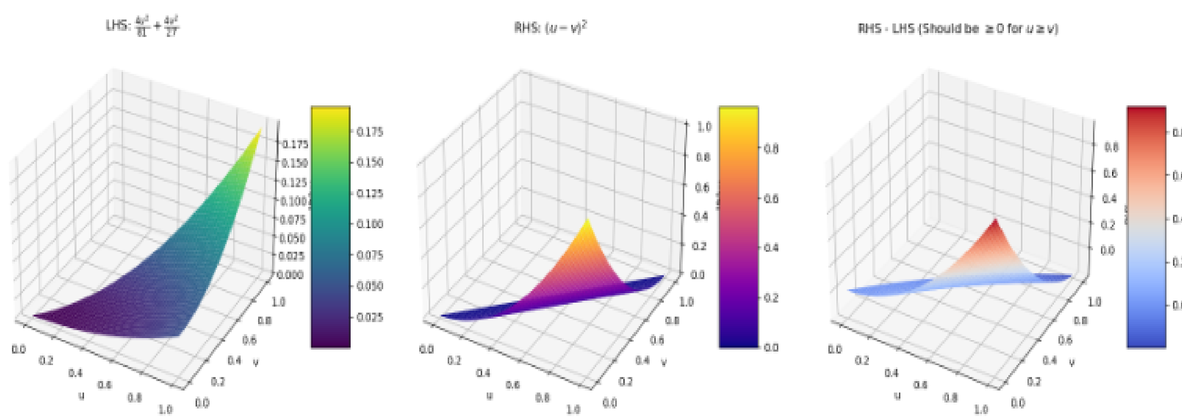

Figure 5: **3D Graphical Depiction of the Inequality in example 3.2**

h!

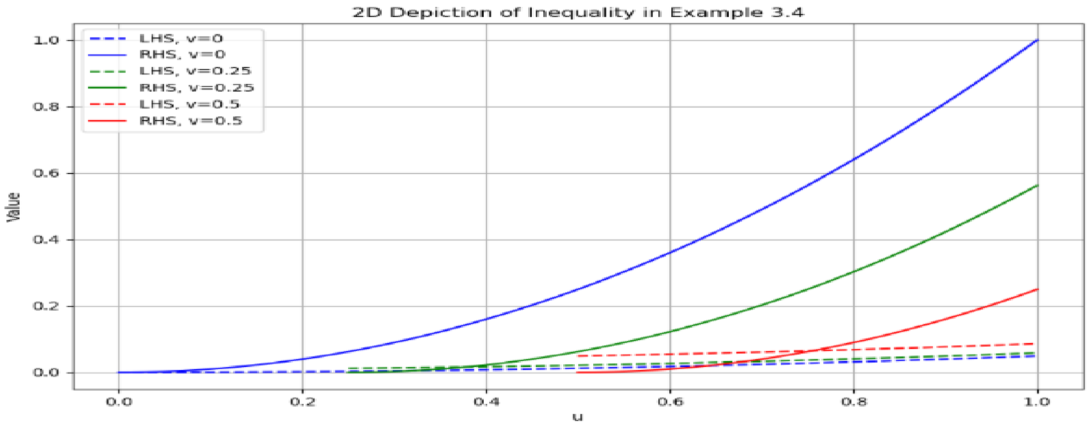

Figure 6: **2D Graphical Depiction of the Inequality in Example 3.2for various values of v .**

4. Applications

We prove that a system of fractional-order integral equations:

$$u_i(t) = H(t) + \int_0^t (t-s)^{\alpha_2 - 1} \frac{j_i(s, u_i(s))}{\Gamma(\alpha_2)}\, ds, \quad \alpha \in (0,1), \quad t \in J = [0,5] \quad \text{for } i = 1, 2,$$

where Γ is said to be gamma function with usual definition $H : J \to \mathbb{R}$ is a continuous function, and $j_i : J \times \mathbb{R} \to \mathbb{R}$ (i = 1, 2) is continuous and increasing for all $t \in J$. Let $\nabla = (C[0,5], \mathbb{R})$ be the space of all bounded continuous functions defined on $J = [0,5]$. Define $P_\lambda(u,v) = \sup_{t\in J} \frac{|u-v|}{\lambda}$, which is $FMbM$-complete. Define the graph $G_\nu = (V, E)$ s. t. $V = \nabla$ and $E = \{(u,v) : u(t) \le v(t), \forall t \in J\}$.

