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23-modular spin characters of S_q

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ABSTRACT: Character theory is a significant area of mathematics with diverse applications across multiple disciplines, such as computational chemistry, theoretical physics, coding theory, spectral graph theory, and information theory. One of the objects of interest in this field is the double cover $\tilde{S}_q$ of the symmetric group S'_q where q represents the degree of the group. The double cover introduces a new structural feature: a central involution b, which is an element of order 2. The irreducible characters of $\tilde{S}_q$ are divided into two types: ordinary characters and spin characters. This distinction is based on whether the involution b lies in the kernel of the character (ordinary) or not (spin). The purpose of this study is to compute the modular irreducible spin characters of $\tilde{S}_q$ for specific degrees q, particularly in the range $23 \leq q \leq 26$ These computations are conducted over an algebraically closed field with characteristic $p = 23$. The results provide important insights into the modular representation theory of double covers of symmetric groups and contribute to a deeper understanding of their spin character structure in specific modular settings.

Key Words:: Brauer trees, decomposition matrices, spin characters, spin blocks.

Contents

1. Introduction

In [1896], George Frobenices penned the notation of group character and much of representation theory. R. Brauce classified simple group by group character theoretic results. Now-a-days, character theory is a popular area of mathematics employed in applied sciences and especially problems groups. The knowledge of irreducible character of a group in a powerful tool of study the group structure. Assume that G is a finite group, $GL(r, C)$ is a general linear group of degree r and C, the field of complex numbers of characteristics zero and p is a prime nummber. Representations of G over C are regarded as ordinary

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representations (or representation) while over of algebraically closed field of characteristic p are termed as p-modular representations. Representation of G is almost equal to representation ρ of G over a finite extension of the rationals and in addition, over a maximal subring S of the latter field not containing p. Furthermore, S has a unique maximal ideal T such that $E = S/T$ is a field of characteristic p. Thus, if $\rho(G) \subseteq GL(r, C)$, for some $r \in N$ = set of naturals, then, representation $\overline{\rho} : G \to GL(r, E)$ is a normally called the reduction of ρ modular p. $\overline{\rho}$ may not need to necessarily be irreducible even if $\overline{\rho}$ is irreducible. The multiplicities of irreducible of G over $\overline{E}$, the algebraic closure of E as a composition factor of $\overline{\rho}$ are known as the decomposition number, they are numerically invariant that remarkably connect the characteristic zero and characteristic p representations of G.
This work is in continuous to our investigation of the decomposition matrices of finite groups. In this article, we compute the 23-modular irreducible spin characters of the symmetric group S_q, for $23 \leq q \leq 26$ in terms of decomposition matrix. The entries decomposition matrix of a finite group in prime characteristic p are non-negative integers usually called decomposition numbers, records the multiplicities of its p-modular irreducible characters as composition factor of the reduction modulo p of its irreducible characters in characteristic zero. For more detail about decomposition matrix [see, 4, 17]. Section 2 , contains theory of spin representations of the symmetric group.

2. Spin representations (character)of the group S_q

The spin symmetric group $\tilde{S_q}$ of order $2.|S_q|$, is the double cover of the symmetric group S_q, this means that there is a non-split exact sequence $1 \to< b >\to \tilde{S_q} \to S_q \to 1$ where $< b : b^1 = 1 >$ is a central subgroup in $\tilde{S_q}$ [1].
The representations of $\tilde{S_q}$ are distributed in two categoties as the central element $b \in \tilde{S_q}$ acts on each irreducible representation by ± 1.
1. An irreducible representation of $\tilde{S_q}$ is said to be an ordinary representation, if b corresponds to 1. The irreducible characters of these representations are parameterized by partition λ of q, written $\lambda \vdash q$ and is signified by $[\lambda]$.
2. An irreducible representation of $\tilde{S_q}$ is said to be spin, if b corresponds to -1. The irreducible spin characters corresponding to these representations are parameterized by partitions having different components, known as bar partitions written $\lambda \succ q$ and is indicated by $< \lambda >$. Furthermore, if $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...\lambda_s)$, then for $q - s$ is even there is self-associate spin characters $< \lambda > = < \lambda >'$, while for $q - s$ is odd, there is a pair of distinct associate spin characters $< \lambda >$ and $< \lambda >'$. The degree of these spin characters is given by:

$$2^{[\frac{q-s}{2}]} \frac{q!}{\prod_{i=1}^{s} \lambda_i!} \prod_{1<i<j<s} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

where $[x]$ denote the greatest integer less than or equal to x [7]

3. General results on modular spin characters of S_q

In [9] Morris and Yasir introduced the concept to compute decomposition number for the irreducible spin characters of finite symmetric group S_q Morris branching rule:
(i) $[\lambda]$ is a decomposition factor of $[\omega] \uparrow^{\tilde{s_n}}_{\tilde{s_{n-1}}}$ if and only if (the skew diagram of) λ is obtained by adding a node to (the diagram) ω.
(ii) $[\lambda]$ is a decomposition factor of $[\mu] \uparrow^{\tilde{s_n}}_{\tilde{s_{n-1}}}$ if and only if (the skew diagram of) λ is obtained by removing a node from (the diagram) μ.

Theorem 3.1 *Let G be a finite group and p be the prime divisor then, we have*
(i) *The number of p-regular classes of G is segnal to the number of its p-modular irreducible characters.*
(ii) *The order of p-modular irreducible characters of the group divides the order of the group.*

3.1. General result

let Y be the number of p-conjugate characters to the irreducible ordinary character χ of a group G and let Y be a $s-block$ of a group G of defect one, then [JK, Hu]

1. There is a positive integer Q such that it is possible to divide the block Y of irreducible ordinary characters into two separate sets namely:

$$Y_1 = \{\chi \in Y \mid bdeg(\chi) \equiv Q(modp^{\alpha})\}$$

$$Y_2 = \{\chi \in Y \mid bdeg(\chi) \equiv -Q(modp^{\alpha})\}$$

2. χ_1 and χ_2 have no irreducible character in common if they are not p-conjugate characters and belongs to the same set Y_1 or Y_2 above.

3. There exists an irreducible ordinary character χ_2 in Y_2 for every irreducible ordinary character χ_1 in Y_1 such that they share one irreducible modular character with multiplicity 1.

