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A Study of Semilinear Elliptic Triharmonic Equations Involving Singular Potentials

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ABSTRACT: The purpose of this work is to investigate the existence and nonexistence of solutions for the following semilinear sixth-order elliptic problem with a singular potential

$$\begin{cases} -\Delta^3 v - \mu \dfrac{\alpha}{|x|^3} v = g(v) + \lambda h(x) \text{ in } \Omega, \\ v > 0 \text{ in } \Omega, \\ v = \Delta v = \Delta^2 v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N (N \geq 7)$ is a smooth bounded domain. $\Delta^3 v = \Delta(\Delta^2 v)$, h, g are nonnegative functions. $h \in L^2(\Omega)$, $h \not\equiv 0$. μ, λ are positive parameters.

Key Words: Triharmonic operator, hardy potential, Navier boundary problem.

Contents

1. Introduction

In the present work, we study the following semilinear sixth-order elliptic problem with singular potential,

$$\begin{cases} -\Delta^3 v - \mu \dfrac{\alpha}{|x|^3} v = g(v) + \lambda h(x) \text{ in } \Omega, \\ v > 0 \ \text{ in } \Omega, \\ v = \Delta v = \Delta^2 v = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N (N > 6)$ is a smooth bounded domain. $\Delta^3 v = \Delta(\Delta^2 v)$, h, g are nonnegative functions. $h \in L^2(\Omega)$, $h \not\equiv 0$. μ, λ are positive parameters. We assume that

$$g : \mathbf{R}^+ \to \mathbf{R}^+ \text{ is a convex } C^1 \text{ function with } g(0) = 0 = g^{'}(0). \tag{1.2}$$

and satisfying the following growth conditions

$$\lim_{t\to\infty} \frac{g(t)}{t} = \infty, \tag{1.3}$$

$$\int_1^\infty f(s)ds < \infty \text{ and } sf(s) < 1 \text{ for } s > 1, \tag{1.4}$$

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where, for $s \geqslant 1$, we define

$$f(s) := \sup_{t>0} \frac{g(t)}{g(ts)}. \tag{1.5}$$

Thus, it can be easily verified that f is a nonincreasing nonnegative function.
Since by convexity $t \mapsto \frac{g(t)}{t}$ is increasing and $g(0) = 0$, it follows that $s \mapsto sf(s)$ is nonincreasing.

Assume that α is a nonnegative constant satisfying

$$\alpha < \frac{(N-2)(N+2)(N-6)}{8}. \tag{1.6}$$

Thanks to Theorem 2.2, there exists a positive constant $\gamma > 0$ such that

$$\int_\Omega \left(|\nabla \Delta v|^2 - \alpha^2 \frac{v^2}{|x|^6} \right) dx \geqslant \gamma \int_\Omega v^2 dx \quad \forall v \in W_0^{3,2}(\Omega). \tag{1.7}$$

We also assume that

$$0 < \mu < \sqrt{\gamma}. \tag{1.8}$$

In recent years, there is an extensive literature exploring the case $\mu = 0$ (see [1,3,7,11,13]). The differential equation $-\Delta^3 v = h$ models complex bending of beams, plates and shells that involve higher-order interactions and constraints. Moreover, in the study of viscoelastic fluids, the triharmonic equation for the streamfunction describes flow patterns and stress distributions [13].

Similar type of problem with the Laplace operator in much more generalized sense was extensively studied by Dupaigne and Nedev in [10]. In [10], the authors proved a necessary and sufficient condition for the existence of L^1 solution and they have also established an estimate from above and below for the solution. We also refer to [5,6,9] (and the references therein) for the related problems in the second order case.

Higher order problems are distinct from the second and forth order case. In this case a possible failure of the maximum principle which plays a crucial role in proving existence results causes several technical difficulties. Possibly because of this reason the knowledge on higher order nonlinear problems is far from being reasonably complete, as it is in the second-order case.

We will state that the problem (1.1) blows up if the solutions to the truncated problems (where the weight $\mu(|x|^6 + \frac{1}{n})^{-1}$ in place of the Hardy type term $\mu|x|^{-6}$) tend to infinity for all $x \in \Omega$ as $n \to \infty$.

The main purpose of this paper is to illustrate the impact of the Hardy type term on the existence or nonexistence of solutions. The related elliptic semilinear case with the Laplacian and Biharmonic operators were studied in [5,9,15].

This paper is structured as follows:

In Section 2 we briefly provide the functional framework for our problem and the embeddings theorem that will be useful in this paper.

Section 3 is related to certain definitions and preliminary results. First, we characterize the radial solutions to the homogeneous problem that enable us to know the singularity of our supersolutions near the origin and prove nonexistence results. The notion of solution we will treat in the nonexistence results is local, we just ask the regularity required to provide distributional sense to the equation.

In Section 4 we prove our main existence result. More precisely, under some hypothesis on g, we prove that there exists $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the problem (1.1) has a minimal solution u_λ in the Sobolev space $\mathcal{H} := W_0^{3,2}(\Omega)$. Moreover, if $\lambda > \lambda_*$, then (1.1) does not have any solution.