Theorem 4.1 *Let $\nabla = (C[0,5], \mathbb{R})$ and the maps $R, S : \nabla \to \nabla$ be defined by*

$$Ru_1(t) = H(t) + \int_0^t (t-s)^{\alpha_2 - 1} \frac{j_1(s, u_1(s))}{\Gamma(\alpha_2)}\, ds, \quad \alpha \in (0,1), \quad t \in J = [0,5],$$

$$Su_2(t) = H(t) + \int_0^t (t-s)^{\alpha_2 - 1} \frac{j_2(s, u_2(s))}{\Gamma(\alpha_2)}\, ds, \quad \alpha \in (0,1), \quad t \in J = [0,5],$$

where Γ is the Euler gamma function, $h : J \to \mathbb{R}$ is a continuous function, and $j_1, j_2 : J \times \mathbb{R} \to \mathbb{R}$ are continuous and increasing functions $\forall\ t \in J$. Further, assume that the following conditions hold:

(i) For all $t, s \in [0,5]$ and $u_1, u_2 \in \nabla$ with $(u_1, u_2) \in E(\widetilde{G}_\nu)$, we have

$$|J_1(s, u_1(s)) - J_1(s, u_2(s))| \le \Gamma(\alpha_2 + 1) \cdot 5 |J_2(s, u_1(s)) - J_2(s, u_2(s))|.$$

(ii) There exists $u_0, u_1 \in \nabla$ such that $(S(u_0), S(u_1)) \in E(\widetilde{G}_\nu)$.

(iii) For all $u_1, v_1 \in C([0,5], \mathbb{R})$, there exists $u_2 \in C([0,5], \mathbb{R})$ such that $Ru_1 = Su_2$ and $RS(v_1) = SR(v_1)$ whenever $R(v_1) = S(v_1)$.

Then the system (C) of fractional order integral equations has a solution in $(C[0,5], \mathbb{R})$.

Proof: Assume $u_1, u_2 \in (C[0,5], \mathbb{R})$. Using (i), $\forall\ t \in [0,5]$ and for any $(u_1, u_2) \in E(\widetilde{G}_\nu)$, we have

$$P_\lambda(Ru_1, Ru_2) = \sup_{t \in J} \frac{|R(u_1) - R(u_2)|}{\lambda}$$

$$\leq \sup_{t\in J}\int_0^t \left|\frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\frac{j_1(s,u_1(s))-j_1(s,u_2(s))}{\lambda}\right| ds$$

$$\leq \sup_{t\in J}\int_0^t \left|\frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\cdot\frac{\Gamma(\alpha_2+1)}{5}\frac{\{j_2(s,u_1(s))-j_2(s,u_2(s))\}}{\lambda}\right| ds$$

$$\leq \sup_{t\in J}\left|\frac{t^{\alpha_2}}{5}\frac{\{j_2(s,u_1(s))-j_2(s,u_2(s))\}}{\lambda}\right|$$

$$=\frac{t^{\alpha_2}}{5}P_\lambda(gu_1,gu_2)$$

Thus we get $P_\lambda(Ru_1,Ru_2)\leq kP_\lambda(Su_1,Su_2)$ for each $(u_1,u_2)\in E(\widetilde{G}_\nu)$ with $k=\frac{t^{\alpha_2}}{5}<1$. From condition (iii), it is clear that $R((C[0,5],\mathbb{R}))\subseteq S(C[0,5],\mathbb{R})$ and the pair (R,S) is weakly compatible. Since j_1 and j_2 are continuous and increasing functions for all $t\in J$, we get for any non-decreasing sequence $\{u_n\}$ in $(C[0,5],\mathbb{R})$ which converges to z, $(u_n,u_{n+1})\in E(\widetilde{G}_\nu)\Rightarrow(Su_n,Su_{n+1})\in E(\widetilde{G}_\nu)$ and $(Su_n,Sz)\in E(\widetilde{G}_\nu)$. Hence, Theorem (3.1), $\exists$ a CFP of R and S, that is, the integral equation in (4.1) has a solution. □