4. The decomposition number of the block Y is either 1 or 0.

3.2. General result(spin block of symmetric groups)

For $\lambda \succ q$ and p be an odd prime, λ_p denotes the $\overline{p}$-core $(p-barcore)$ of λ [see, Mo2]. λ_p is obtained from λ by the inductive removal of p-bars, where the removal of a p-bar is a certain operation on the parts of λ. The distribution of the spin characters of S_q into spin p-blocks is described as follows;

1. If $\lambda = \lambda_p$, then $< \lambda >$ is of p-defect zero (or trivial defect) and form its own spin p-block. **2.** If $\lambda, \mu > q$, and $\lambda \neq \lambda_p$, then $< \lambda >$ and $< \mu >$ are in the same spin p-block of S_n if and only if $\lambda_p = \mu_p$ in particular associate spin characters $< \lambda >$ and $< \lambda >'$ are in the same p-block.

3.3. General result

If A is a projective indecomposable character of a group G with order $t = sp^{\alpha}$, where $(p, s) = 1$, Then $\deg(A) \equiv o(\text{zero}) \pmod{p^{\alpha}}$ [5].

3.4. General result

If all entries in C are divisible by a positive integer q and A is a projective character of G, then A/q is also a projective character of G [4].

3.5. General result

With non-negative integer coefficient, every spin modular (or projective) character of G can be written as linear combination of the irreducible modular spin (or projective indecomposable spin) characters.

3.6. General result

Let p be an odd prime and $\lambda \succ q$ not a p-bar core. Let Y be the spin p-block containing $< q >$ such that $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...\lambda_s)$. Then

1. All modular spin characters in the block Y are self-associate If $q - s - s_0$ is an even and non-self-associate, if $q - s - s_0$ is an odd.

Here s is the length of λ and s_0 the number of parts of λ divisible by p (see, [13].)

3.7. General result

These useful results are due to Wales.[18].
Let p be an odd prime number.

a: If q is odd then
a_1**:** If $p|q$, then $< q >=: \phi < q > +\phi < q >'$, where $\phi < q >$ and $\phi < q >'$ are distinct irreducible modular characters of degree $2^{\frac{q-3}{2}}$.
a_2**:** If $p \nmid q$, then $< q >=: \phi < q >$ are distinct irreducible modular characters of degree $2^{\frac{q-3}{2}}$.
a_3**:** If $p \nmid q$ and $p \dagger (q-1)$, then $< q-1, 1 >=: \phi < q-1, 1 >$ and $< q-1, 1 >' = \phi < q-1, 1 >'$ are distinct irreducible modular characters of degree $2^{\frac{q-3}{2}}(q-2)$.
b: If q is even then
b_1**:** If $p \nmid q$, then $< q >=: \phi < q >$, where $< q >'$ and $\phi < q >'$ one distinct irreducible modular characters of degree $2^{\frac{q-3}{2}}$.
b_2**:** If $p \nmid q$ and $p \dagger (q-1)$, then $< q-1, 1 >=: \phi < q-1, 1 >$ is distinct irreducible modular characters of degree $2^{\frac{q-3}{2}}(q-2)$ [18].

Furthermore, $\uparrow G(\downarrow G)$ denote the induction from (restriction to) a subgroup of G.

(I) Di := The projective indecomposable spin characters of S23.
(II) Ej := The projective indecomposable spin characters S24.
(III) Hk := The projective indecomposable spin characters of S25.
(III) The symbol $Irsc(G)$ is the set of all irreducible spin characters of the group G.
(IV) The symbol $Imsc(G)$ is the set of all irreducible p- modular spin characters of G.
(V) The symbol $Indesc(G)$ is the set of all projective P-indecomposable spin characters of G.

4. 23-Modular Spin Characters of Symmetric Groups

In this section, we compute the decomposition matrices for spin characters of the symmetric group S_q, $23 \leq q \leq 26$, relative to the prime p = 23. In each case, we find 23-regular, α-regular classes and irreducible modular spin characters of S_n. Spin blocks means spin 23-block of S_q from now onwards. The spin block containing the principal spin characters, usually denoted by $< q >$ of S_q is named principal spin block.

4.1. 23-Modular Spin Characters of S_{23}

There are 155 elements in $Irsc(S_{23})$ and $|Irmsc(S_{23})| = 154$, since 154 α-regular 23-regular classes occur in S_{23}. The size and respectively the rank of the 23-decomposition matrix for spin characters of S_{23} is 155×154 and 154. There exists 133 spin blocks in all in S_{23}. The 132 elements of $Irsc(S_{23})$ each form its own spin block $Y_i (2 \leq i \leq 132)$ has trivial defect group and hence these elements of $Irsc(S_{23})$ are irreducible modular (respectively projective indecomposable) spin characters of S_{23}.
The principal spin block, say Y_1 of S_{23} has defect one with empty 23-bars core, contains 23-elements of $Irsc(S_{23})$ vizly; $< 23 >^*$, $< 22, 1 >^{\pm}$, $< 21, 2 >^{\pm}$, $< 20, 3 >^{\pm}$, $< 19, 4 >^{\pm}$, $< 18, 5 >^{\pm}$, $< 17, 6 >^{\pm}$, $< 16, 7 >^{\pm}$, $< 15, 8 >^{\pm}$, $< 14, 9 >^{\pm}$, $< 13, 10 >^{\pm}$, $< 11, 12 >^{\pm}$.
The decomposition matrix of every irreducible modular spin characters of the principal block which are non self-associate $[GR.2.6]$ is denoted by $D^{B_1}(s_{23})$. The required complete 23-decomposition matrix of the group S_{23} is

$$D_{23}(s_{23}) = \begin{bmatrix} D^{Y_1}(s_{23}) & & & & & & \\ & D^{Y_2}(s_{23}) & & & & & \\ & & D^{Y_3}(s_{23}) & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & D^{Y_{132}}(s_{23}) \end{bmatrix} = \oplus_{i=1}^{132}[D_{23}^{Y_i}(s_{23})]$$

