Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.77797

A Study on the Modified Fuzzy Normed Space and the Fixed Point Theorem with Application

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ABSTRACT: In this paper, we introduce a new space, the modified fuzzy normed space, by extending the concept of fuzzy normed space to include the basic concepts of normed spaces. We define two types of contractions on this space: fuzzy $\mathfrak{K}$-type contraction and weak fuzzy $\mathfrak{K}$-type contraction. We introduce a fixed-point theorem in this space. Applying our results, we prove the existence of a solution to a nonlinear integral equation using the obtained results.

Key Words: Modified fuzzy normed space, fuzzy (weak) $\mathfrak{K}$-type contraction, fixed point theorem.

Contents

1. Introduction

In modern mathematics, fixed point theory plays an important role, with its wide applications in dynamical systems, integral equations, optimization, cryptography, and many other fields. The famous Banach theorem states that in certain metric spaces, under contractionary conditions, the existence and uniqueness of a fixed point are guaranteed [1]. In the years since the introduction of Banach's theorem, this theory has undergone numerous refinements and studies to provide more abstract frameworks, including fuzzy theory, which studies data imprecision and uncertainty, as described in some sources [2,3,4,5,6,7,8,9]. As a powerful extension of regular normed spaces, fuzzy normed spaces have been found, allowing the study of fixed point properties in irregular spaces [10,11,12,13,14,15,16,17,18].

Our goal in this study is to provide more flexible analytical structures by defining a modified fuzzy normed space. In this approach, we introduce two innovative types of contractions: fuzzy $\mathfrak{K}$-type contraction and weak fuzzy $\mathfrak{K}$-type contraction. To extend the classical results, we introduce the fixed point theory in this new space to include this new and expanded context. This work includes examples and an application of solving a nonlinear integral equation to illustrate the importance of the theoretical results. This makes our work not just about formulating new spaces and mappings.

2. Preliminaries

Definition 2.1 *[19]. Let χ be a vector space over a field K. A norm on χ is a map $\mathbb{N}\colon \chi \to [0, \infty)$ that satisfies the following conditions for all $a,\ b \in \chi,\ \rho \in\ K$:*

$(*_1)$ $\mathbb{N}\ (a) = 0\ \leftrightarrow a = 0$;

$(*_2)$ $\mathbb{N}(\rho a) = |\rho|\mathbb{N}(a)$;

$(*_3)$ $\mathbb{N}(a + b) \leq \mathbb{N}(a)+\mathbb{N}(b)$.

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2010 *Mathematics Subject Classification:* 46S40.
Submitted July 10, 2025. Published September 30, 2025

A normed space is a pair $(\mathbb{N}, \chi)$.

Definition 2.2 *[19]. Let I=[0, 1], a binary operation* $\# : I \times I \to I$ *is a t-norm if for all* $\rho, \sigma, \vartheta, \partial \in I$ *satisfies the following properties:*

$(\wedge_1)$ *Is associative and commutative;*

$(\wedge_2)$ $\rho \# 1 = \rho$;

$(\wedge_3)$ $\rho \# \sigma \leq \vartheta \# \partial$ *whenever* $\rho \leq \vartheta$ *and* $\sigma \leq \partial$.

Definition 2.3 *[19]. Let* χ *be a non-empty set,* $*$ *be a continuous t-norm on I. A function* $\aleph$*:* $\chi \times (0, \infty) \to [0,\ 1]$ *is a fuzzy norm function if satisfied the following conditions for all* $a,\ b \in \chi, s > 0, \rho \in\ K$ *;*

$(*_1)$ $\aleph(a, s) > 0$;

$(*_2)$ $\aleph(a, s) = 1 \leftrightarrow a = 0$;

$(*_3)$ $\aleph(\rho a, s) = \aleph\left(a, \frac{s}{\rho}\right)$;

$(*_4)$ $\aleph(a + b, s + t) \geq \aleph(a, s) * \aleph(b,\ t)$;

$(*_5)$ $\aleph(a, .) : (0,\ \infty) \to [0, 1]$ *is continuous;*

$(*_6)$ $\lim_{s\to\infty} \aleph(a, s) = 1$.

Then we say $(\aleph, \chi, *)$ *is a fuzzy normed space.*

3. Main results

Definition 3.1 *Let* $\mathcal{N} : \chi \to [0,\ \infty]$ *be a given mapping and let* χ *be a nonempty set. Defined the set* $\Delta(\mathcal{N},\ \chi, \mathfrak{k}\ ) = \{\{\mathfrak{k}_n\} \subset \chi,\ \lim_{n\to\infty}\mathcal{N}(\mathfrak{k}_n - \mathfrak{k}) = 0\}$ *, then* $\mathcal{N}$ *is a modified normed (briefly* $\mathfrak{S}\mathcal{N}$*)on* χ *if it meets the following requirements:*
$\mathfrak{S}\mathcal{N}_1 : \mathcal{N}(\mathfrak{k}) = 0 \Longrightarrow \mathfrak{k} = 0\ \forall \mathfrak{k} \in \chi$
$\mathfrak{S}\mathcal{N}_2$ *: Has* $a > 0$ *such that* $\mathcal{N}(\mathfrak{k}) \leq a \lim_{n\to\infty} sup \mathcal{N}\ (\mathfrak{k}_n)$ *if* $\mathfrak{k} \in \chi$ *and* $\{\mathfrak{k}_n\} \in \Delta(\ \mathcal{N},\ \chi, \mathfrak{k}\ )$*, then the pair* $(\ \chi, \mathcal{N})$ *is a modified normed space (briefly* $\mathfrak{M}\mathcal{N}\mathfrak{H}$*).*

Example 3.1 Let $\mathcal{N} : \chi \to [0,\ \infty]$ be the mapping described by $\begin{cases} \mathcal{N}(\mathfrak{k} -\ \mathfrak{h}) = \mathfrak{k} +\ \mathfrak{h}\ \ if\ \mathfrak{k},\ \mathfrak{h} \neq 0 \\ \mathcal{N}(0,\ \mathfrak{h}) = \mathcal{N}(\ \mathfrak{h},\ 0) = \frac{\mathfrak{h}}{2}\ \ \forall\ \ \mathfrak{h}\ \in \chi \end{cases}$ and let $\chi = [0,\ 1]$.Then $(\chi, \mathcal{N})$is not normed space, but it is a $\mathfrak{W}\mathcal{N}\mathfrak{H}$.