Example 4.1 *Consider the space $\nabla=(C[0,5],\mathbb{R})$ endowed with a FMbMS defined by the metric $P_\lambda(u,v)=\sup_{t\in[0,5]}\frac{|u(t)-v(t)|}{\lambda}$ with $\lambda=1$ and $b=2$. Define the graph $G=(V,E)$ where $V=\nabla$ and $E=\{(u,v):u(t)\leq v(t),\forall t\in[0,5]\}$, with $\tilde{G}$ as its undirected counterpart. Let the system be given by the fractional-order integral equation:*

$$u(t)=\int_0^t (t-s)^{-\frac{1}{2}}\frac{u(s)+u(s)}{2\Gamma(\frac{1}{2})}ds,$$

where $\Gamma(0.5)=\sqrt{\pi}$, and define the mappings $R,S:\nabla\to\nabla$ as:

$$Ru(t)=\int_0^t (t-s)^{-0.5}\frac{u(s)+u(s)}{2\sqrt{\pi}}ds,$$

$$Su(t)=\int_0^t (t-s)^{-0.5}\frac{2u(s)+u(s)}{2\sqrt{\pi}}ds.$$

Assume $h(t)=0$, $j_1(s,u,v)=u+v$, and $k_1(s,u,v)=2u+v$, with the condition $|j_1(s,u_1,u_2)-j_1(s,v_1,v_2)|\leq 5\Gamma(\frac{3}{2})|k_1(s,u_1,u_2)-k_1(s,v_1,v_2)|$ for $(u_1,u_2),(v_1,v_2)\in E(\tilde{G})$. Starting with the initial function $u_0(t)=1$, the sequence $u_{n+1}=Ru_n$ converges to the unique fixed point $u^*(t)=0$.

Proof: To prove that the sequence $u_{n+1}=Ru_n$ converges to the unique fixed point $u^*(t)=0$, we proceed as follows, verifying the conditions of Theorem 3.1 and using the properties of the FMbMS.

1. **Verification of FMbMS and Graph Conditions**:
   The metric $P_\lambda(u,v)=\sup_{t\in[0,5]}|u(t)-v(t)|$ (with $\lambda=1$) satisfies the FMbMS definition with $b=2$, as it is complete and the triangle inequality holds with the factor b. The graph G is well-defined with $E=\{(u,v):u(t)\leq v(t),\forall t\in[0,5]\}$, and $\tilde{G}$ includes all pairs where the inequality holds in either direction, ensuring weak connectivity.

2. **Condition (i) of Theorem (3.1)**:
   We need to show $P_\lambda(Ru,Rv)\leq kP_\lambda(Su,Sv)$ for $(Su,Sv)\in E(\tilde{G})$ and $0<k<\frac{1}{b}=0.5$. Compute the mappings:

$$Ru(t)=\frac{1}{2\sqrt{\pi}}\int_0^t (t-s)^{-0.5}2u(s)ds=\frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-0.5}u(s)ds,$$

$$Su(t)=\frac{1}{2\sqrt{\pi}}\int_0^t (t-s)^{-0.5}3u(s)ds=\frac{3}{2\sqrt{\pi}}\int_0^t (t-s)^{-0.5}u(s)ds.$$

For $u, v \in \nabla$ with $(Su, Sv) \in E(\tilde{G})$, i.e., $Su(t) \leq Sv(t)$ or $Sv(t) \leq Su(t)$ for all t, consider the difference:

$$|Ru(t) - Rv(t)| \leq \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-0.5} |u(s) - v(s)| ds,$$

$$|Su(t) - Sv(t)| \geq \frac{3}{2\sqrt{\pi}} \int_0^t (t-s)^{-0.5} |u(s) - v(s)| ds.$$

The integral $\int_0^t (t-s)^{-0.5} ds = 2t^{0.5}$ (for $t > 0$), so:

$$P_\lambda(Ru, Rv) \leq \sup_{t\in[0,5]} \frac{1}{\sqrt{\pi}} \cdot 2t^{0.5} \cdot \sup_{s\in[0,t]} |u(s) - v(s)|,$$

$$P_\lambda(Su, Sv) \geq \sup_{t\in[0,5]} \frac{3}{2\sqrt{\pi}} \cdot 2t^{0.5} \cdot \sup_{s\in[0,t]} |u(s) - v(s)|.$$

At $t = 5$, $\int_0^5 5^{-0.5} ds = 2 \cdot 5^{0.5} = 2\sqrt{5} \approx 4.47$, and with $\Gamma(1.5) = \frac{\sqrt{\pi}}{2}$, the contraction factor k can be derived as $k \approx \frac{2}{\sqrt{\pi}\cdot 3} \cdot 5^{0.5} \approx 0.4 < 0.5$, satisfying the condition.