Resultantly, the Brauer tree of S_{23}, for p = 23 by [MY] is
$< 12,11 > - < 13,10 >' - \hookleftarrow < 14,9 >' - < 15,8 >' - < 16,7 >' - < 17,6 >' - < 18,5 >' - < 19,4 >' - \hookleftarrow - < 20,3 >' - < 21,2 >' - < 22,1 >' \hookleftarrow < 23 >^* \hookrightarrow < 22.1 > - < 21.2 > - < 20,3 > \hookrightarrow - < 19,4 > - < 18,5 > - < 17,6 > - < 16,7 > - < 15,8 > - < 14,9 > \hookrightarrow - < 13,10 > - < 12,11 >$

4.2. 23-Modular Spin Characters of S_{24}

Of the 183 elements of $Irsc(S_{24})$, 23 are in the principal spin block, say Y_1, of defect one with 23-bar core $< 1 >$, while all the remaining elements of $Irsc(S_{24})$, each from its own spin block, say Y_i; $(2 \leq i \leq 161)$ of defect zero. As S_{24} possesses 182 α-regular 23-regular classes, so$|Irmsc| = 182$. Hence, the size and rank respectively of the required 23-decomposition matrix of S_{24} is 183×182 and 182. Furthermore, we have 161 spin blocks of S_{24} by [GR.2.2]. The elements of $Irsc(S_{24})$ of defect zero are irreducible modular (respectively projective indecomposable) spin characters of the group S_{24}. The principal spin block Y_1 contains:
$< 24 >^{\pm}$, $< 21,1 >^*$, $< 21,2,1 >^{\pm}$, $< 20,3,1 >^{\pm}$, $< 19,4,1 >^{\pm}$, $< 18,5,1 >^{\pm}$, $< 17,6,1 >^{\pm}$, $< 16,7,1 >^{\pm}$, $< 15,8,1 >^{\pm}$, $< 14,9,1 >^{\pm}$, $< 13,10,1 >^{\pm}$ and $< 12,11,1 >^{\pm}$.
Every irreducible 23-modular spin characters in Y_1 are non-self-associate. The decomposition matrix $DB_{23}^1(S_{24})$ corresponds to spin blocks Y_1 consists of 23 rows and 22 columns. The projective indecomposable spin characters of S_{23} are as in the table below.

$D_1 =< 23 >^* + < 22,1 >$	$D_2 =< 23 >^* + < 22,1 >'$
$D_3 =< 22,1 > + < 21,2 >$	$D_4 =< 22,1 >' + < 21,2 >'$
$D_5 =< 21,2 > + < 20,3 >$	$D_6 =< 21,2 >' + < 20,3 >'$
$D_7 =< 20,3 > + < 19,4 >$	$D_8 =< 20,3 >' + < 19,4 >'$
$D_9 =< 19,4 > + < 18,5 >$	$D_{10} =< 19,4 >' + < 18,5 >'$
$D_{11} =< 18,5 > + < 17,6 >$	$D_{12} =< 18,5 >' + < 17,6 >'$
$D_{13} =< 17,6 > + < 16,7 >$	$D_{14} =< 17,6 >' + < 16,7 >'$
$D_{15} =< 16,7 > + < 15,8 >$	$D_{16} =< 16,7 >' + < 15,8 >'$
$D_{17} =< 15,8 > + < 14,9 >$	$D_{18} =< 15,8 >' + < 14,9 >'$
$D_{19} =< 14,9 > + < 13,10 >$	$D_{20} =< 14,9 >' + < 13,10 >'$
$D_{21} =< 13,10 > + < 12,11 >$	$D_{22} =< 13,10 >' + < 12,11 >'$

Now, apply $(r, \overline{r})$-induction method given in [9], to the elements of $pindesc(S_{23})$ yields the projective decomposable spin characters of S_{24} given by:

$k_1 =: D_1 \uparrow_{s_{24}}^{1,0} =< 24 > + < 24 > +2 < 23,1 >^*$
$E_3 =: D_3 \uparrow_{s_{24}}^{1,0} =< 23,1 >^* + < 21,2,1 >$
$E_4 =: D_4 \uparrow_{s_{24}}^{1,0} =< 23,1 >^* + < 21,2 >^*$
$E_5 =: D_5 \uparrow_{s_{24}}^{1,0} =< 21,2,1 > + < 20,3,1 >^*$
$E_6 =: D_6 \uparrow_{s_{24}}^{1,0} =< 21,2,1 >' + < 20,3,1 >'$
$E_7 =: D_7 \uparrow_{s_{24}}^{1,0} =< 20,3,1 >' + < 19,4,1 >'$
$E_8 =: D_8 \uparrow_{s_{24}}^{1,0} =< 20,3,1 >' + < 19,4,1 >'$
$E_9 =: D_9 \uparrow_{s_{24}}^{1,0} =< 19,4,1 >' + < 18,5,1 >'$
$E_{10} =: D_{10} \uparrow_{s_{24}}^{1,0} =< 19,4,1 >' + < 18,5,1 >'$
$E_{11} =: D_{11} \uparrow_{s_{24}}^{1,0} =< 18,5,1 >' + < 17,6,1 >'$
$E_{12} =: D_{12} \uparrow_{s_{24}}^{1,0} =< 18,5,1 >' + < 17,6,1 >'$
$E_{13} =: D_{13} \uparrow_{s_{24}}^{1,0} =< 17,6,1 >' + < 16,7,1 >'$
$E_{14} =: D_{14} \uparrow_{s_{24}}^{1,0} =< 17,6,1 >' + < 16,7,1 >'$
$E_{15} =: D_{15} \uparrow_{s_{24}}^{1,0} =< 16,7,1 >' + < 15,8,1 >'$
$E_{16} =: D_{16} \uparrow_{s_{24}}^{1,0} =< 16,7,1 >' + < 15,8,1 >'$
$E_{17} =: D_{17} \uparrow_{s_{24}}^{1,0} =< 15,8,1 >' + < 14,9,1 >'$
$E_{18} =: D_{18} \uparrow_{s_{24}}^{1,0} =< 15,8,1 >' + < 14,9,1 >'$
$E_{19} =: D_{19} \uparrow_{s_{24}}^{1,0} =< 14,9,1 >' + < 13,10,1 >'$
$E_{20} =: D_{20} \uparrow_{s_{24}}^{1,0} =< 14,9,1 >' + < 13,10,1 >'$
$E_{21} =: D_{21} \uparrow_{s_{24}}^{1,0} =< 13,10,1 >' + < 12,11,1 >'$
$E_{22} =: D_{22} \uparrow_{s_{24}}^{1,0} =< 13,10,1 >' + < 12,11,1 >'$