Definition 3.2 *Examine a mapping* $\mathfrak{Y} : \chi \times (0,\ \infty) \to [0,\ 1]$ *and a nonempty set* χ*. A set* $\Delta(\ \mathfrak{Y},\ \chi, \mathfrak{k}\ ) = \{\{\mathfrak{k}_n\} \subset \chi,\ \lim_{n\to\infty} \mathfrak{Y}(\mathfrak{k}_n - \mathfrak{k},\ s) = 1\ \forall s > 0\}\ \ \ \forall\ \mathfrak{k} \in \chi$ *then* $\mathfrak{Y}$*is considered a modified fuzzy normed space (briefly* $\mathfrak{W}\mathfrak{Y}\mathfrak{H}$*) if, for any* $\mathfrak{k} \in \chi$ *and* $s > 0$*, it satisfies the following criteria:*

$\mathfrak{W}\mathfrak{Y}\mathfrak{H}_1 : \mathfrak{Y}(\mathfrak{k},\ s) > 0$

$\mathfrak{W}\mathfrak{Y}\mathfrak{H}_2 : \mathfrak{Y}(\mathfrak{k},\ s) = 1 \Longrightarrow \mathfrak{k} = 0$

$\mathfrak{W}\mathfrak{Y}\mathfrak{H}_3 :$ *If* $a \geq 1$*, if* $\{\mathfrak{k}_n\} \in \Delta(\mathfrak{Y},\ \chi, \mathfrak{k}\ )$*, then* $\mathfrak{Y}(\mathfrak{k},\ s) \geq \lim_{n\to\infty} \sup \mathfrak{Y}\left(\mathfrak{k}_n - \mathfrak{k},\ \frac{s}{a}\right)$

$\mathfrak{W}\mathfrak{Y}\mathfrak{H}_4 : \lim_{s\to\infty} \mathfrak{Y}(\mathfrak{k},\ s) = 1$ *and* $\mathfrak{Y}(\mathfrak{k},\ .) : (0,\ \infty) \to [0,\ 1]$ *is continuous.*

Then, a $\mathfrak{W}\mathfrak{Y}\mathfrak{H}$ is defined as $(\mathfrak{Y},\ \chi,\ \#)$.

Example 3.2 Let $(\chi,\ \mathcal{N})$ be a $\mathfrak{MNH}$. Establish a mapping $\mathfrak{Y}:\chi\times(0,\ \infty)\to[0,1]$ by

$$\Delta(\mathfrak{Y},\chi,\mathfrak{k}\ )=\{\{\mathfrak{k}_n\}\subset\chi,\ \lim_{n\to\infty}\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{k},\ s\right)=1\}\ \ \forall\ \mathfrak{k}\in\chi \text{ and } s>0 \text{ and } \mathfrak{Y}\left(\mathfrak{k},\ s\right)=e^{-\frac{\mathcal{N}(\mathfrak{k})}{s}} \tag{3.1}$$

The $\mathfrak{MYH}$, $(\mathfrak{Y},\ \chi,\ \#)$is then used, and the t-norm "#" is interpreted as the product norm.

Solution: We only demonstrate that $\mathfrak{Y}$ meets Definition 3.2 property $\mathfrak{WYH}_3$ if $\mathfrak{k}\in\chi$ and $\{\mathfrak{k}_n\}\in\Delta(\mathfrak{Y},\chi,\mathfrak{k}\ )$. Given that $\mathcal{N}$ is a modified norm, it is evident from equation (3.1) and condition $\mathfrak{MN}_2$ of Definition 3.1 that $\{\mathfrak{k}_n\}$ also belongs to $\Delta(\ \mathfrak{Y},\chi,\mathfrak{k}\ )$. Consequently,

$$\mathfrak{Y}\left(\mathfrak{k},\ s\right)=e^{-\frac{\mathcal{N}(\mathfrak{k})}{s}}\geq e^{-\frac{a\lim_{n\to\infty}sup\mathcal{N}(\mathfrak{k}_n)}{s}}=e^{-\frac{\lim_{n\to\infty}sup\mathcal{N}(\mathfrak{k}_n)}{\frac{s}{a}}}=\lim_{n\to\infty}supe^{-\frac{\mathcal{N}(\mathfrak{k}_n)}{\frac{s}{a}}}=\lim_{n\to\infty}sup\mathfrak{Y}\left(\mathfrak{k}_n,\ \frac{s}{a}\right),$$

Hence $\mathfrak{Y}\left(\mathfrak{k},\ s\right)\geq\lim_{n\to\infty}sup\ \mathfrak{Y}\left(\mathfrak{k}_n,\ \frac{s}{a}\right)$.

Proposition 3.1 $\mathfrak{Y}\left(0,\ s\right)=1$ *if and only if* $\Delta(\ \mathfrak{Y},\ \chi,\mathfrak{k})$ *is not empty set.*

Proof: For every $s\geq0$, there exists a sequence $\{\mathfrak{k}_n\}\subset\chi$ such that $\lim_{n\to\infty}\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{k},\ s\right)=1$ if $\Delta(\ \mathfrak{Y},\chi,\mathfrak{k}\ )\neq\emptyset$. By applying Definition 3.2 property $\mathfrak{WYH}_3$, we obtain $\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},\ s\right)=\mathfrak{Y}\left(0,\ s\right)\geq\lim_{n\to\infty}sup\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{k},\ \frac{s}{a}\right)$, then $\Delta(\ \mathfrak{Y},\ \chi,\mathfrak{k}\ )\neq\emptyset$. □

Remark 3.1 It is important to note that the class of fuzzy normed spaces, as illustrated by the example that follows, is smaller than the class of $\mathfrak{WYH}'s$.

