Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.77798

Maximal Global Random Attractors of Gradient-Like Random Dynamical Systems by Stochastic Lyapunov Functions

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ABSTRACT: This paper investigates the existence and construction of the maximal attractors of gradient-like random dynamical systems (RDSs), assuming the system is asymptotically compact, admits a Lyapunov function, and the set of equilibrium points is bounded. The study begins by presenting a stochastic version of LaSalle's Invariance Principle, followed by an analysis of the random Levinson center for gradient RDSs with finite number of equilibrium points.

Key Words: Random dynamical systems, Lyapunov function, asymptotically compact, gradient-like, compact global attractors, LaSalle's invariance principle, Levinson center, maximal compact invariant set, random norm.

Contents

1. Introduction

The long-term behavior of gradient (deterministic) dynamical systems has been the subject of much investigation. In this context, for instance, we cite several studies (see, for example, [3,4,6,12,15,11,17], and references therein). The long-term behavior of gradient-like deterministic dynamical systems is examined by David Cheban [7,8] and the references therein. He demonstrated how to construct the Levinson Center for deterministic dynamic systems. Furthermore, Caraballo, J. A. Langa, and Z. Liu in [5] describe the gradient RDSs as a random semi-flow through the random version of the Lyapunov function.

In [14], A. Hmissi, F. Hmissi, and M. Hmissi give multiple explanations of continuous random dynamical systems (RDS) that have gradient-like characteristics. In [13], H. A. Hashim and I.J. Kadhim use the uniformly unbounded Lyapunov functions to investigate the global stability of compact random sets in random dynamical systems constructed on Banach space. They also showed that compact random sets can be contained in an attraction region, and they give the necessary condition for a compact connected random set to have asymptotic stability is given. In [16] I. J. Kadhim , and A. A. Yasir, use the stochastic Lyapunov functions to study the parallelizable RDSs and investigate the stability the random Lorenz system.

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2010 *Mathematics Subject Classification:* 37Hxx, 15B52, 34Fxx, 37Nxx.
Submitted July 10, 2025. Published December 19, 2025

In [18], the random version of the Levinson center is introduced. Additionally, as in the deterministic case, researchers have noted the close relationship between Levinson's center and the prolongation of the Omega-limit set. in its two random versions.

In this article, we examine how to construct compact global and maximal attractors for gradient-like RDSs using the Lyapunov function. Section 2 is devoted to listing some concepts and facts that are needed in our work. Section 3 is dedicated to investigating the long-term behavior for gradient-like RDSs. Gradient RDSs with an infinite or finite number of equilibrium points are examined. The random variety of Lorenz system is introduced in Section 4 as an application on gradient-like RDS.

2. Preliminaries and Mathematical Framework

In this section, we review some basic concepts related to stochastic dynamic systems associated with the subject of this research. Including the definition of a stochastic dynamic system, the stochastic process, and the stochastic path of the stochastic process. We also touched on some types of stochastic processes and the stability of stochastic processes. Additionally, we described stability using random Lyapunov functions in a manner similar to what is familiar in deterministic dynamical systems.

From now on, we will denote the probability space by the symbol Ω instead of the triplet $(\Omega,\ \mathcal{F},\ \mathbb{P})$, and we will denote the Banach space by the symbol $\mathbb{X}$ instead of the pair $(\mathbb{X}, \cdot)$. As for $\mathcal{D}$, it symbolizes the universe of random sets [5]. And the symbol $\mathbb{X}^{\Omega}$ represents the set of random variables in $\mathbb{X}$.

Definition 2.1. *[1,9] The measurable dynamical system is an action $\theta : \mathbb{R}\times\Omega \longrightarrow \Omega$ which is measurable and it is verified $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for every $B \in \mathcal{F}$ and $t \in \mathbb{R}$. It is denoted by (MDS).*

Definition 2.2. *[1,9] A co-cycle over a MDS θ is a measurable function $\varphi : \mathbb{R} \times \Omega \times \mathbb{X} \longrightarrow \mathbb{X}$, with the property that*

$$\varphi(0, \omega, x) = x, \varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x)) \tag{2.1}$$

for every $t,\ s \in \mathbb{R}, x \in \mathbb{X}$ and $\omega \in \Omega$. The couple (θ, φ) is called random dynamical system (RDS), if $\varphi(\cdot, \omega, \cdot)$ is continuous.

Definition 2.3. *[1,9] The affine RDS is a pair (θ, φ) such that*

$$\varphi\,(t, \omega)\,x = \Phi\,(t, \omega)\,x + \psi\,(t, \omega) \tag{2.2}$$

where $\Phi(t, \omega)$ verify 2.1 and $\psi\colon \mathbb{R} \times \Omega \longrightarrow \mathbb{X}$ is measurable.

Definition 2.4. *[1,9] Let $(\mathbb{X}, d)$ be a metric space.*

a. A random set is the multi-valued map $D : \Omega \longrightarrow 2^{\mathbb{X}}/\,\{\emptyset\}$ with the property that, the positive real-valued function $\rho\,(\omega\ ) := dist_{\mathbb{X}}(x, D(\omega))$ is measurable for every $x \in \mathbb{X}$. The random set D is called compact (closed) if $D(\omega) \subset X$ is compact (closed) every $\omega \in \Omega$.

b. A tempered random variable (TRV) is measurable function $\varepsilon : \Omega \to \mathbb{R}$ with the property that

$$\lim_{t\to+\infty} \frac{1}{|t|} \log |\varepsilon(\theta_t\omega)| = 0.$$

Definition 2.5. *[9] The random trajectory starting from the a random set $D(\omega)$ is defined by $\gamma^t_D\,(\omega) := \cup_{\tau\geq t}\varphi\,(\tau, \theta_{-\tau}\omega)\,D(\theta_{-\tau}\omega)$.*

Throughout, if $t \in \mathbb{R}^+$, then $\gamma^t_D\,(\omega) \equiv \gamma^+_D(\omega)$ denote the forward semi-trajectory. Also if $t \in \mathbb{R}^-$, then $\gamma^t_D\,(\omega) \equiv \gamma^-_D(\omega)$ denote the backward semi-trajectory.