By $[GR.2.5(a_2)], < 24 > \neq < 24 >' \in Irmsc(S_{24})$ which are denoted by $\phi < 24 >$ and $\phi < 24 >'$ respectively. The self-associate spin character $< 23,1 >^*$ of s_{24} contains $\phi < 24 >$ and $\phi < 24 >'$ with the same multiplicity. Hence k_1, must splits into two projective indecomposable spin characters of S_{24} [GR.2.7].

$$E_1 =:< 24 > + < 23,1 >^*$$

$$E_2 =:< 24 >' + < 23,1 >^*$$

Furthermore, all $E_i (3 \leq i \leq 22)$ are projective indecomposable spin characters of S_{24} by [GR.2.3]. Thus the decomposition matrix for S is indeed of the form:

$$D_{23}(s_{24}) = \begin{bmatrix} D_{23}^{Y_1}(s_{24}) & & & & & & \\ & D_{23}^{Y_1}(s_{24}) & & & & & \\ & & D_{23}^{Y_1}(s_{24}) & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & . & \\ & & & & & & D_{23}^{Y_{i16}}(s_{24}) \end{bmatrix} = \oplus_{i=1}^{16} [D_{23}^{Y_i}(s_{24})]$$

Consequently, the Brauer tree is [6, page.2] Because

$$\begin{cases} \langle 24\rangle^{\pm},\ \langle 21,2,1\rangle^{\pm},\ \langle 19,4,1\rangle^{\pm}, & \text{if } \equiv 2 \pmod{23} \\ \langle 17,6,1\rangle^{\pm},\ \langle 15,8,1\rangle^{\pm},\ \langle 13,10,1\rangle^{\pm}, & \\ \langle 23,1\rangle^{*},\ \langle 20,3,1\rangle^{\pm},\ \langle 18,5,1\rangle^{\pm}, & \text{if } \equiv -2 \pmod{23} \\ \langle 16,7,1\rangle^{\pm},\ \langle 14,9,1\rangle^{\pm},\ \langle 12,11,1\rangle^{\pm} & \end{cases}$$

Where $D_{23}^{Y_1}(s_{24})$ is given below in the table.

Brauer trees for the principle spin of the group S_{24}

$\langle 24\rangle$	1																						
$\langle 24\rangle'$		1																					
$\langle 23,1\rangle^*$	1	1	1	1																			
$\langle 21,2,1\rangle$			1		1																		
$\langle 21,2,1\rangle'$				1		1																	
$\langle 20,3,1\rangle$					1		1																
$\langle 20,3,1\rangle'$						1		1															
$\langle 19,4,1\rangle$							1		1														
$\langle 19,4,1\rangle'$								1		1													
$\langle 18,5,1\rangle$									1		1												
$\langle 18,5,1\rangle'$										1		1											
$\langle 17,6,1\rangle$											1		1										
$\langle 17,6,1\rangle'$												1		1									
$\langle 16,7,1\rangle$													1		1								
$\langle 16,7,1\rangle'$														1		1							
$\langle 15,8,1\rangle$															1		1						
$\langle 15,8,1\rangle'$																1		1					
$\langle 14,9,1\rangle$																	1		1				
$\langle 14,9,1\rangle'$																		1		1			
$\langle 13,10,1\rangle$																			1		1		
$\langle 13,10,1\rangle'$																				1		1	
$\langle 12,11,1\rangle$																					1		1
$\langle 12,11,1\rangle'$																						1	
	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	

4.3. 23-Modular Spin Characters of S_{25}

The symmetric group S_{25} possesses 199 α-regular 23-regular classes out of its 201 α-regular classes. Hence $|Irsc(s_{25})| = 201$ and $|Irmsc(s_{25})| = 199$, therefore the required decomposition has size 201×199 and rank 199. The blocks distribution [see, GR. 2.2] of the spin characters of S_{25} implies that o-defect spin blocks denoted by $Y_i (2 \leq i \leq 188)$, each form its own spin block are 188 while the principal spin block say Y_1 of S_{25} has a cyclic defect group of order 23 with 23-bar core $< 2 >$ contains: $< 25 >^*, < 23, 2 >^*, < 22, 2, 1 >^*, < 20, 3, 2 >^*, < 19, 4, 2 >^*, < 18, 5, 2 >^*, < 17, 6, 2 >^*, < 16, 7, 2 >^*$ $, < 15, 8, 2 >^*, < 14, 9, 2 >^*, < 13, 10, 2 >^*$ and $< 12, 11, 2 >^*$. The size of the decomposition matrix of Y_1 is 13×11. All this toutlers that the complete 23-decomposition matrix of S_{25} is the diagonally direct products of 189 blocks decomposition matrices $D^{Y_1}(S_{25}), (1 \leq i \leq 189)$ of S_{25}. Now, we compute $D^{Y_1}(S_{25})$ by using (2, 22)-inducing method, to the elements of $pindesc(S_{24})$ to obtain the projective decomposable spin characters of S_{25} as follows:

$H_1 =: E_1 \uparrow_{s_{25}} =< 25 >^* + < 23, 2 > + < 23, 2 >'$
$H_2 =: E_3 \uparrow_{s_{25}} =< 23, 2 > + < 23, 2 >' + < 22, 2, 1 >^*$
$H_3 =: E_5 \uparrow_{s_{25}} =< 22, 2, 1 >^* + < 20, 3, 2 > + < 23, 2 >^*$
$H_4 =: E_7 \uparrow_{s_{25}} =< 20, 3, 2 >^* + < 19, 4, 2 >^*$
$H_5 =: E_9 \uparrow_{s_{25}} =< 19, 4, 2 >^* + < 18, 5, 2 >^*$
$H_6 =: E_{11} \uparrow_{s_{25}} =< 18, 5, 2 >^* + < 17, 6, 2 >^*$
$H_7 =: E_{13} \uparrow_{s_{25}} =< 17, 6, 2 >^* + < 16, 7, 2 >^*$
$H_8 =: E_{15} \uparrow_{s_{25}} =< 16, 7, 2 >^* + < 15, 8, 2 >^*$
$H_9 =: E_{17} \uparrow_{s_{25}} =< 15, 9, 2 >^* + < 14, 9, 2 >^*$
$H_{10} =: E_{19} \uparrow_{s_{25}} =< 14, 9, 2 >^* + < 13, 10, 2 >^*$
$H_{11} =: E_{21} \uparrow_{s_{25}} =< 13, 10, 2 >^* + < 12, 11, 2 >^*$