Proposition 3.2 *Every fuzzy normed space*$(\aleph,\ \chi,\ \#)$ *is a modified fuzzy normed space.*

Proof: We only prove that the property $\mathfrak{MYH}_3$ of Definition 3.2. Now for $\mathfrak{k}\in\chi$, $\{\mathfrak{k}_n\}\in\Delta(\ \mathfrak{Y},\chi,\mathfrak{k}\ )$ then by using triangular inequality $\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{h},\ s\right)\geq\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{k}\ ,\frac{s}{2}\right)\#\mathfrak{Y}\left(\mathfrak{k}_n-\ \mathfrak{h},\ \frac{s}{2}\right)=\ 1\#\lim_{n\to\infty}sup\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{h},\ \frac{s}{2}\right)=\lim_{n\to\infty}sup\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{h},\ \frac{s}{a}\right)$ where $a=2$. □

Example 3.3 Assume that $\chi=[0,\ 1]$. Describe a set $\Delta(\ \mathfrak{Y},\chi,\mathfrak{k}\ )=\{\{\mathfrak{k}_n\}\subset\chi,\ \lim_{n\to\infty}\mathfrak{Y}\left(\mathfrak{k}_n-\mathfrak{k},\ s\right)=1\}$, $\forall s>0$,and $\mathfrak{k}\in\chi$, where $\mathfrak{Y}:\chi\times(0,\ \infty)\to[0,1]$.

Is defined as follows:

$$\begin{cases}\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{h},s\right)=e^{-\frac{\mathfrak{k},+\mathfrak{h}}{s}} & if\quad \mathfrak{k},\ \mathfrak{h}\neq0\\ \mathfrak{Y}\left(\ \mathfrak{h},\ s\right)=e^{-\frac{\mathfrak{h}}{2s}} & \forall\ \ \mathfrak{h}\ \in\chi\end{cases}$$
, and the t-norm "#" is interpreted as the product norm.

Then, $(\mathfrak{Y},\ \chi,\ \#)$ is the $\mathfrak{MY}$ but not fuzzy normed space.
Given Proposition 3.3, we must only confirm $\mathfrak{WYH}_3$for elements $\mathfrak{k}$ in χ such that $\mathfrak{Y}\left(\mathfrak{k}-\alpha,\ s\right)=1$, indicating that $\mathfrak{k}=\alpha$. Consider the sequence $\{\mathfrak{k}_n\}\subset\chi$ such that $\lim_{n\to\infty}\mathfrak{Y}\left(\mathfrak{k}_n-0,\ s\right)=1\ \}$. For every $n\in\mathrm{N}$ and every $\alpha\in\chi$, we have:

$$\mathfrak{Y}\left(\mathfrak{k}_n-\alpha,s\right)=\begin{cases}e^{-\frac{\mathfrak{k}_n+\alpha}{s}} & if\ \ \mathfrak{k}_n,\alpha\neq0\\ e^{-\frac{\alpha}{2s}} & if\ \ \mathfrak{k}_n=0\end{cases}$$
. Given that $\mathfrak{k}_n+\alpha\geq\alpha,\ s>0$, we have

$$\frac{\alpha}{2\mathrm{s}}<\frac{\mathfrak{k}_\mathrm{n}+\alpha}{\mathrm{s}}$$

$-\frac{\alpha}{2s}>-\frac{\mathfrak{k}_n+\alpha}{s}$, $e^{-\frac{\alpha}{2s}}>e^{-\frac{\mathfrak{k}_n+\alpha}{s}}$, then, $\mathfrak{Y}\left(0-\alpha,s\right)=e^{-\frac{\alpha}{2s}}\geq\lim_{n\to\infty}\mathrm{sup}e^{-\frac{\mathfrak{k}_n+\alpha}{s}}=\lim_{n\to\infty}\mathrm{sup}\mathfrak{Y}\left(\mathfrak{k}_n-\alpha,\ s\right)$, which suggests that $\mathfrak{Y}\left(\alpha,\ s\right)=\mathfrak{Y}\left(0-\alpha,\ s\right)\geq\lim_{n\to\infty}\mathrm{sup}\mathfrak{Y}\left(\mathfrak{k}_n-\alpha,\frac{s}{a}\right)$, where $a=1$.Because the triangle inequality does not hold, $(\mathfrak{Y},\ \chi,\ \#)$ is a $\mathfrak{WYH}$ but not a fuzzy normed space. Given that $\mathfrak{Y}\left(\mathfrak{k}-\alpha,s\right)=e^{-\frac{\mathfrak{k}+\alpha}{s}}$ and $\mathfrak{Y}\left(\mathfrak{k},\ \frac{s}{2}\right)\#\mathfrak{Y}\left(\alpha,\ \frac{s}{2}\right)=e^{-\frac{\mathfrak{k}}{4s}}\#e^{-\frac{\alpha}{4s}}$. So

$$\mathfrak{Y}\left(\mathfrak{k}-\alpha,s\right)\leq\ \mathfrak{Y}\left(\mathfrak{k},\ \frac{s}{2}\right)\#\mathfrak{Y}\left(\alpha,\ \frac{s}{2}\right)$$

Definition 3.3 *Assume that a* $\mathfrak{WYH}$ *is* $(\mathfrak{Y},\ \chi,\ \#)$. *For any* $s>0$, $\{\mathfrak{k}_n\}$*in* χ *and* $\{\mathfrak{k}_n\}\in\Delta(\ \mathfrak{Y},\ \chi,\mathfrak{k}\ )$, *then a sequence* $\{\mathfrak{k}_n\}$ *is* $\mathfrak{Y}$*-convergent sequence if* $\lim_{n\to\infty}\mathfrak{Y}(\mathfrak{k}_n-\mathfrak{k},\ s)=1$.

Definition 3.4 *Assume that a* $\mathfrak{WYH}$ *is* $(\mathfrak{Y},\ \chi,\ \#)$. *For any* $\mathfrak{k}\in\chi, s>0$, *a sequence* $\{\mathfrak{k}_n\}$ *in* χ *is considered a* $\mathfrak{Y}$*-Cauchy sequence if* $\lim_{n,p\to\infty}\mathfrak{Y}(\mathfrak{k}_n-\mathfrak{k}_{n+p},\ s)=1$. *And* $\mathfrak{Y}$*-complete* $\mathfrak{WYH}$ *where all* $\mathfrak{Y}$ *-Cauchy sequences are* $\mathfrak{Y}$*-convergent.*

Remark 3.2 A $\mathfrak{Y}$-convergent sequence might not be a $\mathfrak{Y}$ -Cauchy sequence in $\mathfrak{SYH}$.