The following theorem motivates us to construct the random norm.

Theorem 2.6. *[1] Suppose that the linear cocycle Φ fulfills the integrability condition of the multiplicative ergodic theorem. Take a constant $k > 0$. Present on the invariant set $\tilde{\Omega}$ of the multiplicative ergodic theorem*

$$\langle x, y\rangle_{k,\omega} := \sum_{i=1}^{p} \langle x_i, y_i\rangle_{k,\omega}\,,$$

and for every $x = \bigoplus_{i=1}^{p} x_i$ *and* $y = \bigoplus_{i=1}^{p} y_i$, *with* $x_i, y_i \in E_i(\omega)$. *For* $u_i, v_i \in E_i(\omega)$, *defined*

$$\langle u_i, v_i \rangle_{k,\omega} := \begin{cases} \int_{-\infty}^{\infty} \frac{\langle \Phi(t,\omega)u_i, \Phi(t,\omega)v_i \rangle}{e^{2(\lambda_i t + k|t|)}} dt, & \mathbb{T} = \mathbb{R}, \\ \sum_{n\in\mathbb{Z}} \frac{\langle \Phi(t,\omega)u_i, \Phi(t,\omega)v_i \rangle}{e^{2(\lambda_i t + k|t|)}}, & \mathbb{T} = \mathbb{Z}. \end{cases} \tag{2.3}$$

Put $\langle x, y \rangle_{k,\omega} := x, y$ *for* $\omega \notin \tilde{\Omega}$. *Then*

i. $\langle \cdot, \cdot \rangle_{k,\omega}$ *is a random scalar product in* $\mathbb{R}^d$ *which depends measurably on* ω *and under which the* $E_i(\omega)$ *are orthogonal.*

ii. *For each* $\varepsilon > 0$ *there exists an* ε*-slowly varying random variable* $B_\varepsilon : \Omega \to [1, \infty)$ *such that*

$$\frac{1}{B_\varepsilon(\omega)} \| \cdot \| \leq \|\cdot\|_{k,\omega} \leq B_\varepsilon(\omega) \| \cdot \|,$$

where

$$\|x\|_{k,\omega}^2 = \langle x, x \rangle_{k,\omega} = \sum_{i=1}^{p} \|x_i\|_{k,\omega}^2,$$

and

$$\|x_i\|_{k,\omega}^2 = \begin{cases} \int_{-\infty}^{\infty} \frac{\|\Phi(t,\omega)x_i\|^2}{e^{2(\lambda_i t + k|t|)}} dt, & \mathbb{T} = \mathbb{R}, \\ \sum_{n\in\mathbb{Z}} \frac{\|\Phi(t,\omega)x_i\|^2}{e^{2(\lambda_i t + k|t|)}}, & \mathbb{T} = \mathbb{Z}. \end{cases}$$

defines the square of the random norm corresponding to $\langle \cdot, \cdot \rangle_{k,\omega}$

iii. *Define* $F : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T} \longrightarrow \mathbb{R}$, *by* $F(x, y, t) = \langle x, y \rangle_{k,\theta(t)\omega}$. *This function is continous.*

iv. *For every* $i = 1, \ldots, p, x \in E_i(\omega), t \in \mathbb{T}$, *the inequlity*

$$e^{\lambda_i t - k|t|} \|x\|_\omega \leq \|\Phi(t, \omega) x\|_{k,\theta(t)\omega} \leq e^{\lambda_i t + k|t|} \|x\|_\omega. \tag{2.4}$$

is vailed.

Up to now, $B_\varepsilon^\omega(x_0) := \{x \in \mathbb{X} : \|x - x_0\|_\omega < \varepsilon(\omega)\}$ *represent to the random open ball with respect to the random norm* $\|\cdot\|_\omega$. *For simplicity, we will write,* $\tilde{\theta} \equiv \theta_t \omega$ *and* $\tilde{x} \equiv \varphi(t, \omega) x$.

Definition 2.7. *Let* (θ, φ) *be an RDS.*

1. *The omega-limit set (see [9]) of* $D(\omega)$ *is defined by*

$$\Gamma_D(\omega) = \left\{ x \in \mathbb{X} : \exists \{t_n\} \subset \mathbb{R}^+, \ \{y_n\} \subset D(\omega) \ , \lim_{n\to+\infty} \|\tilde{y}_n - x\|_{\theta_t\omega} = 0 \right\},$$

   where $\tilde{\theta}_n \equiv \theta_{t_n}\omega$ *and* $\tilde{y}_n \equiv \varphi(t_n, \omega) y$.

2. *The forward limit prolongation (see [17]) of* $D(\omega)$ *defined by the random set*

$$J_D^+(\omega) := \Big\{ y \in \mathbb{X} : \exists \ \{x_n\} \text{ and } \{t_n\}, t_n \to +\infty,$$
$$\lim_{n\to+\infty} \inf_{z\in D(\omega)} \|x_n - z\|_\omega = 0 \text{ and } \lim_{n\to+\infty} \|\tilde{x}_n - y\|_{\tilde{\theta}_n} = 0 \Big\}.$$

   where $\tilde{\theta}_n \equiv \theta_{t_n}\omega$ *and* $\tilde{x}_n \equiv \varphi(t_n, \omega) x$.