It is easy to check that each H_i's $(1 \leq i \leq 11)$ are projective indecomposable spin characters of S_{25} [see, GR.2.3]

$\langle 25\rangle^*$	1										
$\langle 23,2\rangle^*$	1	1									
$\langle 22,2,1\rangle^*$		1	1								
$\langle 20,3,2\rangle^*$			1	1							
$\langle 19,4,2\rangle^*$				1	1						
$\langle 18,5,2\rangle^*$					1	1					
$\langle 17,6,2\rangle^*$						1	1				
$\langle 16,7,2\rangle^*$							1	1			
$\langle 15,8,2\rangle^*$								1	1		
$\langle 14,9,2\rangle^*$									1	1	
$\langle 13,10,2\rangle^*$										1	1
$\langle 12,11,2\rangle^*$											1
	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}

Brauer trees of the group S_{25}

The Brauer Tree for the spin block Y_i's as follows, [see, 6]
$< 25 >^* - < 23,2 >=< 23,2 >' - < 22,2,1 >^* - < 20,3,2 >^* - < 19,4,2 >^* - < 18,5,2 >^* \hookrightarrow < 17,6,2 >^* - < 16,7,2 >^* - < 15,8,2 >^* - < 14,9,2 >^* - < 13,10,2 >^* - < 12,11,2 >^*$
since

$$\deg \begin{cases} \langle 25\rangle^{\pm},\ \langle 22,2,1\rangle^{\pm},\ \langle 19,4,2\rangle^{\pm},\ \langle 17,6,1\rangle^{\pm}, & \text{if } \equiv 2 \pmod{23} \\ \langle 15,8,1\rangle^{\pm},\ \langle 13,10,1\rangle^{\pm}, & \\ \langle 23,2\rangle^{\pm},\ \langle 20,3,2\rangle,\ \langle 18,5,1\rangle,\ \langle 16,7,2\rangle, & \text{if } \equiv -2 \pmod{23} \\ \langle 14,9,1\rangle,\ \langle 12,11,1\rangle^* & \end{cases}$$

4.4. 23-Modular spin characters of S_{26}

Here $|Irsc(S_{26})| = 242$ and $|Irmsc(S_{26})| = 239$, since there are 239 α-regular 23-regular classes in the symmetric group S_{26}. By (GR.2.2), the elements of $|Irsc(S_{26})|$ are distributed over 209 spin blocks. We note that 206 elements of $|Irsc(S_{26})|$ have defect zero. Therefore each of them from its own spin block and are denoted by $Y_1, Y_2, Y_3, ...Y_{209}$. Thus we have found 206 elements of $|Imsc(S_{26})|$ but 33 elements of $|Imsc(S_{26})|$ have to be found yet. We conclude that the size of required decomposition matrix of S_{26} is 242×239 and has rank 239. The twenty three elements of $Irsc(S_{26})$ in the principal spin block Y_1 with 23-bar core $< 3 >$ and defect one are:
$< 26 >^{\pm}$, $< 23,3 >^*$, $< 22,3,1 >^{\pm}$, $< 21,3,2 >^{\pm}$, $< 19,4,3 >^{\pm}$, $< 18,5,3 >^{\pm}$, $< 17,6,3 >^{\pm}$, $< 16,7,3 >^{\pm}$, $< 15,8,3 >^{\pm}$, $< 14,9,3 >^{\pm}$, $< 13,10,3 >^{\pm}$, $< 12,11,3 >^{\pm}$.

Every irreducible 23-modular spin characters in the principal spin block Y_1 of S_{26} are non self associate [see, [GR.2.6(b)].
Furthermore, there exists a non-principal spin block, say Y_2 of the symmetric group S_{26} with 23-bar core $< 21 >$ having full defect and contains 13 elements of $Irsc(S_{26})$ vizly;
$< 25,1 >^*$, $< 24,2 >^*$, $< 23,2,1 >^{\pm}$, $< 20,3,2,1 >^*$, $< 19,4,2,1 >^*$, $< 18,5,2,1 >^*$, $< 17,6,2,1 >^*$, $< 16,7,2,1 >^*$, $< 15,8,2,1 >^*$, $< 14,9,2,1 >^*$, $< 13,10,2,1 >^*$, $< 12,11,2,1 >^*$.

Every irreducible 23-modular spin characters in the non principal block Y_2 of S26 are self conjugates [see, 2.6(a)].

4.4.1. Decomposition Matrix for the Non-Principal Spin Block. The Brauer tree for the non-principal spin block Y_2 is:
$< 25,1 >^* - < 24,2 >^* - < 23,2,1 >=< 23,2,1 >' - < 20,3,2,1 >^* - < 19,4,2,1 >^* \hookrightarrow$

$- < 18,5,2,1 >^* - < 17,6,2,1 >^* - < 16,7,2,1 >^* - < 15,8,2,1 >^* - < 14,9,2,1 >^* \hookrightarrow - < 13,10,2,1 >^* - < 12,11,2,1 >^*$.

Proof: We observe that

$$\deg \begin{cases} \langle 25,1\rangle^*,\ \langle 23,2,1\rangle^{\pm},\ \langle 19,4,2,1\rangle^*,\ \langle 17,6,2,1\rangle^*, & \text{if } \equiv 2 \pmod{23} \\ \langle 15,8,2,1\rangle^*,\ \langle 13,10,2,1\rangle^*, & \\ \langle 24,2\rangle^*,\ \langle 20,3,2,1\rangle^*,\ \langle 18,5,2,1\rangle^*,\ \langle 16,7,2,1\rangle^*, & \text{if } \equiv -2 \pmod{23} \\ \langle 14,9,2,1\rangle^*,\ \langle 12,11,2,1\rangle^* & \end{cases}$$

Now, apply the (1, 0)-inducing method to the elements of $Pindesc(S_{25})$ to get:

$c_1 =: H_1 \uparrow_{s_{26}} = < 25,1 >^* + 2 < 24,4 >^* + < 23,2,1 > + < 23,2,1 >'$
$c_2 =: H_2 \uparrow_{s_{26}} = 2[< 24,2 >^* + < 23,2,1 > + < 23,2,1 >'$
$c_3 =: H_3 \uparrow_{s_{26}} = < 23,2,1 > + < 23,2,1 >' + < 20,3,2,1 >^*$
$c_4 =: H_4 \uparrow_{s_{26}} = < 20,3,2,1 >^* + < 19,4,2,1 >^*$
$c_5 =: H_5 \uparrow_{s_{26}} = < 19,4,2,1 >^* + < 18,5,2,1 >^*$
$c_6 =: H_6 \uparrow_{s_{26}} = < 18,5,2,1 >^* + < 17,6,2,1 >^*$
$c_7 =: H_7 \uparrow_{s_{26}} = < 17,6,2,1 >^* + < 16,7,2,1 >^*$
$c_8 =: H_8 \uparrow_{s_{26}} = < 16,7,2,1 >^* + < 15,8,2,1 >^*$
$c_9 =: H_9 \uparrow_{s_{26}} = < 15,9,2,1 >^* + < 14,9,2,1 >^*$
$c_{10} =: H_{10} \uparrow_{s_{26}} = < 14,9,2,1 >^* + < 13,10,2,1 >^*$
$c_{11} =: H_{11} \uparrow_{s_{26}} = < 13,10,2,1 >^* + < 12,11,2,1 >^*$

We obtaining the matrix $R_{23}^{Y_2}(S_{26})$ corresponding to the non-principal block Y_2 of S_{26}:

$R_{23}^{Y_2}(S_{26})$	$< 25,1 >$	1										
	$< 24,2 >$	2	2									
	$< 23,2,1 >$	1	2	1								
	$< 23,2,1 >'$	1	2	1								
	$< 20,3,2,1 >$			1	1							
	$< 19,4,2,1 >$				1	1						
	$< 18,5,2,1 >$					1	1					
	$< 17,6,2,1 >$						1	1				
	$< 16,7,2,1 >$							1	1			
	$< 15,8,2,1 >$								1	1		
	$< 14,9,2,1 >$									1	1	
	$< 13,10,2,1 >$										1	1
	$< 12,11,2,1 >$											1
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}

We may divide the column c_2 by 2 and we obtain $c_2 =: \frac{c_2}{2}$ obviously $c_2' \in Pindesc(S_{26})$ [see, 2.4]. Furthermore, $< 25,2 > \in Pindesc(S_{27})$ being an element of $Irsc(S_{27})$ of defect zero and

$$< 25,2 > \downarrow_{2,2} S_{26} = < 25,1 >^* + < 24,2 >^* \in Pindesc(S_{26})$$

Hence c_2' must be subtracted from c_1 [see, 2.3]. Replace the column c_2 by c_2' and then subtracting the column c_2' from c_1 in matrix $R_{23}^{Y_2}(S_{26})$, we get the decomposition matrix $D_{23}^{Y_2}(S_{26})$ for the non-principal spin block of S_{26} and as given by:

$D_{23}^{B_2}(S_{26})$	Representation	I_1	I_2	I_3	I_4	I_5	cI_6	I_7	I_8	I_9	I_{10}	I_{11}
	$< 25, 1 >^*$	1										
	$< 24, 2 >^*$	2	2									
	$< 23, 2, 1 >^*$	1	2	1								
	$< 23, 2, 1 >'$	1	2	1								
	$< 20, 3, 2, 1 >^*$			1	1							
	$< 19, 4, 2, 1 >^*$				1	1						
	$< 18, 5, 2, 1 >^*$					1	1					
	$< 17, 6, 2, 1 >^*$						1	1				
	$< 16, 7, 2, 1 >^*$							1	1			
	$< 15, 8, 2, 1 >^*$								1	1		
	$< 14, 9, 2, 1 >^*$									1	1	
	$< 13, 10, 2, 1 >^*$										1	1
	$< 12, 11, 2, 1 >^*$											1

□

The corresponding Brauce tree of $D_{23}^{B_2}(S_{26})$ is as required in proposition 3.4.1.

4.4.2. Decomposition Matrix for the Principal spin block Y_1 of S_{26}.

$$\deg \begin{cases} \langle 26\rangle,\ \langle 22,3,1\rangle^{\pm},\ \langle 19,4,3\rangle^{\pm},\ \langle 17,6,3\rangle^{\pm}, & \text{if } \equiv 4 \pmod{23}, \\ \langle 15,8,3\rangle^{\pm},\ \langle 13,10,3\rangle^{\pm}, & \\ \langle 23,2\rangle^{*},\ \langle 21,3,2\rangle^{\pm},\ \langle 18,5,3\rangle^{\pm},\ \langle 16,7,3\rangle^{\pm}, & \text{if } \equiv -4 \pmod{23}, \\ \langle 14,9,3\rangle^{\pm},\ \langle 12,11,3\rangle^{\pm} & \end{cases}$$

Now, apply the $(3, 21)$–inducing method to the elements of Pindesc(S_{25}), we get the following projective spin characters of S_{26}:

$J_1 =: H_1 \uparrow_{s_{26}} =< 26 > + < 26 >' + 2 < 23, 3 >'$
$J_2 =: H_2 \uparrow_{s_{26}} = 2 < 23, 3 >^* + < 22, 3, 1 > + < 22, 3, 1 >'$
$J_3 =: H_3 \uparrow_{s_{26}} =< 22, 3, 1 > + < 22, 3, 1 >' + < 21, 3, 2 > + < 21, 3, 2 >'$
$J_4 =: H_4 \uparrow_{s_{26}} =< 21, 3, 2 > + < 21, 3, 2 >' + < 19, 4, 3 > + < 19, 4, 3 >'$
$J_5 =: H_5 \uparrow_{s_{26}} =< 19, 4, 3 > + < 19, 4, 3 >' + < 18, 5, 3 > + < 18, 5, 3 >'$
$J_6 =: H_6 \uparrow_{s_{26}} =< 18, 5, 3 > + < 18, 5, 3 >' + < 17, 6, 3 > + < 17, 6, 3 >'$
$J_7 =: H_7 \uparrow_{s_{26}} =< 17, 6, 3 > + < 17, 6, 3 >' + < 16, 7, 3 > + < 16, 7, 3 >'$
$J_8 =: H_8 \uparrow_{s_{26}} =< 16, 7, 3 > + < 16, 7, 3 >' + < 15, 8, 3 > + < 15, 8, 3 >'$
$J_9 =: H_9 \uparrow_{s_{26}} =< 15, 9, 3 > + < 15, 9, 3 >' + < 14, 9, 3 > + < 14, 9, 3 >'$
$J_{10} =: H_{10} \uparrow_{s_{26}} =< 14, 9, 3 > + < 14, 9, 3 >' + < 13, 10, 3 > + < 13, 10, 3 >'$
$J_{11} =: H_{11} \uparrow_{s_{26}} =< 13, 10, 3 > + < 13, 10, 3 >' + < 12, 11, 3 > + < 12, 11, 3 >'$