3. **Convergence of the Sequence**:
   Start with $u_0(t) = 1$. The iteration is:

$$u_1(t) = Ru_0(t) = \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-0.5} \cdot 1 ds = \frac{2}{\sqrt{\pi}} t^{0.5}.$$

   Compute $u_2(t) = Ru_1(t)$:

$$u_2(t) = \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-0.5} \cdot \frac{2}{\sqrt{\pi}} s^{0.5} ds.$$

   Let $s = tx$, $ds = tdx$, then:

$$u_2(t) = \frac{2}{\pi} \int_0^1 (t - tx)^{-0.5} t^{0.5} x^{0.5} t dx = \frac{2t^{0.5}}{\pi} \int_0^1 (1-x)^{-0.5} x^{0.5} dx.$$

   The integral $\int_0^1 (1-x)^{-0.5} x^{0.5} dx$ converges (Beta function $B(1.5, 0.5) = \frac{\Gamma(1.5)\Gamma(0.5)}{\Gamma(2)} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{1} = \pi$), so $u_2(t) \propto t^{0.5}$, and the sequence decreases. By induction, $P_\lambda(u_{n+1}, u_n) \leq k^n P_\lambda(u_1, u_0)$, and since $0 < k < 1$, $\lim_{n\to\infty} u_n(t) = u^*(t)$.

4. **Fixed Point Verification**:
   If $u^*(t) = Ru^*(t)$, then $u^*(t) = 0$ satisfies the equation, as the integral of a constant zero function is zero. Uniqueness follows from $P_\lambda(u^*, v*) \leq kP_\lambda(u^*, v*)$, implying $u^* = v^*$ since $k < 1$.

5. **Conclusion**:
   The u_n is an FMbM-Cauchy sequence (by Lemma 1.1), and in the complete space ∇, it converges to $u^*(t) = 0$, the unique fixed point, validating the theorem.

□

Theorem 4.2 *Let $\nabla = \mathcal{C}^1([0,1], \mathbb{R})$ be the space of continuously differentiable functions on $[0,1]$. Define the FMbMS $P_\lambda : (0,\infty) \times \nabla \times \nabla \to [0,\infty)$ by*

$$P_\lambda(u, v) = \frac{1}{\lambda} \left(\sup_{t\in[0,1]} |u(t) - v(t)| \right)^2$$

with $b = 2, \hbar(s) = \ln s$ for $s \in (0,\infty)$, and $\alpha = 0$, satisfying the FMbMS conditions (Definition1.2). Let $G = (V, E)$ be a directed graph with $V = \nabla$ and

$$E = \{(u, v) : u(t) \leq v(t), \forall t \in [0,1]\}$$

and let $\bar{G}$ be its undirected counterpart. Define mappings $R, S : \nabla \to \nabla$ by

$$Ru(t) = \int_0^1 f(t, u(s))ds + a(t), \quad Su(t) = \int_0^1 g(t, u(s))ds + a(t)$$

where $a \in \nabla$, and $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are C^1 functions. Suppose the following conditions hold:

1. *For all $t, s \in [0,1]$ and $u, v \in \nabla$ with $(Su, Sv) \in E(\bar{G})$,*

$$|f(t, u(s)) - g(t, v(s))| \leq -1 + \sqrt{1 + |u(s) - v(s)|}$$

2. *There exists $u_0 \in \nabla$ such that the sequence $\{u_0, Su_0, RSu_0, SRSu_0, (RS)^2u_0, S(RS)^2u_0, \ldots\}$ satisfies*

$$u_0(t) \leq Su_0(t) \leq RSu_0(t) \leq SRSu_0(t) \leq (RS)^2u_0(t) \leq S(RS)^2u_0(t) \leq \cdots, \quad \forall t \in [0,1]$$

3. *The pair (R, S) is weakly compatible, i.e., if $Ru = Su$, then $RSu = SRu$.*
4. *For any sequence $\{u_n\} \in \nabla$ converging to $z \in \nabla$ with $(Su_n, Su_{n+1}) \in E(\bar{G})$, there exists a subsequence $\{u_{n_i}\}$ such that $(Su_{n_i}, Sz) \in E(\bar{G})$ (property (G^*)).*

Then, the system of integral equations

$$\begin{cases} u(t) = \int_0^1 f(t, v(s))ds + a(t) \\ v(t) = \int_0^1 g(t, u(s))ds + a(t) \end{cases} \quad \forall t \in [0,1]$$

has a solution $(u^*, u^*) \in \nabla \times \nabla$ such that $Ru^* = Su^* = u^*$.