Example 3.4 Assume that $\chi=\mathbb{R}^+\cup\{0\}$. For each $\mathfrak{k}\in\chi$ and $s>0$, define a set $\Delta(\ \mathfrak{Y},\ \chi,\mathfrak{k}\ )=\{\{\mathfrak{k}_n\}\subset\chi,\ \lim_{n\to\infty}\mathfrak{Y}(\mathfrak{k}_n-\mathfrak{k},\ s)=1\ \}$, where $\mathfrak{Y}:\chi\times(0,\ \infty)\to[0,1]$ is described by $(\mathfrak{Y},\ \chi,\ \#)$ is a $\mathfrak{SYH}$, where $\mathfrak{Y}(\mathfrak{k}-\alpha,s)=\begin{cases} e^{-\frac{\mathfrak{k}+\alpha}{s}} & \text{if at least one of } \mathfrak{k} \text{ or } \alpha=0\\ e^{-\frac{1+\mathfrak{k}+\alpha}{s}} & o.w.\end{cases}$. Take the sequence $\{\mathfrak{k}_n\}$ as follows:

$\mathfrak{k}_n=\frac{1}{n},\ \forall\ n\in N.$ $\lim_{n\to\infty}\mathfrak{Y}(\mathfrak{k}_n,\ s)=\lim_{n\to\infty}\mathfrak{Y}(\mathfrak{k}_n-0,\ s)=\lim_{n\to\infty}e^{-\frac{\mathfrak{k}_n}{s}}$ $=1$. $\{\mathfrak{k}_n\}$ $\mathfrak{Y}$-converges to 0. Currently, $\lim_{n,p\to\infty}\mathfrak{Y}(\mathfrak{k}_n-\mathfrak{k}_{n+p},\ s)=\lim_{n,p\to\infty}e^{-\frac{1+\mathfrak{k}_n+\mathfrak{k}_{n+p}}{s}}$

$=\lim_{n,p\to\infty}e^{-\frac{1+\frac{1}{n}+\frac{1}{n+p}}{s}}$ $=\lim_{n,p\to\infty}e^{-\frac{1}{s}}.\ e^{-\frac{1}{ns}}e^{-\frac{1}{(n+p)s}}=e^{-\frac{1}{s}}.1.1\neq1$, is therefore not a $\mathfrak{Y}$ -Cauchy sequence.

Proposition 3.3 *Let* $\{\mathfrak{k}_n\}$ *in* χ, $\mathfrak{k}\in\chi$ *and* $(\mathfrak{Y},\ \chi,\#)$ *is a* $\mathfrak{WYH}$. $\mathfrak{k}=\alpha$ *if* $\{\mathfrak{k}_n\}$ $\mathfrak{Y}$ *-converges to* $\mathfrak{k}$ *and* $\{\mathfrak{k}_n\}$ $\mathfrak{Y}$ *-converges to* α.

Proof: Using the property $\mathfrak{WYH}_3$ of Definition 3.2, we obtain

$$\mathfrak{Y}(\mathfrak{k}-\alpha,s)\geq\lim_{n\to\infty}sup\mathfrak{Y}\left(\mathfrak{k}_n-\alpha,\ \frac{s}{a}\right)=1\ .$$

□

Definition 3.5 *Assume that a* $\mathfrak{WYH}$ *is* $(\mathfrak{Y},\ \chi,\ \#)$. *A self-mapping* $\mathcal{G}:\ \chi\to\ \chi$ *is fuzzy* $\mathfrak{K}$*-type contraction (briefly* $\mathfrak{FK}-\mathfrak{C}$*) if* $\mathfrak{Y}(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\alpha),\mathfrak{K}s)\geq\ \mathfrak{Y}(\mathfrak{k}-\alpha,s)$, $\forall s>0$ *for any* $\mathfrak{k},\alpha\in\chi$ *and for some* $\mathfrak{K}\in(0,\ 1)$.

Proposition 3.4 *Let* $\mathcal{G}$ *be a* $\mathfrak{FK}-\mathfrak{C}$ *and* $(\mathfrak{Y},\ \chi,\ \#)$ *be a* $\mathfrak{WYH}$. $\mathfrak{Y}(0,s)=\mathfrak{Y}(\mathfrak{k}-\mathfrak{k},s)=1$ *if any fixed point* $\mathfrak{k}$ *of* $\mathcal{G}$ *fulfills* $\mathfrak{Y}(\mathfrak{k}-\mathfrak{k},s)>0$.

Proof: Let $\mathfrak{k}$ be a fixed point of $\mathcal{G}$ such that $\mathfrak{k}\in\chi$. Since $\mathcal{G}$ is a $\mathfrak{FK}-\mathfrak{C}$,
$\mathfrak{Y}(0,s)=\mathfrak{Y}(\mathfrak{k}-\mathfrak{k},s)=\mathfrak{Y}(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{k}),s)\geq\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},\frac{s}{\mathfrak{K}}\right)\geq\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},\frac{s}{\mathfrak{K}^2}\right)\cdots\geq\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},\frac{s}{\mathfrak{K}^n}\right)\to 1$ *as* $n\to\infty$. Consequently, $\mathfrak{Y}(0,s)=1$. □

The Banach fixed point theorem is now established as follows in the context of $\mathfrak{WYH}'s$:

Theorem 3.1 *Let* $\mathcal{G}$ *be a* $\mathfrak{FK}-\mathfrak{C}$ *and* $(\mathfrak{Y},\ \chi,\ \#)$ *be a* $\mathfrak{Y}$ *-complete* $\mathfrak{WYH}$. *The existence of* $c_1\in\chi$ *such that* $\mathcal{Q}(\mathfrak{Y},\mathcal{G}-c_1,s)>0$. *For* $\mathfrak{i},\mathfrak{j}\in N, s>0$,

$$\mathcal{Q}(\mathfrak{Y},\mathcal{G}-c_1,s)=\sup\{\ \mathfrak{Y}(\mathcal{G}^{\ \mathfrak{i}}(c_1)-\mathcal{G}^{\mathfrak{j}}(c_1),s\}.$$

One fixed point of $\mathcal{G}$ *is then reached by*$\{\ \mathcal{G}^{\ n}(c_1)\}$.