In section 5, deals with the case for which (1.1) does not have any solution even in the very weak sense. In this case we establish complete blow-up phenomenon.

2. Functional Framework

We briefly describe the natural framework to treat the solutions to the problem considered. Let $\Omega \subset \mathbb{R}^N$ denote a smooth domain. We define the Sobolev space

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega),\ D^\beta v \in L^p(\Omega) \text{ for all } 1 \leqslant |\beta| \leqslant k\},$$

equipped with the norm

$$\|v\|_{W^{k,p}(\Omega)} = \left(\int_\Omega |v|^p \, dx + \int_\Omega \sum_{1\leqslant|\beta|\leqslant k} |D^\beta v|^p \, dx\right)^{1/p}. \tag{2.1}$$

Taking the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$ gives rise to the Sobolev space $W_0^{k,p}(\Omega)$, with the norm

$$\|v\|_{W_0^{k,p}(\Omega)} = \left(\int_\Omega \sum_{1\leqslant|\beta|\leqslant k} |D^\beta v|^p \, dx\right)^{1/p},$$

equivalent to (2.1).

Indeed, using interpolation theory one can dispose of the intermediate derivatives and get that

$$\|v\|_{W_0^{k,p}(\Omega)} = \left(\int_\Omega |D^k v|^p \, dx\right)^{1/p}, \tag{2.2}$$

defines a norm which is equivalent to (2.1), see for example [2].

Theorem 2.1 *(Rellich–Kondrachov's Theorem)([15])*Let $k \in \mathbb{N}$ and $1 \leqslant p < \infty$. Suppose that $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain. Then,

$$W^{k+j,p}(\Omega) \hookrightarrow W^{j,p}(\Omega) \text{ and } W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \text{ for all } 1 \leqslant q \leqslant \frac{Np}{N-kp} = p^*,$$

Let us briefly discuss that for certain domains, $\mathcal{H} := W_0^{3,2}(\Omega)$ is a Hilbert space, endowed with the following scalar product

$$< v, u >_{\mathcal{H}} = \int_\Omega < \nabla\Delta v, \nabla\Delta u > dx,$$

which induces the norm

$$\|v\|_{\mathcal{H}} = |\nabla\Delta v|_{L^2(\Omega)},$$

equivalent to (2.2) with $k = p = 2$.

Theorem 2.2 *([14])* For any $v \in \mathcal{H}$ and $N > 6$, it holds that

$$\bar{\mu} \int_\Omega \frac{v^2}{|x|^6} \, dx \leqslant \int_\Omega |\nabla\Delta v|^2 \, dx, \tag{2.3}$$

where $\bar{\mu} = (\frac{(N-2)^2(N+2)^2(N-6)^2}{64})$ is optimal.

Proposition 2.1 *Let $u \in \mathcal{H}$. Then, there exist unique $u_1, u_2 \in \mathcal{H}$ such that $u = u_1 + u_2$, satisfying $u_1 \geq 0$, and $u_2 \leq 0$ in Ω and $\int_\Omega < \nabla\Delta u_1, \nabla\Delta u_2 > dx = 0$.*

Proof: Consider the cone of the a.e. positive functions defined in Ω,

$$\mathcal{C} := \{v \in \mathcal{H} : \; v(x) \geq 0 \text{ a.e in } \Omega\}$$

the corresponding dual cone with respect to the scalar product above is defined as

$$\mathcal{C}^* := \{w \in \mathcal{H} : \int_\Omega < \nabla\Delta w, \nabla\Delta u > dx \leq 0, \text{ for all } u \in \mathcal{C}\}.$$

Let us take as u_1 the orthogonal projection of $u \in \mathcal{H}$ on $\mathcal{C}$, namely, let u_1 be such that

$$\int_\Omega |\nabla\Delta(u-u_1)|^2 dx = \min_{v\in\mathcal{C}} \int_\Omega |\nabla\Delta(u-v)|^2 dx.$$

Letting $u_2 = u - u_1$, for all $t \geq 0$ and $v \in \mathcal{C}$, it holds that

$$\int_\Omega |\nabla\Delta(u-u_1)|^2 dx \leq \int_\Omega |\nabla\Delta(u-(u_1-tv))|^2 dx$$
$$= \int_\Omega |\nabla\Delta(u-u_1)|^2 dx + 2t\int_\Omega < \nabla\Delta(u-u_1), \nabla\Delta v > dx + t^2\int_\Omega |\nabla\Delta v|^2 dx.$$

Therefore

$$2t\int_\Omega < \nabla\Delta(u-u_1), \nabla\Delta v > dx \leq t^2\int_\Omega |\nabla\Delta v|^2 dx. \tag{2.4}$$

Choosing $t > 0$, simplifying and making then $t \to 0$, we obtain that

$$\int_\Omega < \nabla\Delta u_1, \nabla\Delta v > \ dx \leq 0, \ \text{for any } v \in \mathcal{C}, \ \text{hence } u_2 \in \mathcal{C}^*.$$