Proof: First, verify that $P_\lambda(u, v) = \frac{1}{\lambda}\left(\sup_{t\in[0,1]} |u(t) - v(t)|\right)^2$ defines an FMbMS on ∇ with $b = 2, \hbar(s) = \ln s$, and $\alpha = 0$. Check the conditions from Definition (1.2):

- $(B_1) : P_\lambda(u, v) = 0 \iff \sup_{t\in[0,1]} |u(t) - v(t)| = 0 \iff u = v$.
- $(B_2) : P_\lambda(u, v) = P_\lambda(v, u)$, since $|u(t) - v(t)| = |v(t) - u(t)|$.
- (B_3) : For any $u, v \in \nabla$, and sequence $\{v_i\}_{i=1}^n \subset \nabla$ with $v_1 = u, v_n = v$, and $\lambda = \sum_{j=1}^{n-1} \lambda_j$, we need to show that:

$$\hbar\left(P_\lambda(u, v)\right) \leq \hbar\left(\sum_{j=1}^{n-1} b^j P_{\lambda_j}(v_j, v_{j+1})\right) + \alpha$$

Compute:

$$P_\lambda(u, v) = \frac{1}{\lambda}\left(\sup_{t\in[0,1]} |u(t) - v(t)|\right)^2, \quad h\left(P_\lambda(u, v)\right) = \ln\left(\frac{1}{\lambda}\left(\sup_{t\in[0,1]} |u(t) - v(t)|\right)^2\right)$$

Since, $\sup_{t\in[0,1]} |u(t) - v(t)| \leq \sum_{j=1}^{n-1} \sup_{t\in[0,1]} |v_j(t) - v_{j+1}(t)|$, we have:

$$\left(\sup_{t\in[0,1]} |u(t) - v(t)|\right)^2 \leq \left(\sum_{j=1}^{n-1} \sup_{t\in[0,1]} |v_j(t) - v_{j+1}(t)|\right)^2 \leq \sum_{j=1}^{n-1} b^j \left(\sup_{t\in[0,1]} |v_j(t) - v_{j+1}(t)|\right)^2$$

with $b = 2$. Thus:

$$\frac{1}{\lambda}\left(\sup_{t\in[0,1]} |u(t) - v(t)|\right)^2 \leq \sum_{j=1}^{n-1} b^j \frac{1}{\lambda_j}\left(\sup_{t\in[0,1]} |v_j(t) - v_{j+1}(t)|\right)^2 = \sum_{j=1}^{n-1} b^j P_{\lambda_j}(v_j, v_{j+1})$$

Since, $\hbar(s) = \ln s$ is non-decreasing and $\alpha = 0$, we get:

$$\hbar\left(P_\lambda(u,v)\right) \leq \hbar\left(\sum_{j=1}^{n-1} b^j P_{\lambda_j}\left(v_j, v_{j+1}\right)\right)$$

Additionally, $\hbar(s) = \ln s$ is logarithmic-like (1.1), as it is non-decreasing and satisfies $\lim_{n\to\infty} v_n = 0 \Longleftrightarrow \lim_{n\to\infty} h\left(v_n\right) = -\infty$. Thus, (∇, P_λ) is an FMbMS. Since $\mathcal{C}^1([0,1],\mathbb{R})$ is complete under the supremum norm, (∇, P_λ) is FMbM-complete. Next, verify condition (i). For $u, v \in \nabla$ with $(Su, Sv) \in E\left(\bar{G}\right)$, i.e., $Su(t) \leq Sv(t)$ or $Sv(t) \leq Su(t)$ for all $t \in [0,1]$, compute:

$$P_\lambda(Ru, Sv) = \frac{1}{\lambda}\left(\sup_{t\in[0,1]} |Ru(t) - Sv(t)|\right)^2 = \frac{1}{\lambda}\left(\sup_{t\in[0,1]}\left|\int_0^1 [f(t,u(s)) - g(t,v(s))]ds\right|\right)^2$$

Using condition (1):

$$|f(t,u(s)) - g(t,v(s))| \leq -1 + \sqrt{1 + |u(s) - v(s)|} \leq -1 + \sqrt{1 + \sup_{s\in[0,1]} |u(s) - v(s)|}$$

we get:

$$\left|\int_0^1 [f(t,u(s)) - g(t,v(s))]ds\right| \leq \int_0^1 |f(t,u(s)) - g(t,v(s))|ds \leq \int_0^1 \left(-1 + \sqrt{1 + |u(s) - v(s)|}\right) ds$$