Definition 2.8. *Consider the RDS* $(\theta, \ \varphi)$.

i. $D(\omega)$ *is called a forward invariant random set if*

$$\varphi(t, \omega) D(\omega) \subseteq D(\tilde{\theta}), t > 0 \text{ and } \omega \in \Omega.$$

ii. *$D(\omega)$ is called an attractor of (θ, φ) if for some TRV δ,*

$$\lim_{t\to+\infty} \sup_{y\in S(\omega)} \inf_{x\in D(\omega)} \|\varphi(t,\omega)\,y - x\|_{\theta_t\omega} = 0, \omega \in \Omega,$$

where

$$S(\omega) := S(D(\omega), \delta(\omega)) = \left\{x \in \mathbb{X} : \inf_{z\in D(\omega)} \|x - z\|_\omega < \delta(\omega)\right\}.$$

iii. *A random variable $x \in \mathbb{X}^\Omega$ is an equilibrium of (θ, φ) if*

$$\varphi(t, \omega, x) = x(\theta_t\omega), forall t \geq 0.$$

The set of all equilibrium points of (θ, φ) will be indicated by $Fix(\varphi)$.

Definition 2.9. *[1] The solution $\varphi(\cdot, \omega)x_0$ is called*

i. *stable if $\forall\varepsilon(\omega) > 0, \exists\delta(\omega) > 0$,*

$$\sup_{0\leq t<\infty} \|\tilde{x} - \tilde{x}_0\|_{\tilde{\theta}} < \varepsilon(\omega)$$

whenever $x \in B^\omega_\delta(x_0)$.

ii. *asymptotically stable if it is stable and*

$$\lim_{t\to\infty} \|\tilde{x} - \tilde{x}_0\|_{\tilde{\theta}} = 0,$$

wherever $x \in B^\omega_\delta(x_0)$.

iii. *exponentially stable if it is stable, and for every x, there is a constant $c < 0$ such that*

$$\|\tilde{x} - \tilde{x}_0\|_{\tilde{\theta}} < e^{ct} \; for \; t \geq T(\omega, x),$$

wherever $x \in B^\omega_\delta(x_0)$..

Up to now the random sets will be considered invariant and compact.

Definition 2.10. *[2] The Lyapunov function for a random set $A(\Omega)$ is a function $\mathcal{V} : \omega \times \mathbb{X} \longrightarrow \mathbb{R}^+$ satisfying the following conditions:*

i. *For every $\Omega \in \Omega$, the function $\mathcal{V}(\omega, \cdot) : \mathbb{X} \longrightarrow \mathbb{R}^+$ is continuous and the function $\mathcal{V}(\cdot, x) : \Omega \longrightarrow \mathbb{R}^+$ is measurable for every $x \in \mathbb{X}$.*

ii. *For every $\omega : \mathcal{V}(\omega, x) \longrightarrow \infty$ as $\|x\|_\omega \longrightarrow \infty$.*

iii. *$\mathcal{V}(\omega, x)$ is vanished on $A(\omega)$, and $\mathcal{V}(\omega, x)$ is positive on for $A^c(\omega)$ (the complement of $A(\omega)$).*

iv. *$\mathcal{V}$ is reducing alongside the trajectory of φ, that is*

$$\mathcal{V}(\tilde{\theta}, \tilde{x}) \leq \mathcal{V}(\omega, x), t > 0 \text{ and } x \notin A(\omega).$$

Definition 2.11. *If the diameter of a random set is finite ,then it is called bounded.*

In the following definition we will make a small change to the Definition 1.4.4 in [9].

Definition 2.12. *An RDS (θ, φ) is called asymptotically compact in $\mathcal{D}$, if for every $D \in \mathcal{D}$ and $\omega \in \Omega$, there is compact random set $B_0(\omega)$ such that*

$$\lim_{t\longrightarrow+\infty} \sup_{x\in D(\theta_{-t}\omega)} \inf_{b\in B_0(\omega)} \|\tilde{x} - b\|_{\theta_t\omega} = 0.$$

In this case $B_0(\omega)$ is called an attracting compact random set.

In the following definition, we will define point dissipative and compact dissipative with respect to random norm. This idea is similar to that given in [19].

Definition 2.13. *An RDS (θ, φ) is called:*

1. ***Point dissipative*** *if*
$$\lim_{t\to+\infty} \inf_{y\in K(\omega)} \|\widetilde{x} - y\|_{\tilde{\theta}} = 0, \tag{2.5}$$
for some random set $K(\omega)$ and every $x \in \mathbb{X}$.

2. ***Compact dissipative*** *if*
$$\lim_{t\to+\infty} \sup_{x\in D(\theta_{-t}\omega)} \inf_{y\in K(\omega)} \|\widetilde{x} - y\|_{\tilde{\theta}} = 0, \tag{2.6}$$

for some random set $K(\omega)$ and for all compact random set $D(\omega)$ and $x \in \mathbb{X}$ [19].

Definition 2.14. *[19] If K is nonvoid compact random set attract the compact random sets in a compact dissipative RDS (θ, φ). Then*

$$J_{\mathbb{X}}(\omega) := \Gamma_K(\omega) = \cap_{t\in\mathbb{R}} \cup_{y\in K(\theta_{-t}\omega)} \varphi(t, \theta_{-t}\omega)\, y$$

is the random Levinson Center (RLC) of (θ, φ).

3. Gradient-like Random Dynamical Systems

This section represents the important topic of this article where and is the formation of compact maximal attractors for the gradient RDS's. This section is divided into two parts. In the first sub-section we proved the principle of invariance of LaSalle's in the random case. In the second sub-section, we proved the existence of the greatest attractor (Levinson Center) for gradient RDSs.

3.1. Attractors of Gradient-like Random Dynamical Systems

In this subsection, we use the Lyapunov function to examine the asymptotic behavior of gradient-like RDSs. Note that in [14] the authors define a gradient-like RDS by using sections and then characterize this concept by Lyapunov function.

Definition 3.1. *Lyapunov function $\mathcal{V} : \Omega \times \mathbb{X} \to \mathbb{R}$ for (θ, φ) is called global if*

$$\mathcal{V}(\widetilde{\theta}, \widetilde{x}) \ \leq \mathcal{V}(\omega, x),$$

for every x in $\mathbb{X}$ and t in $\mathbb{R}$.