In particular, we can put $v = u_1$, thus

$$\int_\Omega < \nabla\Delta u_1, \nabla\Delta u_2 > dx \leq 0.$$

Arguing analogously, for some $t \in [-1, 0)$ in (2.4), and letting t tend to 0, we get the reverse inequality, and hence

$$\int_\Omega < \nabla\Delta u_1, \nabla\Delta u_2 > dx = 0.$$

Let us show next the uniqueness. Assume that $u = u_1 + u_2 = v_1 + v_2$ with $u_1, v_1 \in \mathcal{C}$ and $u_2, v_2 \in C^*$. Then

$$0 = \|u_1 - v_1 + u_2 - v_2\|^2 = \|u_1 - v_1\| + \|u_2 - v_2\| - 2 < u_1, v_2 >_{\mathcal{H}} - 2 < v_1, u_2 >_{\mathcal{H}}$$
$$\geq \|u_1 - v_1\| + \|u_2 - v_2\|.$$

This implies that $u_1 = v_1$ and $u_2 = v_2$ as desired. To conclude the proof we show that every function $w \in \mathcal{C}^*$ is nonpositive and, in particular, $u_2 \leq 0$. For every arbitrary nonnegative $h \in C_0^\infty(\Omega)$, consider the solution to the following problem

$$\begin{cases} -\Delta^3 v = h \ \text{ in } \Omega, \\ \ v = 0, \ -\Delta v = 0, \ \Delta^2 v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.5}$$

By the Maximum Principle $v \in \mathcal{C}$. But then,

$$0 \geq \int_\Omega < \nabla\Delta v, \nabla\Delta w > dx = \int_\Omega hw \, dx,$$

for every nonnegative function $h \in C_0^\infty(\Omega)$. By density, we conclude that $w(x) \leq 0$ a.e. $x \in \Omega$ as we wanted to prove. □

3. Some Definitions and Auxiliary Results

Definition 3.1 *We state that $v \in \mathcal{H}$ is a supersolution (subsolution) to (1.1) if $v > 0$ a.e. in Ω, $g(v) \in L^2(\Omega)$ and*

$$\int_\Omega \left(< \nabla\Delta v, \nabla\Delta\phi > -\mu\frac{\alpha}{|x|^3} v\phi \right) \, dx \geqslant (\leqslant) \int_\Omega (g(v) + \lambda h(x))\phi \, dx \quad \forall \phi \in \mathcal{H}.$$

We say that $v \in \mathcal{H}$ is a solution to (1.1) if v is both a supersolution and subsolution to (1.1).

Definition 3.2 *We state that $v \in L^1(\Omega)$ is a very weak solution to problem (1.1) if $v > 0$ a.e. in Ω, $\mu\frac{\alpha}{|x|^3}v + g(v) \in L^1_{loc}(\Omega)$ and v satisfies (1.1) in the distributional sense, i.e.,*

$$\int_\Omega v\left(\Delta^3\phi - \mu\frac{\alpha}{|x|^3}\phi\right)\,dx = \int_\Omega (g(v) + \lambda h(x))\phi\,dx \quad \forall \phi \in C_0^\infty(\Omega).$$

Lemma 3.1 (Strong Maximum Principle) Let v to be a nontrivial supersolution to the problem

$$\begin{cases} -\Delta^3 v = 0 & \text{in } \Omega, \\ v = -\Delta v = \Delta^2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

then $\Delta^2 v > 0$, $-\Delta v > 0$ and $v > 0$ in Ω.

Proof: Set $-\Delta v = u$, if v is a supersolution to (3.1), then u is a supersolution to the problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = -\Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

Utilizing the Strong Maximum Principle to the Biharmonic operator (see [15]) yields that $-\Delta u > 0$, $u > 0$ in Ω, as a result $\Delta^2 v > 0$, $-\Delta v > 0$ and $v > 0$ in Ω. □

Lemma 3.2 (Comparison Principle) Let u and v fulfill the following

$$\begin{cases} -\Delta^3 u \geqslant -\Delta^3 v & \text{in } \Omega, \\ u \geqslant v & \text{on } \partial\Omega, \\ -\Delta u \geqslant -\Delta v & \text{on } \partial\Omega, \\ \Delta^2 u \geqslant \Delta^2 v & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

Then, $\Delta^2 v \leqslant \Delta^2 u$, $-\Delta v \leqslant -\Delta u$ and $v \leqslant u$ in $\bar{\Omega}$.

Proof: Consider the change of variables $w = u - v$, thus w is a supersolution to (3.1). So it suffices to apply to w the Strong Maximum Principle 3.1. □

Note that from the corresponding system (3.2), we can also get a weak Harnack's type inequality.