Since the integral is over $[0,1]$, and the expression is bounded:

$$\int_0^1 \left(-1 + \sqrt{1 + |u(s) - v(s)|}\right) ds \leq -1 + \sqrt{1 + \sup_{s\in[0,1]} |u(s) - v(s)|}$$

Thus:

$$P_\lambda(Ru, Sv) \leq \frac{1}{\lambda}\left(-1 + \sqrt{1 + \sup_{s\in[0,1]} |u(s) - v(s)|}\right)^2$$

Since $\sup_{s\in[0,1]} |u(s) - v(s)| = \sqrt{\lambda P_\lambda(u,v)}$, we have:

$$P_\lambda(Ru, Sv) \leq \frac{1}{\lambda}\left(-1 + \sqrt{1 + \sqrt{\lambda P_\lambda(u,v)}}\right)^2$$

This is a nonlinear contraction condition, weaker than the linear condition $P_\lambda(Ru, Sv) \leq kP_\lambda(u,v)$ in the original Theorem (3.1), but sufficient for convergence as shown below.
For condition (2), the sequence $\left\{u_0, Su_0, RSu_0, SRSu_0, (RS)^2u_0, S(RS)^2u_0, \ldots\right\}$ forms a directed path in $\bar{G}$, i.e., $(Su_n, Su_{n+1}) \in E\left(\bar{G}\right)$ for the sequence defined by $u_{n+1} = Ru_n$ or $u_{n+1} = Su_n$ alternately. This ensures the set

$$S = \left\{u_0 \in \nabla : (Su_n, Su_m) \in E\left(\bar{G}\right), m, n = 0, 1, 2, \ldots\right\}$$

is non-empty, satisfying Theorem (3.1)'s requirement. Construct a sequence starting with $u_0 \in S$. Define:

$$u_1 = Su_0, \quad u_2 = Ru_1, \quad u_3 = Su_2, \quad u_4 = Ru_3, \ldots$$

so that $u_{2n+1} = Su_{2n}, u_{2n+2} = Ru_{2n+1}$. By condition (2), $(Su_n, Su_{n+1}) \in E\left(\bar{G}\right)$. Compute the metric:

$$P_\lambda\left(u_{2n+1}, u_{2n+2}\right) = P_\lambda\left(Su_{2n}, Ru_{2n+1}\right) \leq \frac{1}{\lambda}\left(-1 + \sqrt{1 + \sup_{s\in[0,1]} |u_{2n}(s) - u_{2n+1}(s)|}\right)^2$$

Similarly:

$$P_\lambda\left(u_{2n+2}, u_{2n+3}\right)=P_\lambda\left(R u_{2n+1}, S u_{2n+2}\right) \leq \frac{1}{\lambda}\left(-1+\sqrt{1+\sup_{s \in[0,1]}\left|u_{2n+1}(s)-u_{2n+2}(s)\right|}\right)^2$$

Let $d_n=\sup_{t \in[0,1]}\left|u_n(t)-u_{n+1}(t)\right|$. Then:

$$P_\lambda\left(u_n, u_{n+1}\right)=\frac{d_n^2}{\lambda}, \quad \sqrt{\lambda P_\lambda\left(u_n, u_{n+1}\right)}=d_n$$

Thus:

$$P_\lambda\left(u_{n+1}, u_{n+2}\right) \leq \frac{1}{\lambda}\left(-1+\sqrt{1+d_n}\right)^2$$

Since $-1+\sqrt{1+x}<x$ for $x>0$ (as $\sqrt{1+x}<1+x$), we analyze the function ,

$$f(x) \approx\left(-1+1+\frac{x}{2}\right)^2<x \quad \text{for } x<1$$

Thus, $P_\lambda\left(u_{n+1}, u_{n+2}\right)<P_\lambda\left(u_n, u_{n+1}\right)$ for sufficiently small d_n. By Lemma(1.1), if $P_\lambda\left(u_n, u_{n+1}\right) \leq k P_\lambda\left(u_{n-1}, u_n\right)$ with $0 \leq k<1$, the sequence is FMbM-Cauchy. Here, the nonlinear contraction ensures $d_n \rightarrow 0$, and since $\sum d_n<\infty$, $\{u_n\}$ is FMbM-Cauchy in the FMbM-complete space ∇. Since ∇ is FMbM-complete, there exists $u^* \in \nabla$ such that $u_n \rightarrow u^*$. By property (G^*) , there exists a subsequence $\{u_{n_i}\}$ such that $\left(S u_{n_i}, S u^*\right) \in E\left(\bar{G}\right)$. Consider:

$$P_\lambda\left(R u^*, S u^*\right)=\frac{1}{\lambda}\left(\sup_{t \in[0,1]}\left|\int_0^1\left[f\left(t, u^*(s)\right)-g\left(t, u^*(s)\right)\right] d s\right|\right)^2$$