Definition 3.2. *An RDS (θ, φ) with the Lyapunov function $\mathcal{V}$ is called gradient if for all $t \geq 0$ the equality*

$$\mathcal{V}\left(\widetilde{\theta}, \widetilde{x}\right) = \mathcal{V}(\omega, x) \ \text{implies} x \in Fix(\varphi).$$

The following differential equation, the most basic illustration of a gradient-like RDS

$$\dot{x} = -\nabla\mathcal{V}(\omega, x), x \in \mathbb{R}^n, \tag{3.1}$$

where $\mathcal{V} :\ \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be function such that $\mathcal{V}(\omega, \cdot) :\ \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and $\mathcal{V}(\cdot, x) :\ \Omega \longrightarrow \mathbb{R}$ is measurable and $\nabla :\ = \ \partial x_1, \ldots,\ \partial x_n$. Assuming that $\varphi(t, \omega)x$ is defined on $\mathbb{R}^+$ and passes through $x \in \mathbb{R}^n$ at $t = 0$, then

$$\frac{d}{dt}\mathcal{V}\left(\widetilde{\theta}, \widetilde{x}\right) = -\left|\nabla\mathcal{V}\left(\widetilde{\theta}, \widetilde{x}\right)\right|^2 \leq 0 \tag{3.2}$$

for $x \in \mathbb{R}^n$ and $t > 0$. By (3.2) we have

$$\mathcal{V}\left(\widetilde{\theta}, \widetilde{x}\right) \leq \mathcal{V}\left(\widetilde{\theta}, \widetilde{x}_i\right) = \mathcal{V}(\omega, x), t \geq 0, \ \text{where}\ \widetilde{x}_i \equiv \varphi(t,\ \omega)\, x_i.$$

By (3.1) we have $x \in Fix(\varphi)$, that is, (θ, φ) (the RDS produced by Eq. (3.1)) is a gradient RDS.

Definition 3.3. *An RDS (θ, φ) is called a gradient-like if (θ, φ) admits a global Lyapunov function.*

Theorem 3.4. ***(LaSalle's Invariance Principle in the Random Case)*** *If $\mathcal{V}$ is a Lyapunov function for a gradient-like RDS (θ, φ), then the following claims are true.*

1. *If $\gamma_x^+(\omega)$ is precompact, then for every $x \in \mathbb{X}$ and every $p \in \Gamma_x^+(\omega)$ we have such $\mathcal{V}(\omega, p) = C_x$, for some $C_x \in \mathbb{R}$.*
2. *If $\gamma_x^-(\omega)$ is precompact, $\gamma \in \Phi_x andq \in \Gamma_\gamma^-(\omega)$, then $\mathcal{V}(\omega, q) = c_\gamma$ for some $c_\gamma \in \mathbb{R}$.*

Proof. Define $\psi : \mathbb{R}^+ \to \mathbb{R}$ by equality $\psi(t) := \mathcal{V}(\theta_t\omega, \varphi(t, \omega)x)$. Clearly that ψ is continuous. Since for all $t_2 \geq\ t_1$ we have $\psi(t_2) \leq \psi(t_1)$, then

$$\mathcal{V}(\widetilde{\theta}, \widetilde{x}) \longrightarrow C_x \ as \ t \longrightarrow +\infty.$$

If $p \in \Gamma_x(\omega), \exists \{t_n\} \subset \mathbb{R}$ with $t_n \longrightarrow\ +\infty$ hence

$$x_n \longrightarrow p \ as \ n \longrightarrow +\infty, \ where \ x_n \equiv \varphi(t_n, \theta_{-t_n}\ \omega)x$$

hence,

$$\mathcal{V}(\omega, p) = \lim_{n \longrightarrow +\infty} \mathcal{L}(\omega, x_n) = C_x.$$

Defined $\psi : \mathbb{R}^- \to \mathbb{R}$, by $\psi(s) := \mathcal{V}(\omega, \gamma(s))\ , s \in \mathbb{R}^-, \gamma \in \Phi_x$. Because $\psi(s_1) \geq \psi(s_2)$ for every $s_1 \leq s_2(s_1, s_2 \in \mathbb{R}^-)$ and ψ is upper-bounded on $\mathbb{R}^-$, then $\lim_{n\to+\infty} \mathcal{V}(\omega, \gamma(s_n)) = c_\gamma$. If $q \in \Gamma_\gamma^-(\omega)$, then there is $s_n \subseteq \mathbb{R}^-$ with $s_n \to -\infty$ so that $q = \lim_{n\to+\infty} \gamma(s_n)$ and

$$\mathcal{V}(\omega, q) = \lim_{n\to+\infty} \mathcal{V}(\omega, \gamma(s_n)) = c_\gamma.$$

□

Theorem 3.5. *Consider a gradient-like RDS (θ, φ) with a Lyapunov function $\mathcal{V}$. If $x \in\ J_x^+(\omega)$, then*

$$\mathcal{V}\ (\omega, x) = \mathcal{V}(\widetilde{\theta}, \widetilde{x}), \forall t \in [0, \infty).$$

Proof. Let $p \in J_x^+(\omega)$. Because $J_x^+(\omega) \subseteq J_{\tilde{x}}^+(\omega)$ for all $t \in \mathbb{R}$, then

$$p = \lim_{n\to+\infty} \varphi(t_n, \theta_{-t_n}\ \omega)\tilde{x}_n, t_n \to +\infty \ and \ \widetilde{x}_n \longrightarrow \widetilde{x}.$$