Lemma 3.3 (Weak Harnack inequality)([15]) Let v be a positive distributional supersolution to the problem (3.1), then for each $B_R(x_0) \subset\subset \Omega$, there is a constant $C = C(\theta, \rho, q, R) > 0$, $0 < q < \frac{N}{N-2}$ and $0 < \theta < \rho < 1$, such that

$$|v|_{L^q(B_{\rho R}(x_0))} \leqslant C \text{ ess} \inf_{B_{\theta R}(x_0)} v.$$

Lemma 3.4 *Let $h \in L^2(\Omega)$, $h \geqslant 0$ a.e. in Ω, $h \not\equiv 0$; α be a positive constant verifying (1.6) and μ satisfy (1.8). Then the problem*

$$\begin{cases} -\Delta^3 v - \mu\frac{\alpha}{|x|^3}v = h & \text{in } \Omega, \\ v = \Delta v = \Delta^2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

admits a positive solution $v \in \mathcal{H}$.

Proof: Given $h \in L^2(\Omega)$, $h \geqslant 0$, we know that there exists unique solution $v_1 \in \mathcal{H}$ to the problem

$$\begin{cases} -\Delta^3 v_1 = h & \text{in } \Omega, \\ v_1 = \Delta v_1 = \Delta^2 v_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

Applying the Strong Maximum Principle 3.1, it follows that $v_1 > 0$ in Ω. Recursively, for $n > 1$ we define

$$\begin{cases} -\Delta^3 v_n = \mu \dfrac{\alpha}{|x|^3} v_{n-1} + h \ \text{ in } \Omega, \\ v_n = \Delta v_n = \Delta^2 v_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.6}$$

Thanks to (1.7) we have $\mu \frac{\alpha}{|x|^3} v_{n-1} \in L^2(\Omega)$, which in turn yields the existence of unique solution $v_n \in \mathcal{H}$ to (3.6). Moreover, invoking the Comparison Principle 3.2 we arrive at

$$0 < v_1 \leqslant \cdots \leqslant v_{n-1} \leqslant v_n \leqslant \cdots .$$

Claim: $\{v_n\}$ is a Cauchy sequence in $\mathcal{H}$. Indeed, notice that

$$-\Delta^3 (v_{n+1} - v_n) - \mu \frac{\alpha}{|x|^3} (v_n - v_{n-1}) = 0 \ \text{ in } \Omega, \tag{3.7}$$

and $v_{n+1} - v_n = \Delta(v_{n+1} - v_n) = \Delta^2 (v_{n+1} - v_n) = 0$ on $\partial\Omega$. By taking $v_{n+1} - v_n$ as a test function in (3.7), then using (1.7) and Hölder's inequality we have

$$\begin{aligned} \int_\Omega |\nabla\Delta(v_{n+1} - v_n)|^2 \, dx &= \mu \int_\Omega \frac{\alpha}{|x|^3} (v_n - v_{n-1})(v_{n+1} - v_n) \, dx \\ &\leqslant \mu \left(\int_\Omega \left(\frac{\alpha}{|x|^3} \right)^2 (v_n - v_{n-1})^2 \, dx \right)^{1/2} \left(\int_\Omega (v_{n+1} - v_n)^2 \, dx \right)^{1/2} \\ &\leqslant \frac{\mu}{\sqrt{\gamma}} \left| \nabla\Delta(v_n - v_{n-1}) \right|_{L^2(\Omega)} \left| \nabla\Delta(v_{n+1} - v_n) \right|_{L^2(\Omega)} . \end{aligned}$$

i.e.,

$$|\nabla\Delta(v_{n+1} - v_n)|_{L^2(\Omega)} \leqslant \frac{\mu}{\sqrt{\gamma}} |\nabla\Delta(v_n - v_{n-1})|_{L^2(\Omega)} \leqslant \cdots \leqslant (\frac{\mu}{\sqrt{\gamma}})^{n-1} |\nabla\Delta(v_2 - v_1)|_{L^2(\Omega)}.$$

As $\mu < \sqrt{\gamma}$, the previous estimate yields that $\{v_n\}$ is a Cauchy sequence in $\mathcal{H}$. Hence, there exists $v \in \mathcal{H}$ such that $v_n \to v$ in $\mathcal{H}$. Moreover, $v > 0$ since $v_n > 0$ for all $n \geqslant 1$.

Since $v_n \in \mathcal{H}$ solves (3.6), passing to the limit, we conclude that v is a solution to (3.4). □

Lemma 3.5 *Let $h \in L^2(\Omega)$, $h \geqslant 0$ and let (1.2) be satisfied. Let λ, μ and α be positive constants with α satisfies* (1.6) *and μ fulfills (1.8). Suppose that there exists a nonnegative supersolution $\bar{v} \in \mathcal{H}$ to (1.1) (respectively for (3.4)). Then, there exists a unique positive minimal solution $\underline{v} \in \mathcal{H}$ to the problem (1.1) (which satisfies $\underline{v} \leqslant \bar{w}$ for any positive supersolution $\bar{w}$ to (1.1)) (respectively for (3.4)).*