Proof: Because $\mathcal{G}$ is a $\mathfrak{FK}-\mathfrak{C}$, this proves it. So for all $\mathfrak{i},\mathfrak{j}\in N$,we have $\mathfrak{Y}(\mathcal{G}^{\ n+\mathfrak{i}}(c_1)-\mathcal{G}^{n+\mathfrak{j}}(c_1),s)\geq\mathfrak{Y}(\mathcal{G}^{\ n-1+\mathfrak{i}}(c_1)-\mathcal{G}^{n-1+\mathfrak{j}}(c_1),\frac{s}{\mathfrak{K}})$

$$\sup\{\mathfrak{Y}\left(\mathcal{G}^{\ n+\mathfrak{i}}(c_1)-\mathcal{G}^{n+\mathfrak{j}}(c_1),s\right)\}\geq\ \sup\{\mathfrak{Y}\left(\mathcal{G}^{\ n-1+\mathfrak{i}}(c_1)-\mathcal{G}^{n-1+\mathfrak{j}}(c_1),\frac{s}{\mathfrak{K}}\right)\}$$

$$\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-\mathcal{G}^{\ n}\left(c_1\right),s\right)\geq\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-\mathcal{G}^{\ n-1}\left(c_1\right),\frac{s}{\mathfrak{K}}\right),\ \text{we have}$$
$$\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-\mathcal{G}^{\ n}\left(c_1\right),s\right)\geq\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-c_1,\frac{s}{\mathfrak{K}^n}\right)\ \text{for every}\ n\in N \tag{3.2}$$

We use (3.2) to get $\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right)-\mathcal{G}^{n+p}\left(c_1\right),s\right)\geq\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-\mathcal{G}^{\ n}\left(c_1\right),s\right)\geq\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-c_1,\frac{s}{\mathcal{K}^n}\right)$ for each $n,p\in N$.

Given that $\mathcal{Q}\left(\mathfrak{Y},\mathcal{G}-c_1,\frac{s}{\mathfrak{K}^n}\right)>0$ and $\mathfrak{K}\in(0,1)$, $\{\mathcal{G}^{\ n}\left(c_1\right)\}$is a $\mathfrak{Y}$-Cauchy sequence since $\lim_{n,p\to\infty}\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right)-\mathcal{G}^{n+p}\left(c_1\right),\ s\right)=1$. Since $(\mathfrak{Y},\ \chi,\ \#)$ is a $\mathfrak{Y}$ -complete $\mathfrak{WYH}$, $\{\mathcal{G}^{\ n}\left(c_1\right)\}$ converge to γ at some point $\gamma\in\chi$

$$\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right)-\mathcal{G}\left(\gamma\right),s\right)\geq\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right)-\gamma,\frac{s}{\mathfrak{K}}\right)$$
$$\geq\cdots\geq\ \mathfrak{Y}\left(\mathcal{G}\left(c_1\right)-\gamma,\frac{s}{\mathfrak{K}^n}\right)$$

We obtain $\lim_{n\to\infty}\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right)-\mathcal{G}\left(\gamma\right),\ s\right)=1$ as $n\to\infty$, indicating that $\mathcal{G}^{\ n}\left(c_1\right)$ converges to $\mathcal{G}\left(\gamma\right)$. Proposition 3.3 gives us $\mathcal{G}\left(\gamma\right)=\gamma$, meaning that γ is a fixed point of $\mathcal{G}$.

Uniqueness: Let $\aleph\left(\gamma-\alpha,s\right)>0$ and let $\alpha\in\chi$ be another fixed point of $\mathcal{G}$. As $\mathcal{G}$ is a $\mathfrak{F}\mathfrak{K}-\mathfrak{C}$, we obtain $\mathfrak{Y}\left(\gamma-\alpha,s\right)=\mathfrak{Y}\left(\mathcal{G}(\gamma)-\mathcal{G}(\alpha),s\right)\geq\mathfrak{Y}\left(\gamma-\alpha,\frac{s}{\mathfrak{K}}\right)\geq\mathfrak{Y}\left(\gamma-\alpha,\frac{s}{\mathfrak{K}^2}\right)\geq\cdots\geq\mathfrak{Y}\left(\gamma-\alpha,\frac{s}{\mathfrak{K}^n}\right)\to1$ as $n\to\infty$. Consequently, $\gamma=\alpha$. □

Corollary 3.1 *Let* $(\aleph,\ \chi,\#)$ *be a complete fuzzy normed space and let* $\mathcal{G}:\ \chi\to\ \chi$ *be such that, for some* $\mathfrak{K}\in(0,1)$, $\aleph\left(\mathcal{G}(\gamma)-\mathcal{G}(\alpha),\mathfrak{K}s\right)\ \geq\aleph\left(\gamma-\alpha,s\right)$ *for all* $\gamma,\alpha\in\chi$ *and* $s>0$*. If* $c_1\in\chi$ *such that* $\sup\{\ \aleph\left(\mathcal{G}^{\ \mathrm{i}}(c_1)-\mathcal{G}^{\ \mathrm{j}}(c_1),s\right);\ \mathrm{i},\mathrm{j}\in N,\ s>0\}>0$*. So, the sequence* $\{\ \mathcal{G}^{\ n}\left(c_1\right)\}$ $\mathfrak{Y}$*-convergen to a unique fixed point.*

Example 3.5 Given $\chi=[0,1]$, define $\mathfrak{Y}:\chi\times(0,\ \infty)\to[0,1]$ by $\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{h},s\right)=e^{-\frac{|\mathfrak{k}+\mathfrak{h}|}{s}}$ for all $\mathfrak{k},\ \mathfrak{h}\in\chi$ and $s>0$. Verifying that $(\mathfrak{Y},\ \chi,\ \#)$ is a $\mathfrak{Y}$ -complete $\mathfrak{WYH}$ is simple. For $\mathfrak{K}\in(0,1)$, we define $\mathcal{G}:\ \chi\to\ \chi$ by $\mathcal{G}\left(\mathfrak{k}\right)=\frac{\mathfrak{K}\mathfrak{k}}{n}$. For $n\geq1$, we obtain $\mathfrak{Y}\left(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{h}),\mathfrak{K}s\right)=\mathfrak{Y}\left(\frac{\mathfrak{K}\mathfrak{k}}{n}-\frac{\mathfrak{K}\mathfrak{h}}{n},\mathfrak{K}s\right)=e^{-\frac{\left|\frac{\mathfrak{K}\mathfrak{k}}{n}+\frac{\mathfrak{K}\mathfrak{h}}{n}\right|}{\mathfrak{K}s}}=e^{-\frac{\mathfrak{K}\ |\mathfrak{k}+\mathfrak{h}|}{\mathfrak{K}ns}}\geq e^{-\frac{|\mathfrak{k}+\mathfrak{h}|}{s}}=\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{h},s\right)$. $\mathfrak{Y}$ is a $\mathfrak{F}\mathfrak{K}-\mathfrak{C}$ as a result. Additionally, $\mathcal{Q}\left(\mathfrak{Y}-\mathcal{G},0,s\right)=\sup\{\ \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{i}}\left(0\right)-\mathcal{G}^{\ \mathrm{j}}\left(0\right),s\right);\ \mathrm{i},\mathrm{j}\in N,\ s>0\}$ $=\sup\{e^{-\frac{\left|\mathcal{G}^{\ \mathrm{i}}(0)-\mathcal{G}^{\ \mathrm{j}}(0)\right|}{s}}\}=1>0$ for $\mathfrak{k}=0\in\chi$. Hence, $\mathfrak{k}=0\in[0,\ 1]$ is a unique fixed point of $\mathcal{G}$, and all the requirements of Theorem 3.1 are met.