Hence

$$\begin{aligned} \mathcal{V}\ (\omega, p) &= \lim_{n\to+\infty} \mathcal{V}\ (\omega, \varphi(t_n, \theta_{-t_n}\ \omega)\tilde{x}_n) \\ &\leq \lim_{n\longrightarrow+\infty} \mathcal{V}\ (\omega, \widetilde{x}_n) = \mathcal{V}(\omega, \bar{x}), \ where \ \bar{x} \equiv \varphi(t, \theta_{-t}\ \omega)x \end{aligned}$$

Especially, $\mathcal{V}(\omega, x) \leq \mathcal{V}(\omega, \bar{x})$, because x in $J_x^+(\omega)$. However,

$$\mathcal{V}(\omega, x) \leq \mathcal{V}(\omega, \varphi(t, \theta_{-t}\omega)x), \ because \ x \in J_x^+(\omega).$$

However,

$$\mathcal{V}(\omega, x) \geq \mathcal{V}(\omega, \bar{x}), x \ in \ \mathbb{X} \ and \ t \geq 0,$$

So, from the above discussion, we get the result. □

Corollary 3.6. *An RDS (θ, φ) is gradient-like iff and only if the following statement holds: $J_x(\omega) \neq \emptyset$ implies that x is an equilibrium and vice versa.*

3.2. Maximal Attractor Gradient Random Dynamical Systems

For gradient RDSs with a finite number of random equilibrium points, we provide a comprehensive description of maximal compact invariant random attractor (random Levinson center).

Lemma 3.7. *If (θ, φ) is a gradient RDS. Then $\Omega_{\mathbb{X}}(\omega) = \Omega(\varphi) = Fix(\varphi)$, where $\Omega_{\mathbb{X}}(\omega) := \overline{\cup\{\Gamma_x^+(\omega) : x \in \mathbb{X}\}}$ and $\Omega(\varphi) = \{x \in \mathbb{X} : x \in J_x^+(\omega)\}$.*

Proof. For any RDSs, we have $Fix(\varphi) \subseteq \Omega_{\mathbb{X}}(\omega) \subseteq \Omega(\varphi)$. It is enough to show that $\Omega(\varphi) \subseteq Fix(\varphi)$ for gradient RDSs. Now $x \in J_x^+(\omega)$ whenever $x \in \Omega(\varphi)$, so, from Theorem 3.4 (item 2), yield is an unchanged along the positive trajectory. Thus $x \in Fix(\varphi)$. □

Theorem 3.8. *If (θ, φ) is a gradient asymptotically compact RDS and $Fix(\varphi)$ is bounded. Suppose further that $\mathcal{V}$ is a Lyapunov function such for any sequence $\{x_n\}$ with $\|x_n - x_0\|_\omega \longrightarrow +\infty$ as $n \longrightarrow \infty$ implies that $\mathcal{V}(\omega, x_n) \to +\infty$, for some $x_0 \in \mathbb{X}$. Then*

1. *The given RDS is compact dissipative.*
2. *If $\mathcal{V}$ is bounded,*
$$\lim_{t\to+\infty} \sup_{x\in M(\theta_{-t}\omega)} \inf_{y\in J_{\mathbb{X}}(\omega)} \varphi\left(t, \theta_{-t}\omega\right) x - y_{\theta_t\omega} = 0.$$
for every bounded random set $M(\omega)$.

Proof. Let $x \in \mathbb{X}$. Then $\gamma_x^+(\omega)$ is bounded. Suppose, if possible, that it is not bounded, so there is $\{t_n\}$, $t_n \to +\infty$ and a point $x_0 \in \mathbb{X}$ satisfy $\lim_{n\to\infty} \|\widetilde{x}_n - x_0\|_{\widetilde{\theta}_n} = +\infty$, where $\widetilde{\theta}_n \equiv \theta_{t_n}\omega$ and $\widetilde{x}_n \equiv \varphi\left(t_n, \omega\right) x$. By hypothesis, we have
$$\lim_{n\longrightarrow\infty} \mathcal{V}(\widetilde{\theta}_n, \widetilde{x}_n) = +\infty.$$
However,
$$\mathcal{V}(\widetilde{\theta}_n, \widetilde{x}_n) \leq \mathcal{V}(\omega, x), \forall\ n \in \mathbb{N}.$$
Our assertion is proved by this contradiction. Since (θ, φ) is asymptotically compact, hence $\gamma_x^+(\omega)$ is precompact, and so, $\Gamma_x^+(\omega)$ is a nonvoid compact random set. By Lemma 3.7, we have $\Gamma_x^+(\omega) \subseteq Fix(\varphi)$. The set $Fix(\varphi)$ is invariant and closed. By hypothesis (θ, φ) is compact. Therefore $\gamma_x^+(\omega)$ is precompact, and there is a nonvoid compact random set $K(\omega) := Fix(\varphi) \subset \mathbb{X}$ *such that* $\Omega_{\mathbb{X}} \subseteq K(\omega)$. Consequently (θ, φ) is pointwise dissipative. If $M(\omega)$ be a nonvoid compact random set in $\mathbb{X}$. To prove that the $\gamma_M^+(\omega)$ is precompact. Indeed, according to the our assumptions it is enough to show that $\gamma_M^+(\omega)$ is bounded. The set

$$M_{\mathfrak{c}}(\omega) := \{x \in \mathbb{X} : \mathcal{V}(\omega, x) \leq \mathfrak{c}\},\ \textit{where}\ \mathfrak{c} := \max\{\mathcal{V}(\omega, x) : x \in M(\omega)\},$$