Proof: Let $\bar{v} \in \mathcal{H}$ be a nonnegative supersolution to (1.1) and $v_0 \in \mathcal{H}$ be a positive solution to the problem

$$\begin{cases} -\Delta^3 v_0 = \lambda h & \text{in } \Omega, \\ v_0 = \Delta v_0 = \Delta^2 v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Applying the Comparison Principle 3.2 we infer that $0 < v_0 \leqslant \bar{v}$ in Ω. Then, by means of iteration we will show the existence of $v_n \in \mathcal{H}$ for $n = 1, 2, \cdots$ solving the problem

$$\begin{cases} -\Delta^3 v_n = \mu \dfrac{\alpha}{|x|^3} v_{n-1} + g(v_{n-1}) + \lambda h(x) & \text{in } \Omega, \\ v_n = \Delta v_n = \Delta^2 v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Since $\bar{v}$ is a weak supersolution to (1.1), we have $g(\bar{v}) \in L^2(\Omega)$. Thanks to the fact that $0 < v_0 \leqslant \bar{v}$ and g is convex (thus g is nondecreasing), we obtain $g(v_0) \leqslant g(\bar{v})$. Therefore $g(v_0) + \lambda h \in L^2(\Omega)$. Also, by (1.7) it follows that $\frac{\alpha}{|x|^3} v_0 \in L^2(\Omega)$. Therefore v_1 is well defined and by comparison principle $0 < v_0 \leqslant v_1 \leqslant \bar{v}$. By induction method, identically one can show that v_n is well defined and $0 < v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n \leq \cdots \leqslant \bar{v}$. $\square$

Claim: $\{v_n\}$ is uniformly bounded in $\mathcal{H}$. In fact, since v_n verifies (3.9) we can write

$$
\begin{aligned}
|\nabla\Delta v_n|^2_{L^2(\Omega)} &= \int_\Omega \left(\mu\frac{\alpha}{|x|^3}v_{n-1} + g(v_{n-1}) + \lambda h(x)\right) v_n\, dx \leqslant \int_\Omega \left(\mu\frac{\alpha}{|x|^3}\tilde{v} + g(\tilde{v}) + \lambda h(x)\right)\tilde{v}\, dx \\
&\leq \left[\mu\left|\frac{\alpha}{|x|^3}\bar{v}\right|_{L^2(\Omega)} + |g(\bar{v})|_{L^2(\Omega)} + \lambda|h|_{L^2(\Omega)}\right]|\bar{v}|_{L^2(\Omega)} \\
&\leqslant C.
\end{aligned}
$$

As a consequence there exists $\underline{v} \in \mathcal{H}$ such that, up to a subsequence, $v_n \rightharpoonup \underline{v}$ in $\mathcal{H}$ and $v_n \to \underline{v}$ in $L^2(\Omega)$.

By means of (3.9) we have,

$$
\int_\Omega < \nabla\Delta v_n, \nabla\Delta\phi > dx = \int_\Omega \left(\mu\frac{\alpha}{|x|^3}v_{n-1} + g(v_{n-1}) + h\right)\phi dx \quad \forall\phi \in \mathcal{H}.
$$

Using Vitaly's Convergence Theorem we can pass to the limit on the right-hand side and get $\underline{v}$ is a solution to (1.1). Also $\underline{v} > 0$ since $v_n > 0$ for all $n \geqslant 1$.

Let $\bar{w}$ be another supersolution to (1.1), then the comparison principle 3.2 yields that $v_0 \leqslant \bar{w}$ and $v_n \leqslant \bar{w}$ for any $n \geqslant 1$. Passing to the limit, we infer that $\underline{v} \leqslant \bar{w}$. This concludes the proof.

Remark 3.1 *In what follows, we denote the minimal positive solution to (3.4) by ξ_1 and denote $\xi_1 = \mathcal{L}(h)$ where $\mathcal{L} = \left(-\Delta^3 - \mu\frac{\alpha}{|x|^3}\right)^{-1}$.*

4. Existence and Nonexistence Results

Theorem 4.1 *Assume that $h \in L^2(\Omega)$, $h \geqslant 0$ a.e. in Ω, $h \not\equiv 0$; α, μ be nonnegative constants satisfying (1.6) and (1.8) respectively and g be a nonnegative function fulfilling (1.2)-(1.5).*

Suppose there exists two constants $\epsilon > 0$ and $C > 0$ such that

$$
g(\epsilon\xi_1) \in L^2(\Omega) \ \text{ and } \ \mathcal{L}(g(\epsilon\xi_1)) \leqslant C\xi_1 \ \text{ a.e. in } \Omega. \tag{4.1}
$$

Then there exists $0 < \lambda^* = \lambda^*(N;\alpha;h(x);g;\mu)$ such that if $\lambda < \lambda^*$, the problem (1.1) has a minimal positive solution $v_\lambda \in W_0^{3,2}(\Omega)$ satisfying

$$
\lambda\xi_1 \leqslant v_\lambda \leqslant 2\lambda\xi_1 \ \text{ a.e. in } \Omega.
$$

In order to prove this theorem, we first need to prove the next proposition.

Proposition 4.1 *Assume that g, h, α and μ fulfill the prior assumptions of Theorem 4.1, then there exists $\lambda^* > 0$ such that*

$$
g(2\lambda\xi_1) \in L^2(\Omega) \text{ and } \mathcal{L}(g(2\lambda\xi_1)) \leqslant \lambda\xi_1 \ \text{ a.e. in } \Omega, \tag{4.2}
$$

for any $\lambda \in (0, \lambda^*)$.