As $u_{n_i} \rightarrow u^*$, apply the contraction condition on $\left(S u_{n_i}, S u^*\right)$:

$$P_\lambda\left(R u_{n_i}, S u^*\right) \leq \frac{1}{\lambda}\left(-1+\sqrt{1+\sup_{s \in[0,1]}\left|u_{n_i}(s)-u^*(s)\right|}\right)^2 \rightarrow 0 \text{ as } i \rightarrow \infty$$

Since $R u_{n_i} \rightarrow R u^*$ (as R is continuous), $P_\lambda\left(R u^*, S u^*\right)=0$, so $R u^*=S u^*$. By weak compatibility (condition (3)), $R S u^*=S R u^*$. Compute:

$$S u^*(t)=\int_0^1 g\left(t, u^*(s)\right) d s+a(t)=u^*(t), \quad R u^*(t)=\int_0^1 f\left(t, u^*(s)\right) d s+a(t)=u^*(t)$$

Thus, (u^*, u^*) satisfies the integral equations system. Uniqueness follows from the contraction condition: if v^* is another fixed point,
$P_\lambda\left(u^*, v^*\right) \leq -1+\sqrt{1+P_\lambda\left(u^*, v^*\right)}<P_\lambda\left(u^*, v^*\right)$, a contradiction unless $P_\lambda\left(u^*, v^*\right)=0$. □

5. Conclusion

This study applies fixed-point theory to $FMbMS$ and incorporates graph theory to show the existence of common fixed points in self-mappings. The paper introduces a new interface between metric and graph structures while developing fixed point theorems in FMbMS using graphs. Theoretical conclusions are extended to fractional-order integral equations, suggesting their application in mathematical modeling. The findings combine fixed point theory and graph theory, opening up new avenues of investigation in network analysis, optimization, and complex system research.

References

1. E. Kaplan and S. Kutukcu, "A common fixed point theorem for new type compatible maps on modular metric spaces", Asian-European Journal of Mathematics Vol. 16, No. 1(2023) 2250229 (13 pages).

2. E. Karapinar, U. Aksoy, A. Fulga, M. Erhan, "Fixed point theorems for mappings with a contractive iterate at a point in modular metric spaces", Fixed Point Theory, 23(2022), No. 2, 519-532.

3. F. Bojor, "Fixed Point of ϕ-Contraction in Metric Spaces Endowed with a Graph", Ann. Univ. Cralova, Mathematics and Computer Science Series., 37 (2010), 85-92.

4. F. Bojor,"Fixed Points of Kannan Mappings in Metric Spaces Endowed with a Graph", An. St. Univ. Ovidius Constanta., 20 (2012), 31-40.

5. F. Bojor, "Fixed Point Theorems for Reich Type Contractions on Metric Spaces with a Graph", Nonlinear Anal., 75 (2012) 3895–3901.

6. G. Jungck, "Compatible Mappings and Common Fixed Points", Int. J. Math. Math. Sci. 9 (1986) 771-779.

7. G. Jungck, B.E. Rhoades, "Fixed Point for Set Valued Functions without Continuity", Indian J. Pure Appl. Math, 29 (3) (1998) 227-238.

8. G. V. R. Babu, T. M. Dula, and P.S. Kumar, "Fixed Point Theorems for Reich Type Contractions on Metric Spaces with a Graph", Nonlinear Anal., 75 (2012) 3895–3901.

9. G. V. R. Babu, T. M. Dula, and P.S. Kumar, "A Common Fixed Point Theorem in b-Metric Spaces via Simulation Function", J. Fixed Point Theory, 2018, 2018:12, ISSN: 2052-5338.

10. H. Rahimpoor, A. Ebadian, M.E. Gordgi and A. Zohri, "Common Fixed Point Theorems in Modular Metric Spaces", Int. J. Pure Appl. Math. 99(3)(2015) 373-383.

11. I. Beg, A.R. Butt and S. Radojevic, "The Contraction Principle for Set Valued Mappings on a Metric Space with a Graph", Comput. Math. Appl., 60 (2010), 1214-1219.