Definition 3.6 *Consider the* $\mathfrak{WYH}$ $(\mathfrak{Y},\ \chi,\ \#)$*. A mapping* $\mathcal{G}:\chi\to\ \chi$ *is said to be weak fuzzy* $\mathfrak{K}$*-type contraction (briefly* $\mathfrak{F}\mathfrak{K}-\mathfrak{Q}\mathfrak{C}$*) If for any* $\mathfrak{k},\ \mathfrak{h}\in\chi$ *and* $s>0$ *and for some* $\mathfrak{K}\in(0,1)$*, we have*

$$\mathfrak{Y}\left(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{h}),\mathfrak{K}s\right)\geq\min\{\ \mathfrak{Y}\left(\mathfrak{k}-\mathfrak{h},s\right),\mathfrak{Y}\left(\mathfrak{k}-\mathcal{G}\mathfrak{k},s\right),\mathfrak{Y}\left(\mathfrak{h}-\mathcal{G}\mathfrak{h},s\right),\mathfrak{Y}\left(\mathfrak{k}-\mathcal{G}\mathfrak{h},s\right),\mathfrak{Y}\left(\mathfrak{h}-\mathcal{G}\mathfrak{k},s\right)\}.$$

The following finding demonstrates that weak fuzzy $\mathfrak{K}$*-type contraction (briefly* $\mathfrak{F}\mathfrak{K}-\mathfrak{Q}\mathfrak{C}$*) is likewise covered by Proposition 3.4.*

Proposition 3.5 *Assume that* $\mathcal{G}$ *is a* $\mathfrak{F}\mathfrak{K}-\mathfrak{Q}\mathfrak{C}$ *and that* $(\mathfrak{Y},\ \chi,\ \#)$ *is a* $\mathfrak{WYH}$*.* $\mathfrak{Y}\left(0,s\right)=\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},s\right)=1$*if any fixed point* $\mathfrak{k}\in\chi$ *of* $\mathcal{G}$ *fulfills* $\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},s\right)\geq0$.

Proof: Let $\mathfrak{k}$ be a fixed point of $\mathcal{G}$ such that $\mathfrak{k}\in\chi$. Since $\mathcal{G}$ is a $\mathfrak{F}\mathfrak{K}-\mathfrak{Q}\mathfrak{C}$ so $\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},s\right)=\mathfrak{Y}\left(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{k}),s\right)$ $\geq\min\{\ \mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},\frac{s}{\mathfrak{K}}\right),\ \mathfrak{Y}\left(\mathfrak{k}-\mathcal{G}\mathfrak{k},\frac{s}{\mathfrak{K}}\right),\mathfrak{Y}\left(\mathfrak{k}-\mathcal{G}\mathfrak{k},\frac{s}{\mathfrak{K}}\right),\ \mathfrak{Y}\left(\mathfrak{k}-\mathcal{G}\mathfrak{k},\frac{s}{\mathfrak{K}}\right),\ \mathfrak{Y}\left(\mathfrak{k}-\mathcal{G}\mathfrak{k},\frac{s}{\mathfrak{K}}\right)\}=\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},\frac{s}{\mathfrak{K}}\right)$ $=\mathfrak{Y}\left(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{k}),\frac{s}{\mathfrak{K}}\right)\geq\mathfrak{Y}\left(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{k}),\frac{s}{\mathfrak{K}^2}\right)\geq\cdots\geq\mathfrak{Y}\left(\mathcal{G}(\mathfrak{k})-\mathcal{G}(\mathfrak{k}),\frac{s}{\mathfrak{K}^n}\right)\to1$ as $n\to\infty$. Consequently, $\mathfrak{Y}\left(\mathfrak{k}-\mathfrak{k},s\right)=1$. □

Theorem 3.2 *Let $\mathcal{G}: \ \chi \rightarrow \ \chi$ be a $\mathfrak{FK}-\mathfrak{QC}$ and $(\mathfrak{Y},\ \chi,\ \#)$ be a $\mathfrak{Y}$ -complete $\mathfrak{WYH}$. When $c_1 \in \chi$, then $\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G}-c_1, s\right) > 0$. The fixed point $\alpha \in \chi$ is thus reached by $\{\mathcal{G}^{\ n}\left(c_1\right)\}$. Additionally, $\alpha = \beta$ if β is anther fixed point of $\mathcal{G}$ such that $\mathfrak{Y}\left(\alpha - \beta\ , s\right) > 0$.*

Proof: Given that $\mathcal{G}$is a $\mathfrak{FK}-\mathfrak{QC}$,

$$\mathfrak{Y}(\mathcal{G}^{\ \mathrm{n+i}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), s) \geq \min\{\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n-1+i}}\left(c_1\right) - \mathcal{G}^{n-1+\mathrm{j}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right),$$
$$\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n-1+i}}\left(c_1\right) - \mathcal{G}^{\mathrm{n+i}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right), \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n-1+i}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right),$$
$$\mathfrak{Y}\left(\mathcal{G}^{\ n-1+\mathrm{j}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right), \mathfrak{Y}\left(\mathcal{G}^{n-1+\mathrm{j}}\left(c_1\right) - \mathcal{G}^{\mathrm{n+i}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)\}.$$