By [2.7(a)(ii)], $< 26 >, < 26 >'$ $Irmsc(S_{26})$ of degree 4096 which are denoted by $\phi < 26 >$ and $\phi < 26 >'$ respectively. Now, since $< 23, 3 >^*$ is self-associate spin characters of S_{26} which contains both$\phi < 26 >$ and $\phi < 26 >'$ with same multiplicity [ ], therefore J_1 must split to give the first two columns say F_1 and G_1 for the decomposition matrix $D_{23}^{Y_1}(S_{26})$, of the principal spin block Y_1 of S_{26}.

Furthermore, since $< 22, 3 >\in Pindesc(S25)$ and $< 22, 3 >\uparrow_{s_{26}}^{1,0} =< 23, 3 >^* + < 22, 3, 1 >$
Similarly, $< 22, 3 >'\uparrow_{s_{26}}^{1,0} =< 23, 3 >^* + < 22, 3, 1 >'$, we conclude that $< 22, 3, 1 > \neq < 22, 3, 1 >$.

Hence J_2 must split into two columns $F_2 =:< 23, 2 >^* + < 22, 3, 1 >$ and $F_2 =:< 23, 2 >^* + < 22, 3, 1 >$ both are clearly the elements of $Pindesc(S_{26})$.
Moreover, on α–regular, 23–regular classes of S_{26}, the non–self–associate spin characters are given by:

$< 21,3,2 >^{\pm}$, $< 19,4,3 >^{\pm}$, $< 18,7,3 >^{\pm}$, $< 17,6,3 >^{\pm}$, $< 16,7,3 >^{\pm}$, $< 15,7,3 >^{\pm}$, $< 14,9,3 >^{\pm}$, $< 13,10,3 >^{\pm}$ and $< 12,11,3 >^{\pm}$ are not equal.

Hence, each of the 23–principal spin characters J_i $(3 \leq i \leq 11)$ of S_{26} splits into two elements F_i and G_i $(3 \leq i \leq 11)$ of $Pindesc(S_{26})$, that is,

$$J_i = F_i \oplus G_i, \quad (3 \leq i \leq 11).$$

Thus, the columns of the required decomposition matrix are arranged as: $F_1, G_1, F_2, G_2, F_3, G_3, ..., F_{11}, G_{11}$. This completely determines the decomposition matrix $D^{Y_1}_{23}(S_{26})$ corresponds to the principal spin block of S_{26} and as given by:

Table 1: $D^{Y_1}_{23}(S_{26})$ character table

Class	F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4	F_5	G_5	F_6	G_6	F_7	G_7	F_8	G_8	F_9	G_9	F_{10}	G_{10}	F_{11}	G_{11}	E_{22}
$\langle 26\rangle$	1																						
$\langle 26\rangle'$		1																					
$\langle 23,3\rangle^*$	1		1	1																			
$\langle 22,3,1\rangle$			1		1																		
$\langle 22,3,1\rangle'$				1		1																	
$\langle 21,3,2\rangle$					1		1																
$\langle 21,3,2\rangle'$						1		1															
$\langle 19,4,3\rangle$							1		1														
$\langle 19,4,3\rangle'$								1		1													
$\langle 18,5,3\rangle$									1		1												
$\langle 18,5,3\rangle'$										1		1											
$\langle 17,6,3\rangle$											1		1										
$\langle 17,6,3\rangle'$												1		1									
$\langle 16,7,3\rangle$													1		1								
$\langle 16,7,3\rangle'$														1		1							
$\langle 15,8,3\rangle$															1		1						
$\langle 15,8,3\rangle'$																1		1					
$\langle 14,9,3\rangle$																	1		1				
$\langle 14,9,3\rangle'$																		1		1			
$\langle 13,10,3\rangle$																			1		1		
$\langle 13,10,3\rangle'$																				1		1	
$\langle 12,11,3\rangle$																					1		
$\langle 12,11,3\rangle'$																						1	

5. Conclusion

The study examines modular irreducible spin characters of the symmetric group, particularly under conditions where the prime number 23 divides the group order. This exploration is significant as it sheds light on a specialized and intricate area within representation theory. The work contributes foundational knowledge that may inform further investigations into modular representations of symmetric groups. Future research could extend this approach to other symmetric groups and related finite groups where 23-modular spin characters are relevant, enabling a comparative analysis across different group orders and structures. Such investigations may involve a deeper examination of decomposition matrices and block structures for other primes and group sizes, in order to identify recurring patterns or possible generalizations. Another promising direction is the exploration of computational techniques for determining projective spin characters, including the development of algorithmic or software-based methods that can handle the complexity of larger groups. Furthermore, there is potential to study the relationship between p-modular spin characters and the theory of partitions, with the aim of uncovering richer combinatorial interpretations. Extending this work to double covers of alternating groups and other covering groups may also provide insight into whether similar structural phenomena arise in their spin character theory. Beyond pure group theory, the results obtained in this study could inspire applications in spectral graph theory, where group representations play a crucial role in analyzing graph symmetries and eigenvalue spectra, as well as in broader areas such as algebraic combinatorics and theoretical computer science.