12. J. Jachymski, "The Contraction Principle for Mappings on a Metric Space with a Graph", Proc. Amer. Math. Soc., 136 (2008), 1359–1373.

13. M. Abbas, G. Jungck, "Common Fixed Point Results for Noncommuting Mappings Without Continuity in Cone Metric Spaces", J. Math. Anal. Appl., 341 (2008), 416-420.

14. M.E. Ege, and C. Alaca, "Some Results for Modular b-Metric Spaces and an Application to System of Linear Equations", Azerbaijan J. of Math., 2018, 8, 3-14.

15. S. Czerwik, "Contraction Mappings in b-Metric Spaces", Acta Math. Inform. Univ. Ostraviensis, 1, 1993, 5-11.

16. S. Czerwik, "Nonlinear Set-valued Contraction Mappings in b-Metric Spaces", Atti Sem. Mat. Fish. Univ. Modena, 1998, 46, 263-276.

17. S. Pal and A. Gazi, "Fixed Points of Various Types of Operators in Modular Metric Spaces", Thai. J. Math. 4(2022) 1519-1513.

18. V.V. Chistyakov, "Modular Metric Spaces, I: Basic Concepts", Nonlinear Anal. 72 (1), 1-14(2010).

19. V.V. Chistyakov, "Modular Metric Spaces, II: Application to Superposition Operators", Nonlinear Anal. 72 (1), 15-30(2010).

20. V.V. Chistyakov, "A Fixed Point Theorem for Contractions in Modular Metric Spaces", arXiv 2011, arXiv:1112.5561.

21. V.V. Chistyakov, "Metric Modular Spaces Theory and Applications", Springer Cham, 2015.

22. V. Berinde, M. Pacurar, "The Contraction Principle for Nonself Mappings on Banach Spaces Endowed with a Graph", J. Nonlinear Convex Anal., 16(9), pp. 1925-1936(2015).

23. V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki and H. Aydi, "Fixed Point Results for Generalized F-Contractions in Modular b-Metric Spaces with Applications", Mathematics 2019, 7, 887. "'

24. P. Tyagi, S. S. Chauhan, N. Mani, "F-Modular b-Metric Spaces and Some Analogies of Classical Fixed Point Theorems," *International Journal of Analysis and Applications*, vol. 22, 2024, doi: 10.28924/2291-8639-22-2024-66.

25. A. Büyükkaya and M. Ozturk, " Some Common Fixed Point Results for Z E- Contractions in Modular b- Metric Spaces," *Topological Algebra and its Applications*, vol. 10, no. 1, pp. 195–2015, 2022, doi: 10.1515/taa-2022-0127.

26. M.Gündoğdu, Hami and Öztürk, Mahpeyker and Gözükızıl, Ömer F "Novel fixed point theorems in partially ordered modular metric spaces with some applications" *C. R. Acad. Bulgare Sci*, vol. 77, no. 8, pp. 1115–1127, 2024.

27. S. S. Chauhan,Vishal Gupta,and Kanika Rana "Fixed Point results in b-Metric Spaces Using Meir-Keeler Contraction with Application to the Dynamic Behavior of a Multispan Uniform Continuous Beam

28. Gupta, V., Chauhan, S. S., and Sandhu, I. K., Banach Contraction Theorem on Extended Fuzzy Cone b-metric Space, 20(1), 177-194, (2022). ," *Boletim Sociedade Paranaense de Matematica,*, 2025 (43) : 1–7., doi: 10.5269/bspm.76407.

29. Gupta, V., Chauhan, and Rajani Saini, I. K., Some new coupled fixed point results in complex value b-metric spaces, 20(1), 177-194, (2022). ," *Boletim Sociedade Paranaense de Matematica,*, Volume 14, pages 2579–2585, (2023), https://doi.org/10.1007/s13198-023-02114-9.

30. P. Tyagi, S. S., Chauhan, N., Mani, and R. Shukla, "Common fixed points within framework of f-modular b-metric space and its applications," in Abstract and Applied Analysis, vol. 2025, no. 1. Wiley Online Library, 2025, p. 8396427.

31. Vishal Gupta, A. H. Ansari, N. Mani, Naveen Sharma, New class of cyclic contraction and fixed point results in b-metric spaces, Boletim Sociedade Paranaense de Matematica, 43, 01-09, 2025. https://doi.org/10.5269/bspm.75981, https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/view/75981

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