is a bounded. Indeed, suppose the contrary, so there exists $\{x_n\} \subseteq M_c(\omega), x_0 \in \mathbb{X}$ so that $\|x_n - x_0\|_\omega \to +\infty$, and consequently, $\lim_{n\ \to+\infty} \mathcal{V}(\omega, x_n) = +\infty$ as. However, $x \in M_c(\omega)$, and therefore for all $n \in \mathbb{N}$, we get $\mathcal{V}(\omega, x_n) \leq c$. This is a contradiction. So (θ, φ) is point dissipative, and $\gamma_M^+(\omega)$ is precompact for any compact random sets $M(\omega)$. Using Theorem 3.23 in [17], the RDS (θ, φ) is compact dissipative. If $J_{\mathbb{X}}(\omega)$ is a Levinson center of (θ, φ) and $M(\omega)$ any bounded random set. Then by hypothesis, $\{\mathcal{V}(\omega, x) : (\omega, x) \in \Omega \times \mathbb{X}\} \subset \mathbb{R}$ is bounded. We claim that $\gamma_M^+(\omega)$ is bounded random set whenever $M(\omega)$ is. Indeed, if $\mathfrak{c} := \sup \mathcal{V}(\omega, x) : x \in M(\omega)$, then
$$\mathcal{V}(\widetilde{\theta}, \widetilde{x}) \leq \mathcal{V}(\omega, x) \leq c, \forall x \in M(\omega)\ \textit{and}\ t \in \mathbb{R}^+,$$
and so, $\gamma_x^+(\omega) \subseteq M_c(\omega)$. Therefore $\gamma_x^+(\omega)$ is bounded and forward invariant. Because the RDS (θ, φ) is asymptotically compact, the set $\Omega(M)$ is nonvoid and compact, and
$$\lim_{t\to+\infty} \sup_{x\in M(\omega)} \inf_{y\in\Omega(M)} \|\widetilde{x} - y\|_{\widetilde{\theta}} = 0.$$
It follows from the maximality of $J_{\mathbb{X}}(\omega)$ that $\Omega(M) \subseteq J_{\mathbb{X}}(\omega)$, and so, $J_{\mathbb{X}}(\omega)$ attracts $M(\omega)$. □

4. Applications

This section is dedicated to studying the gradient-like RDSs induced by the Lorenz stochastic system and the Stratonovich stochastic differential equations.

First, we propose a Lyapunov function to study the stability of the stochastic version of the Lorenz system. Then we study the stability of affine Stratonovich stochastic differential equations by constructing a Lyapunov function dependent upon random norm.

4.1. The Stochastic Version of Lorenz System

Consider the stochastic Lorenz system (see [10]) as an application of gradient-like RDSs.

$$\begin{aligned} dx &= \sigma\,(y-x)\,dt, \\ dy &= \left(ry - xz - \left(1+\frac{\beta^2}{2}\right)y\right)dt - \beta z dW_t \\ dz &= \left(xy - \left(1+\frac{\beta^2}{2}\right)z\right)dt + \beta y dW_t \end{aligned} \tag{4.1}$$

To show that the system (4.1) is gradient-like, choose the Lyapunov function

$$\mathcal{V}\,(x,y,z) := rx^2 + \sigma y^2 + \sigma (z-2r)^2. \tag{4.2}$$

Dividing $\dot{\mathcal{V}}\,(x,y,z)$ by $2r^2\sigma b$ produces the equation for an ellipsoid

$$\frac{\dot{\mathcal{V}}\,(x,y,z)}{2r^2\sigma b} = -\frac{x^2}{br} - \frac{y^2}{br} - \frac{(z-2r)^2}{r^2} + 1 \tag{4.3}$$

This demonstrates that $\dot{\mathcal{V}}$ is positive inside the ellipsoid provided by

$$\frac{x^2}{br} + \frac{y^2}{br} + \frac{(z-2r)^2}{r^2} = 1. \tag{4.4}$$

and negative outside of it.

Because there is no convergent behavior, the dynamics inside the ellipsoid are unstable. The trajectory converge towards the ellipsoid when $\dot{\mathcal{V}} < 0$ outside of it. Thus, outside of an ellipsoid, $\mathcal{V}\,(x,y,z)$ is a stochastic Lyapunov function. So,

$$\mathcal{V}\left(\widetilde{\theta}, \varphi\,(t,\omega)\,(x,y,z)\right) < \mathcal{V}(\omega,x,y,z). \tag{4.5}$$

Consequently by Definition 3.3 the RDS induced by the stochastic system (4.1) is gradient-like.

The following figures illustrate the convergence of a stochastic trajectory of the Lorenz system toward the boundary of a Lyapunov ellipsoid with different values of γ. The ellipsoid surface corresponds to the level set $V(x,y,z) = 1$, where V is a Lyapunov function of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = V(x,y,z).$$

outside this ellipsoid, the Lyapunov function is strictly decreasing, and thus the system exhibits gradient-like dynamics that drive the trajectory toward the ellipsoid's boundary. The red path shows a stochastic realization approaching and oscillating around this surface.

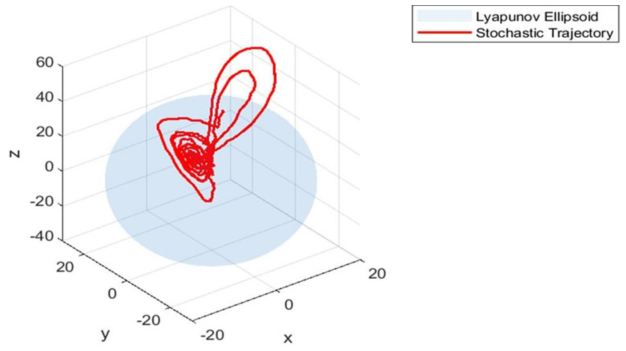

Figure 1: Convergence of the stochastic Lorenz trajectory toward the Lyapunov ellipsoid boundary (RK4 Method) with $\gamma = (2, 2, 2)$.