Proof: By (1.5) we get $f(\frac{\epsilon}{2\lambda}) \geqslant \frac{g(t)}{g(t\frac{\epsilon}{2\lambda})}$ for any $t > 0$. In particular, take $t = 2\lambda\xi_1$, it follows that

$$
g(2\lambda\xi_1) \leqslant f(\frac{\epsilon}{2\lambda})g(\epsilon\xi_1) \text{ in } \Omega. \tag{4.3}
$$

Combining (4.1) with (4.3) yields that $g(2\lambda\xi_1) \in L^2(\Omega)$ and $\mathcal{L}(g(2\lambda\xi_1))$ is well defined. By the minimality of $\mathcal{L}(g(2\lambda\xi_1))$, (4.1) and (4.3) we obtain

$$
\mathcal{L}(g(2\lambda\xi_1)) \leqslant f(\frac{\epsilon}{2\lambda})\mathcal{L}(g(\epsilon\xi_1)) \leqslant Cf(\frac{\epsilon}{2\lambda})\xi_1 \text{ a.e. in } \Omega.
$$

To achieve the proof, it suffices to show that

$$\lim_{\lambda\to 0}\frac{1}{\lambda}f(\frac{\epsilon}{2\lambda})=0 \text{ or equivalently } \lim_{\tau\to\infty}\tau f(\tau)=0.$$

Since $s\to sf(s)$ is nonincreasing, the above limit is well defined, i.e., there exists $c^{'}\geqslant 0$ such that $\lim_{\tau\to\infty}\tau f(\tau)=c^{'}$.

If $c^{'}>0$, then $f(\tau)\sim\frac{c^{'}}{\tau}$ near ∞ and this contradicts (1.4). Hence $c^{'}=0$ and (4.2) holds for $\lambda>0$ small. □

Proof of Theorem 4.1

Let $\lambda\in(0,\lambda^*)$ and define $w:=\mathcal{L}(g(2\lambda\xi_1))+\lambda\xi_1$. Clearly $w>0$ and $w\in\mathcal{H}$ since ξ_1 and $\mathcal{L}(g(2\lambda\xi_1))$ are in $\mathcal{H}$ by Lemma 3.1. Moreover, in view of Proposition 4.1 we know that $w\leqslant 2\lambda\xi_1$ a.e. in Ω and hence $g(w)\in L^2(\Omega)$. Thus we have

$$-\Delta^3(w-\lambda\xi_1)-\mu\frac{\alpha}{|x|^3}(w-\lambda\xi_1)=g(2\lambda\xi_1)\quad\text{in }\Omega.$$

As a result

$$-\Delta^3 w-\mu\frac{\alpha}{|x|^3}w=g(2\lambda\xi_1)+\lambda h\geqslant g(w)+\lambda h\quad\text{in }\Omega,$$

and $w=\Delta w=\Delta^2 w=0$ on $\partial\Omega$. Hence, w is a positive supersolution to (1.1). By applying Lemma 3.5, we obtain the existence of a minimal positive solution $v_\lambda\in\mathcal{H}$ to (1.1). This necessarily implies

$$v_\lambda\leqslant w\leqslant 2\lambda\xi_1 \text{ a.e. in }\Omega. \tag{4.4}$$

On the other hand, from $\mathcal{L}(h)=\xi_1$, one can easily check that $\mathcal{L}(\lambda h)=\lambda\xi_1$. Then it is not difficult to establish that v_λ is a supersolution to the equation satisfied by $\lambda\xi_1$. By minimality of $\lambda\xi_1$, we arrive at

$$v_\lambda\geqslant\lambda\xi_1 \text{ in }\Omega, \tag{4.5}$$

which completes the proof.

5. Nonexistence and Complete Blow-Up Results

Define

$$\tilde{\lambda}^*=\sup\{\lambda>0:\quad\text{the problem (1.1) has a very weak solution}\}.$$

One can easilty check that if $v\in\mathcal{H}$ is a solution to (1.1) in the sense of Definition 3.1, then v is a very weak solution to (1.1) as well. As a result $\tilde{\lambda}^*\geqslant\lambda^*$.

Lemma 5.1 *$\tilde{\lambda}^*$ defined above is finite.*

Proof: Assume that the problem (1.1) has a very weak solution $v\in L^1(\Omega)$, that is to say,

$$\int_\Omega v\left(\Delta^3\varphi-\mu\frac{\alpha}{|x|^3}\varphi\right)dx=\int_\Omega(g(v)+\lambda h(x))\varphi\,dx\quad\forall\varphi\in C_0^\infty(\Omega). \tag{5.1}$$

Let $\tilde{\Omega}\subset\subset\Omega$ and $\psi\in C_0^\infty(\Omega)$, $\psi\geqslant 0$ such that $supp(\psi)\subset\tilde{\Omega}$. Choose φ to verify the problem

$$\begin{cases}-\Delta^3\varphi=\psi & \text{in }\Omega,\\ \zeta=\Delta\varphi=\Delta^2\zeta=0 \text{ on }\partial\Omega.\end{cases}$$