Then for all $\mathfrak{i}, \mathfrak{j} \in N$, we have

$$\sup\ \mathfrak{Y}(\mathcal{G}^{\ \mathrm{n+i}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), s) \geq \min\{\sup \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n-1+i}}\left(c_1\right) - \mathcal{G}^{n-1+\mathrm{j}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right),$$
$$\sup \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n-1+i}}\left(c_1\right) - \mathcal{G}^{\mathrm{n+i}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right), \sup \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n-1+i}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right),$$
$$\sup \mathfrak{Y}\left(\mathcal{G}^{\ n-1+\mathrm{j}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right), sup \mathfrak{Y}\left(\mathcal{G}^{n-1+\mathrm{j}}\left(c_1\right) - \mathcal{G}^{\mathrm{n+i}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)\};$$

$$\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), s\right) \geq \min\left\{\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n-1}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)),\ \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)),\right.$$
$$\left.\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)), \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)), \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right))\right\};$$
$$\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), s\right) \geq \min\{\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n-1}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right),\ \ \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)\};$$
$$\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), s\right) \geq\ \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n-1}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right)\ .$$

Consequently, for every $n > 0$,we have

$$\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), s\right) \geq\ \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - c_1, \frac{s}{\mathfrak{K}^n}\right)\ .$$

Using the above inequality, we obtain

$$\mathfrak{Y}(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), s) \geq \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathcal{G}^{\ \mathrm{n}}\left(c_1\right), s\right) \geq\ \mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - c_1, \frac{s}{\mathfrak{K}^n}\right) \tag{3.3}$$

For each $n, \mathfrak{j} \in N$. Given that $\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - c_1, \frac{s}{\mathfrak{K}^n}\right) > 0$ and $\mathfrak{K} \in (0,1)$,

$$\lim_{n,\mathfrak{j}\rightarrow\infty} \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \mathcal{G}^{n+\mathrm{j}}\left(c_1\right), s\right) = 1\ ,$$

suggesting that $\{\ \mathcal{G}^{\ n}\left(c_1\right)\}$ is a $\mathfrak{Y}$-Cauchy sequence. If $(\mathfrak{Y},\ \chi,\ \#)$ is a $\mathfrak{Y}$ -complete $\mathfrak{WYH}$, then $\{\ \mathcal{G}^{\ n}\left(c_1\right)\}$ converges to α for some $\alpha \in \chi$.

$$\mathfrak{Y}(\mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right) - \mathcal{G}\left(\alpha\right), s) \geq\ \min\left\{\mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \alpha, \frac{s}{\mathfrak{K}}\right),\ \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right),\right.$$
$$\left.\mathfrak{Y}\left(\alpha - \mathcal{G}\alpha, \frac{s}{\mathfrak{K}}\right), \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \mathcal{G}\alpha, \frac{s}{\mathfrak{K}}\right), \mathfrak{Y}\left(\alpha - \mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right), \frac{s}{\mathfrak{K}}\right), \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \alpha, \frac{s}{\mathfrak{K}}\right)\right\}.$$

Thus, we have $\mathfrak{Y}(\mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right) - \mathcal{G}\left(\alpha\right), s) \geq \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n}}\left(c_1\right) - \alpha, \frac{s}{\mathfrak{K}}\right) \geq \ldots \geq \mathfrak{Y}\left(\mathcal{G}\left(c_1\right) - \alpha, \frac{s}{\mathfrak{K}^n}\right)$.

$\lim_{n\rightarrow\infty} \mathfrak{Y}\left(\mathcal{G}^{\ \mathrm{n+1}}\left(c_1\right) - \mathcal{G}\left(\alpha\right), s\right) = 1$ is obtain as $n \rightarrow \infty$, indicating that $\mathcal{G}\left(\alpha\right)$ is a fixed point of $\mathcal{G}^{\ n}\left(c_1\right)$. Proposition 3.3 gives us $\mathcal{G}\left(\alpha\right) = \alpha$, meaning that α is a fixed point of $\mathcal{G}$.

Uniqueness: $\mathfrak{Y}\left(\alpha - \beta\ , s\right) > 0$ if $\beta \in \chi$is another fixed point of $\mathcal{G}$. Given that $\mathcal{G}$is a $\mathfrak{FK}-\mathfrak{QC}$, $\mathfrak{Y}\left(\alpha - \beta\ , s\right) = \mathfrak{Y}\left(\mathcal{G}(\alpha) - \mathcal{G}(\beta)\ , s\right) \geq \min\{\mathfrak{Y}\left(\alpha - \beta\ , \frac{s}{\mathfrak{K}}\right),\ \ \mathfrak{Y}\left(\alpha - \mathcal{G}\alpha\ , \frac{s}{\mathfrak{K}}\right),\ \mathfrak{Y}\left(\beta - \mathcal{G}\beta\ , \frac{s}{\mathfrak{K}}\right),$ $\mathfrak{Y}\left(\alpha - \mathcal{G}\beta\ , \frac{s}{\mathfrak{K}}\right)\ \mathfrak{Y}\left(\beta - \mathcal{G}\alpha\ , \frac{s}{\mathfrak{K}}\right)\} = \mathfrak{Y}\left(\alpha - \beta\ , \frac{s}{\mathfrak{K}}\right) = \mathfrak{Y}\left(\mathcal{G}(\alpha) - \mathcal{G}(\beta)\ , \frac{s}{\mathfrak{K}}\right) \geq \mathfrak{Y}\left(\alpha - \beta\ , \frac{s}{\mathfrak{K}^2}\right) \geq \ldots \geq$ $\mathfrak{Y}\left(\alpha - \beta\ , \frac{s}{\mathfrak{K}^n}\right) \rightarrow 1$ as $n \rightarrow \infty$.Consequently, $\alpha = \beta$.

4. Application

As an application of our primary fixed point finding demonstrated in Theorem 3.1, the existence of the solution to a specific non-linear integral equation has been examined in this section.