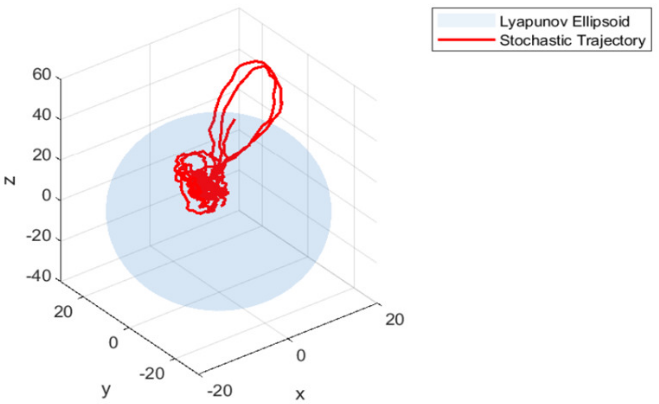

Figure 2: Convergence of the stochastic Lorenz trajectory toward the Lyapunov ellipsoid boundary (RK4 Method) with $\gamma = (5, 0, 5)$.

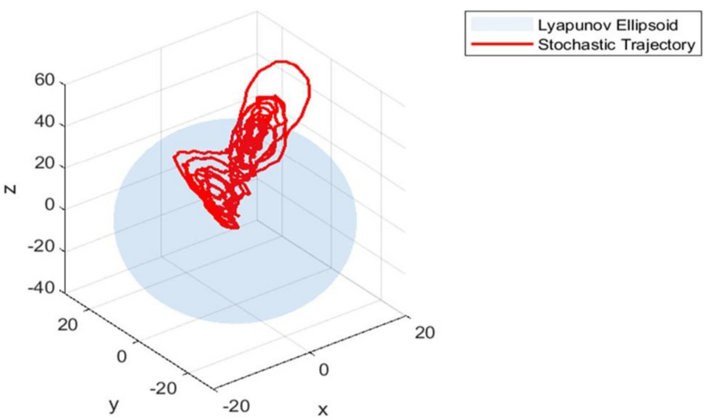

Figure 3: Convergence of the stochastic Lorenz trajectory toward the Lyapunov ellipsoid boundary (RK4 Method) with $\gamma = (0, 9, 0)$.

4.2. Stability of Affine Stratonovich Stochastic Differential Equations

Affine RDSs are usually induced by affine stochastic differential equations(SDEs) or random differential equations(RDEs) ([1,9,3]). In order to sake of conciseness, we ruminate the SDE

$$dx = \sum_{j=0}^{m} (A_j x + b_j) \circ dW^j, \circ dW^0 := dt \tag{4.6}$$

where $A_j \in \mathbb{R}^{d\times d}$ and $b_j \in \mathbb{R}^d, j = 0, \ldots, m$. (4.6) produces an affine RDS represented as

$$\varphi(t,\omega)x = \Phi(t,\omega)\left(x + \sum_{j=0}^{m} \int_0^t \Phi(t,\omega)^{-1} b_j \circ dW_s^j\right) \tag{4.7}$$

where Φ is the linear RDS produced by the analogous linear SDE

$$dx = \sum_{j=0}^{m} A_j x \circ dW^j.$$

If $\lambda := \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\| < 0$ (the Lyapunov exponent of Φ [2]),
then

$$x_0(\omega) := \sum_{j=0}^{m} \int_{-\infty}^{0} \Phi(t,\omega)^{-1} b_j \circ dW_s^j$$

is the initial value of the fixed solution of φ. Thus, $A(\omega) = \{x_0(\omega)\}$ is a random compact set that is invariant. For each random variable X, it acts as an attractor,

$$\varphi(t,\omega)X(\omega) - x_0(\theta_t\omega) = \Phi(t,\omega)(X - x_0(\omega)) \to 0$$

as $t \to \infty$, even $\mathbb{P}$-a.s. The random equilibrium $\{x_0(\omega)\}$ is stable with analogous stochastic Lyapunov function

$$\mathcal{V}(\omega, x) := \|x_0(\omega) - x\|_{\kappa,\omega}, \tag{4.8}$$

wherever $\|\cdot\|_{\kappa,\omega}$ is a random Lyapunov norm (see [1],4.3) such that

$$\|\Phi(t,\omega)\|_{\kappa,\omega,\theta_t\omega} \le e^{(\lambda+\kappa)t} \ for\ t \ge 0, \tag{4.9}$$

whenever $\kappa > 0$ and $\lambda + \kappa < 0$. The function $\mathcal{V}$ defined in (4.8) satisfy the conditions of Definition 2.10 on $x_0(\omega)$. For,

i. the function $x \longmapsto \|x_0(\omega) - x\|_{\kappa,\omega}$ is continuous and the function $\omega \longmapsto \|x_0(\omega) - x\|_{\kappa,\omega}$ is measurable.

ii. For every ω, $\lim_{\|x\|_\omega \longrightarrow \infty} \|x_0(\omega) - x\|_{\kappa,\omega} = \infty$.

iii. $\mathcal{V}(\omega, x_0(\omega)) = \|x_0(\omega) - x_0(\omega)\|_{\kappa,\omega} = 0$, and for every $\omega, \mathcal{V}(\omega, x) = \|x_0(\omega) - x\|_{\kappa,\omega} > 0$.

iv. By (4.9)

$$\begin{aligned}\mathcal{V}(\theta_t\omega, \varphi(t,\omega)x) &= \|\Phi(t,\omega)(x_0(\omega) - x)\|_{\kappa,\omega} \\ &\le \|\Phi(t,\omega)\|_{\kappa,\omega,\theta_t\omega} \mathcal{V}(\omega, x) \\ &\le e^{(\lambda+\kappa)t} \mathcal{V}(\omega, x) < \mathcal{V}(\omega, x) \ \ for\ t \ge 0.\end{aligned}$$

Hence $\mathcal{V}$ is strictly decreasing alongside the trajectory of φ. So, $\mathcal{V}$ is a stochastic Lyapunov function for $A(\omega) = \{x_0(\omega)\}$, and by Theorem 3.8 $A(\omega) = \{x_0(\omega)\}$ is asymptotically stable.