Clearly $\varphi\in C^\infty(\Omega)$. By invoking the Strong Maximum Principle 3.1, it immediately follows that $\varphi>0$ in Ω. Thus we derive that $\varphi\geqslant c>0$ in $\tilde{\Omega}$. Substituting this φ in (5.1) we obtain

$$\mu\int_\Omega\frac{\alpha}{|x|^3}v\varphi\,dx+\int_\Omega g(v)\varphi\,dx+\lambda\int_\Omega h(x)\varphi\,dx=\int_\Omega v\varphi\,dx=\int_{\tilde{\Omega}}v\varphi\,dx. \tag{5.2}$$

By (1.3), given any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$v \leqslant C_\epsilon + \epsilon g(v).$$

Substituting the last inequality in (5.2) we infer

$$\begin{aligned}\mu \int_\Omega \frac{\alpha}{|x|^3} v\varphi + \int_\Omega g(v)\varphi + \lambda \int_\Omega h(x)\varphi = \int_{\tilde\Omega} v\psi &\leqslant C_\epsilon \int_\Omega \psi + \epsilon \int_{\tilde\Omega} g(v)\psi \\ &\leqslant C_\epsilon \int_\Omega \psi + \epsilon \left|\frac{\psi}{\varphi}\right|_{L^\infty(\tilde\Omega)} \int_\Omega g(v)\varphi. \end{aligned} \tag{5.3}$$

Choosing $\epsilon > 0$ such that $\epsilon|\frac{\psi}{\varphi}|_{L^\infty(\tilde\Omega)} < 1/2$ in (5.3), we conclude that

$$\mu \int_\Omega \frac{\alpha}{|x|^3} v\varphi \, dx + \frac{1}{2}\int_\Omega g(v)\varphi \, dx + \lambda \int_\Omega h(x)\varphi \, dx \leqslant C \int_\Omega \psi \, dx \leqslant C'.$$

This necessarily implies that $\tilde\lambda^* < \infty$. In particular there are no solutions to the problem (1.1) when $\lambda > \tilde\lambda^*$ even in the very weak sense. □

Theorem 5.1 *Let $v_n \in \mathcal{H}$ be the minimal nonnegative solution to the problem*

$$\begin{cases} -\Delta^3 v_n - \mu\alpha_n(x) v_n = g_n(v_n) + \lambda h_n & \text{in } \Omega, \\ v_n = \Delta v_n = \Delta^2 v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where $\lambda > \tilde\lambda^*$, $\alpha_n(x) = \alpha(|x|^3 + \frac{1}{n})^{-1}$ and $\{h_n(x)\}$, $\{g_n\}_{n\geq 1}$ be increasing sequence of bounded functions converging pointwise respectively to $h(x)$ and g. Then

$$v_n(x) \to \infty \quad \forall x \in \Omega,$$

i.e., there is a complete blow-up in the problem (1.1) when $\lambda > \tilde\lambda^*$.

Proof: The existence of minimal solution $v_n \in \mathcal{H}$ to (5.4) results from Theorem 6.1. By the monotonicity property of $\{\alpha_n\}$, $\{h_n(x)\}_{n\geq 1}$ and $\{g_n\}$, one can notice that v_{n+1} is a supersolution to the equation satisfied by v_n. Then by minimality of v_n we obtain $v_n \leqslant v_{n+1}$ for any $n \geqslant 1$.

So, to derive the blow-up result, it suffices to establish it for the family of minimal solutions $\{v_n\}$.

Reasoning by contradiction, we suppose there exists $x_0 \in \Omega$ and $C > 0$ such that $v_n(x_0) \leqslant C$. Invoking Lemma 3.3, we know that for each $B_R(x_0) \subset\subset \Omega$, there exists $C' = C'(\theta, \rho, R) > 0$ with $0 < \theta < \rho < 1$, such that

$$|v_n|_{L^1(B_{\rho R}(x_0))} \leqslant C' \operatorname{ess} \inf_{B_{\theta R}(x_0)} v_n \leqslant C' v_n(x_0) \leqslant d, \quad d > 0.$$

Then following the similar argument as in [15], one can prove that there exist $r > 0$ and a positive constant $d = d(r)$ such that

$$\int_{B_r(0)} v_n \, dx \leqslant d, \text{ uniformly in } n \in \mathbb{N}.$$

Therefore, applying the monotone convergence theorem we see that, there exists $v \geqslant 0$ such that $v_n \to v$ in $L^1(B_r(0))$.