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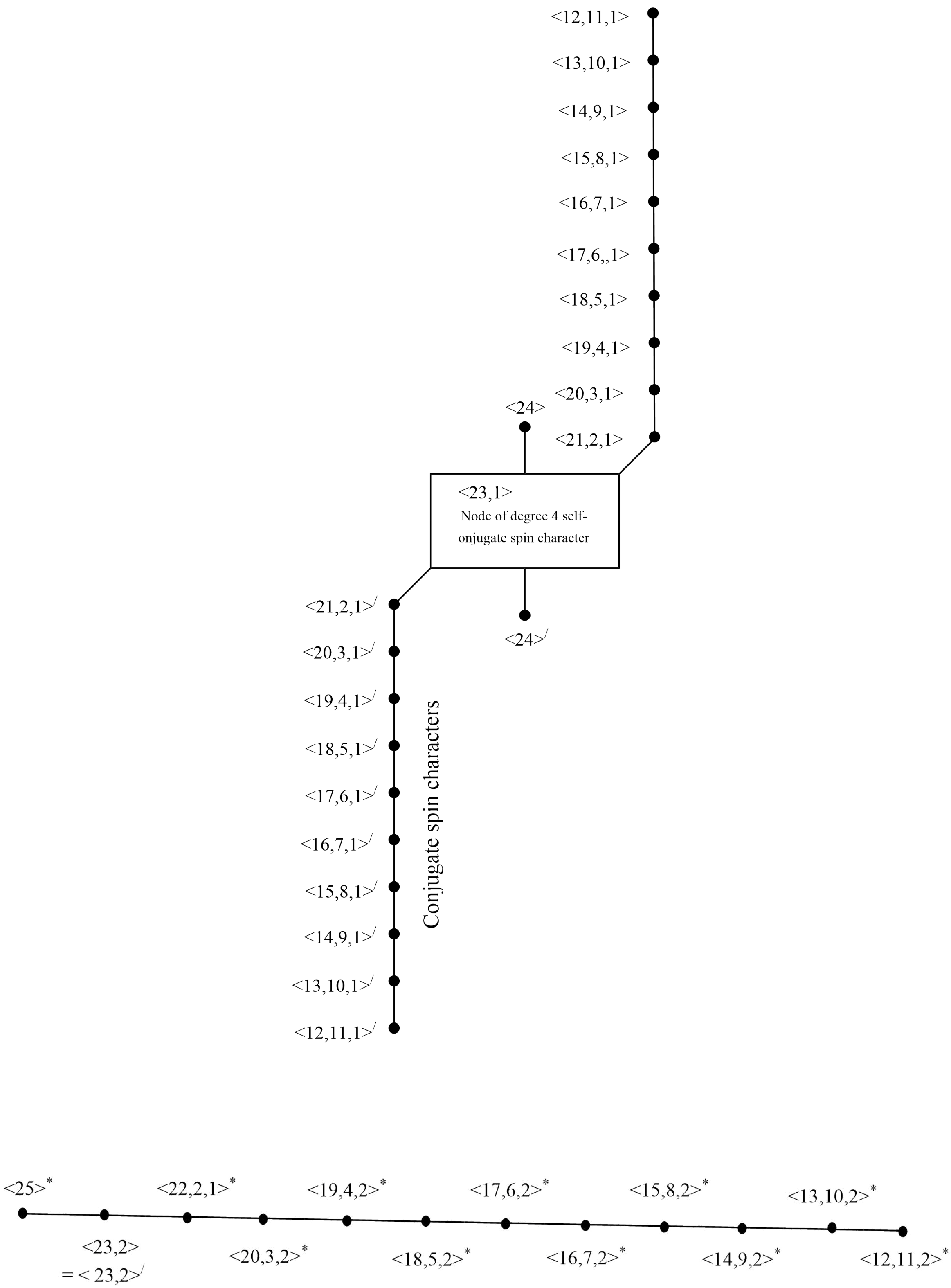
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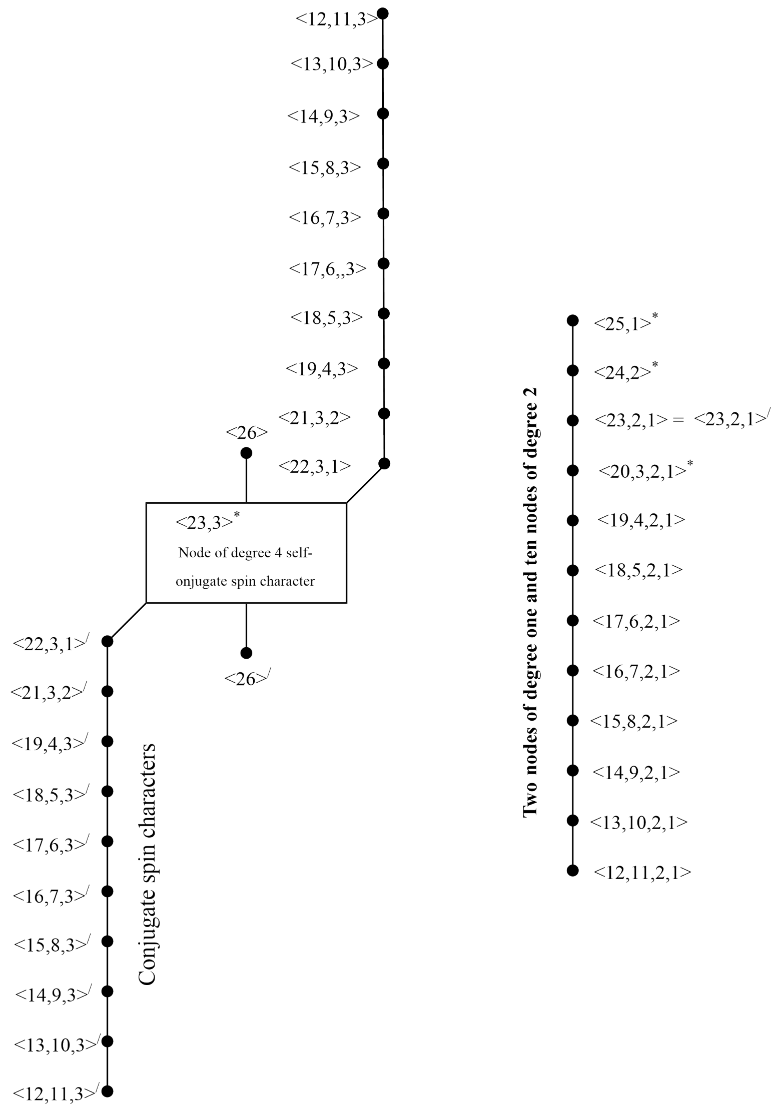
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<12,11,3>′
Conjugate spin characters
Two nodes of degree one and ten nodes of degree 2
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