The following three figures illustrate the effect of different noise intensities γ on the trajectory of a 2-dimensional affine stochastic differential system of the form

$$dx = (Ax + b)dt + \gamma dW(t)$$

where $A \in \mathbb{R}^{2\times 2}$ is a constant matrix, $b \in \mathbb{R}^2$ is a bias vector, and $W(t)$ denotes a 2-dimensional Wiener process. The system was simulated using the 4th-order Runge-Kutta method for the deterministic part and additive Gaussian noise for the stochastic part.

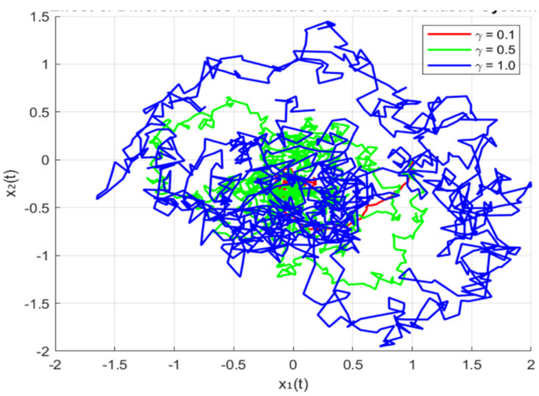

Figure 4: Effect of Noise intensities on Affine Stochastic System with $\gamma = [0.1, 0.5, 1.0], A = [-1, 2; -2, -1]; b = [0.5; -0.3]$.

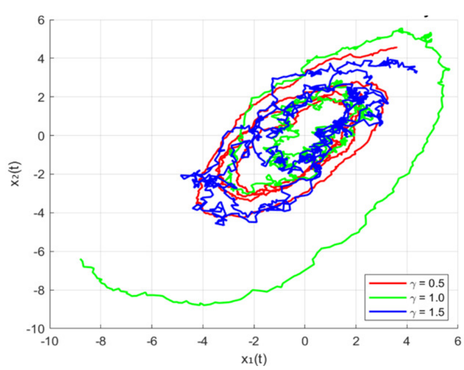

Figure 5: Effect of Noise intensities on Affine Stochastic System with $\gamma = [0.5, 1.0, 1.5], A = [-2, 3; -3, 2]; b = [0.5; -0.3]$.

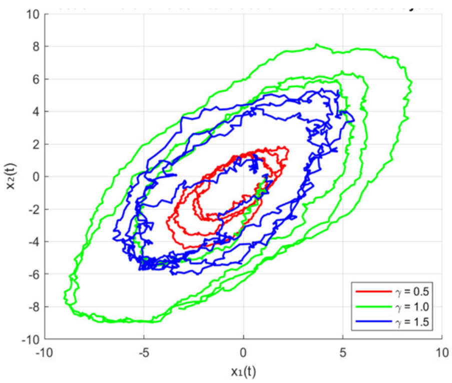

Figure 6: Effect of Noise intensities on Affine Stochastic System with $\gamma = [0.5, 1.0, 1.5], A = [-2, 3; -3, 2]; b = [0.5; -0.3]$.

The plot in Figure 7 shows the evolution of the Lyapunov function $V(t) = \|x(t)\|^2$ for a linear stochastic system integrated using the Runge-Kutta method. The decay of $V(t)$ over time reflects the contracting behavior of the system, indicating asymptotic stability despite stochastic perturbation.

The plot in Figure 8 comparison demonstrates how the Lyapunov norm $V(t) = \|x(t)\|^2$ evolves under different noise intensities. In the low-noise case ($\gamma = 0.1$), the system shows clear convergence and stability. Under high noise ($\gamma = 1.0$), the fluctuations dominate and stability deteriorates, indicating the critical role of stochastic intensity in system behavior.

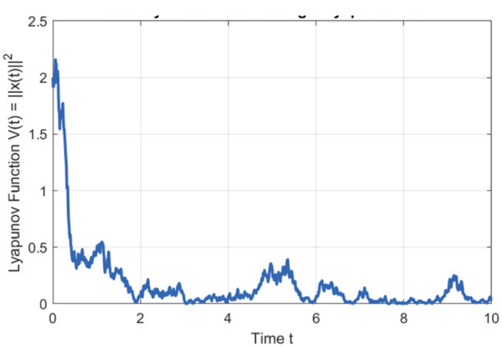

Figure 7: Evolution of the Lyapunov norm $V(t) = \|x(t)\|^2$ under stochastic dynamics (RK4 integration).

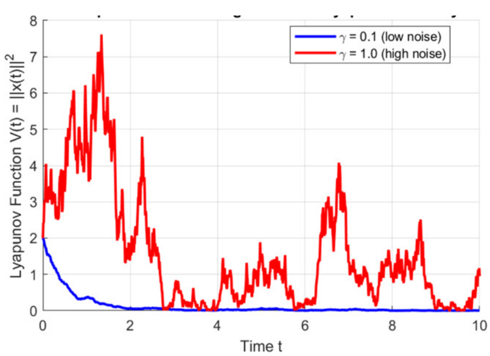

Figure 8: Evolution of the Lyapunov norm $V(t) = \|x(t)\|^2$ under low and high stochastic noise using RK4 integration.

5. Conclusion

The most important implications in this paper are summarized as follows:

1. If the forward (backward) semi-trajectory is pre-compact, then the Lyapunov function for a gradient-like RDS is invariant. Also, the Lyapunov function for a gradient-like RDS is invariant at the non-wandering point.

2. Every gradient-like asymptotically compact RDS with a bonded set of equilibrium is compact dissipative and admits the maximal attractor for any bounded random set.

3. The stochastic Lorenz system defines a random dynamical system that is gradient-like with respect to the Lyapunov function $\mathcal{V}(x, y, z) := rx^2 + \sigma y^2 + \sigma(z - 2r)^2$.

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