Example 4.1 $\chi = C[0,\ I]$, which is the class of all continuous functions with real values specified on $[0, I]$. For any $\mathfrak{k}, \mathfrak{h}\ \in \chi$ and $s > 0$, define a $\mathfrak{Y}$-complete $\mathfrak{WYH}$, $\mathfrak{Y} : \chi \times (0,\ \infty) \to [0, 1]$ by using

$$\mathfrak{Y}\left(\mathfrak{k} - \mathfrak{h}, s\right) = e^{-\frac{sup_{l\in[0,1]}|\mathfrak{k}(l)-\mathfrak{h}(l)|}{s}}.$$

Examine the integral equation

$$\mathfrak{k}\left(s\right) = f\left(s\right) + \int_0^I w\left(s, l\right) F(s - l,\mathfrak{k}(l))dl \tag{4.1}$$

In which $I > 0$ and the continuous functions $f : [0, I] \to R,\ w : [0, I] \times [0, I] \to R$, and $F : [0, I]] \times R \to R$ are all functions.

Theorem 4.1 *Assume that the* $\mathfrak{WYH}$ $(\mathfrak{Y},\ \chi,\ \#)$ *is* $\mathfrak{Y}$ *-complete. Assume that the integral operator* $\mathcal{G} :\ \chi \to\ \chi$ *is defined by*

$$\mathcal{G}\left(\mathfrak{k}\left(s\right)\right) = f\left(s\right) + \int_0^I w\left(s, l\right) F(s - l,\mathfrak{k}(l))dl$$

for every $\mathfrak{k}\ \in \chi$ *and* $s, l \in [0, I]$*. Assume the following conditions are holds:*

1. *We have* $|F(s - l, \mathfrak{k}(l)| < |F(s - l, \mathfrak{h}(l)|$ *for every* $\mathfrak{k}\ ,\mathfrak{h} \in \chi$ *and* $s, l \in [0, I]$*.*
2. *For every* $s, l \in [0, I]$ *,* $sup_{l\in[0,1]} \left|\int_0^I w\left(s, l\right) dl\right| \le \mathfrak{K} < 1$*.*

Then, $\mathfrak{k}^{\#} \in \chi$ *is the solution to the integral equation* (4.1)*.*

Proof: For $\mathfrak{k}^{\#} \in \chi$, we have

$$\mathcal{Q}\left(\mathfrak{Y}, \mathcal{G} - \mathfrak{k}^{\#}, s\right) = \sup\{\ \mathfrak{Y}\left(\mathcal{G}^{\ \mathfrak{i}}\left(\mathfrak{k}^{\#}\right) - \mathcal{G}^{\ \mathfrak{j}}\left(\mathfrak{k}^{\#}\right), s\right); \mathfrak{i}, \mathfrak{j} \in N,\ \ s > 0.$$

All we need to demonstrate is that $\mathcal{G}$ is $\mathfrak{K}$-type contraction. We have

$$\begin{aligned}
\mathfrak{Y}\left(\mathcal{G}\mathfrak{k} - \mathcal{G}\mathfrak{h}, \mathfrak{K}s\right) &= e^{-\frac{sup_{l\in[0,1]}|\mathcal{G}\mathfrak{k}(l)-\mathcal{G}\mathfrak{h}(l)|}{\mathfrak{K}s}} \\
&= e^{-\frac{sup_{l\in[0,1]}\left|\int_0^I w(s,l)F(s-l,\mathfrak{k}(l)dl-\int_0^I w(s,l)F(s-l,\mathfrak{h}(l)dl\right|}{\mathfrak{K}s}} \\
&= e^{-\frac{sup_{l\in[0,1]}\left|\int_0^I w(s,l)(F(s-l,\mathfrak{k}(l)-F(s-l,\mathfrak{h}(l))dl\right|}{\mathfrak{K}s}} \\
&\ge e^{-\frac{sup_{l\in[0,1]}\left|\int_0^I w(s,l)dl\right|\left|\int_0^I (F(s-l,\mathfrak{k}(l)-F(s-l,\mathfrak{h}(l))dl\right|}{\mathfrak{K}s}} \\
&\ge e^{-\frac{\mathfrak{K}\int_0^I |\mathfrak{k}(l)-\mathfrak{h}(l)|dl}{\mathfrak{K}s}} \\
&\ge e^{-\frac{\mathfrak{K}sup_{l\in[0,1]}|\mathfrak{k}(l)-\mathfrak{h}(l)|}{\mathfrak{K}s}} \\
&\ge e^{-\frac{sup_{l\in[0,1]}|\mathfrak{k}(l)-\mathfrak{h}(l)|}{s}} = \mathfrak{Y}\left(\mathfrak{k} - \mathfrak{h}, s\right)
\end{aligned}$$

Given that Theorem 3.16' requirements are all met, $\mathcal{G}$ has a fixed point. Thus, there is a solution to integral equation (4.1).

The following outcome is a direct conclusion of Corollary 3.1 and Theorem 3.1. □

Corollary 4.1 *Assume that* $(\aleph,\ \chi,\ \#)$ *is a fuzzy normed space that is* $\mathfrak{Y}$ *-complete. Let* $\mathcal{G}(\mathfrak{k}\,(s)) = f\,(s) + \int_0^I w\,(s,l)\,F(s-l,\mathfrak{k}(l))dl$ *be the integral operator defined by* $\mathcal{G}:\ \chi \to\ \chi$. *For any* $\mathfrak{k} \in \chi$ *and* $s,l \in [0,I]$. *Assume the following circumstances are met:*

1. *We have* $|F(s-l,\mathfrak{k}\,(l)) - F(s-l,\mathfrak{h}(l)| < |\mathfrak{k}(l) - \mathfrak{h}(l)|$ *for every* $\mathfrak{k},\mathfrak{h}\ \in \chi$ *and* $s,l \in [0,I]$.
2. *For any* $s,l \in [0,I]$, $sup_{l\in[0,1]}\left|\int_0^I w\,(s,l)\,dl\right| \leq\ \mathcal{K} < 1$

Then, $\mathfrak{k}^{\#} \in \chi$ *is the solution to the integral equation* (4.1).

5. Conclusion

In order to expand the classical setting to include a larger class of problems, we created a broader framework in this study by introducing modified fuzzy normed spaces. Several fixed point conclusions that improve on current theirs were found by constructing fuzzy $\mathfrak{K}$-type contractions and weak $\mathfrak{K}$-type fuzzy contractions. To elucidate the theoretical conclusions and show their applicability, an example was provided. Moreover, the conclusion's practical significance is highlighted by their application to a nonlinear integral equation. Further investigation into the solution of intricate functional equations in fuzzy setting is made possible by this study.

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