Let ϕ be the solution to the problem

$$\begin{cases} -\Delta^3 \phi = \mathbf{1}_{B_r(0)} & \text{in } \Omega, \\ \phi = \Delta\phi = \Delta^2\phi = 0 \text{ on } \partial\Omega. \end{cases}$$

Clearly $\phi \in W^{3,p}(\Omega)$ since $\mathbf{1}_{B_r(0)} \in L^p(\Omega)$ for all $p \geqslant 1$. Taking ϕ as a test function in (5.4) we get

$$\int_\Omega \left(\alpha_n(x) v_n \phi + \frac{g(v_n)}{1 + \frac{1}{n} g(v_n)}\phi + \lambda h_n \phi \right) dx = \int_{B_r(0)} v_n \, dx \leqslant C.$$

Utilizing Monotone Convergence Theorem and Fatou's lemma, it results that

$$\alpha_n(x)v_n \uparrow \frac{\alpha}{|x|^3}v \text{ in } L^1_{\text{loc}}(B_r(0)),\ g_n(v_n) \to g(v) \text{ in } L^1_{\text{loc}}(B_r(0)) \text{ and } h_n(x) \uparrow h(x) \text{ in } L^1_{\text{loc}}(B_r(0)).$$

Thus, we conclude that v is a very weak solution to (1.1) in $B_{r_1}(0) \subset\subset B_r(0)$ and this contradicts with $\lambda > \tilde{\lambda}^*$. □

6. Appendix

Theorem 6.1 *Assume that the assumption (1.8) is fulfilled, then the problem (5.4) admits a nonnegative minimal solution for any positive λ.*

Proof: We will first establish that there exists a nonnegative minimal solution $v_n \in \mathcal{H}$ to the following problem

$$\begin{cases} -\Delta^3 v_n - \mu\dfrac{\alpha}{|x|^3}v_n = g_n(v_n) + \lambda h_n(x) & \text{in } \Omega \\ v_n = \Delta v_n = \Delta^2 v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, let $v_n^0 \in \mathcal{H}$ be a positive solution to the problem

$$\begin{cases} -\Delta^3 v_n^0 = \lambda h_n & \text{in } \Omega, \\ v_n^0 = \Delta v_n^0 = \Delta^2 v_n^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Since $\lambda h_n \in L^\infty(\Omega) \subset L^2(\Omega)$ we obtain $v_n^0 \in \mathcal{H}$. Then, using iteration we will show that there exists $v_n^m \in \mathcal{H}$ for $m = 1, 2, \ldots$ such that v_n^m solves the problem

$$\begin{cases} -\Delta^3 v_n^m = \mu\dfrac{\alpha}{|x|^3}v_n^{m-1} + g_n\left(v_n^{m-1}\right) + \lambda h_n(x) & \text{in } \Omega, \\ v_n^m = \Delta v_n^m = \Delta^2 v_n^m = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Thanks to (1.7) and the fact that $h_n, g_n \in L^\infty(\Omega)$ yields that $\mu\frac{\alpha}{|x|^3}v_n^0 + g_n\left(v_n^0\right) + \lambda h_n(x) \in L^2(\Omega)$ which in turn implies that v_n^1 is well defined. Moreover, by comparison principle $0 < v_n^0 \leqslant v_n^1$.

Using the induction method, one can show via the same manner that v_n^m is well defined and $0 < v_n^0 \leqslant v_n^1 \leqslant \cdots \leqslant v_n^m \leqslant \ldots$.

Claim: $\{v_n^m\}$ is uniformly bounded in $\mathcal{H}$. To see this, note that from (6.2) we have

$$|\nabla\Delta v_n^m|^2_{L^2(\Omega)} = \int_\Omega \left(\mu\frac{\alpha}{|x|^3}v_n^{m-1} + g_n\left(v_n^{m-1}\right) + \lambda h_n(x)\right) v_n^m\, dx. \tag{6.3}$$

Using Hölder's inequality, Young's inequality and (1.7), the terms on the right-hand side can be reduced as follows

$$\begin{aligned} \lambda\int_\Omega h_n v_n^m\, dx &\leqslant \lambda|h_n|_{L^\infty(\Omega)}|\Omega|^{1/2}\,|v_n^m|_{L^2(\Omega)} \leqslant \frac{C}{\sqrt{\gamma}}|h_n|_{L^\infty(\Omega)}\,|\nabla\Delta v_n^m|_{L^2(\Omega)} \\ &\leqslant \epsilon\,|\nabla\Delta v_n^m|^2_{L^2(\Omega)} + c(\epsilon)|h_n|^2_{L^\infty(\Omega)}, \end{aligned}$$

$$\int_\Omega g_n\left(v_n^{m-1}\right)v_n^m\, dx \leqslant |g_n|_{L^\infty(\Omega)}|\Omega|^{1/2}\,|v_n^m|_{L^2(\Omega)} \leqslant \frac{C}{\sqrt{\gamma}}|g_n|_{L^\infty(\Omega)}\,|\nabla\Delta v_n^m|_{L^2(\Omega)}\,,$$

$$\mu\int_\Omega \frac{\alpha}{|x|^3}v_n^{m-1}v_n^m\, dx \leqslant \mu\int_\Omega \frac{\alpha}{|x|^3}(v_n^m)^2\, dx \leqslant \mu\left|\frac{\alpha}{|x|^3}v_n^m\right|_{L^2(\Omega)}|v_n^m|_{L^2(\Omega)} \leqslant \frac{\mu}{\sqrt{\gamma}}\,|\nabla\Delta v_n^m|^2_{L^2(\Omega)}\,.$$

Since $\mu/\sqrt{\gamma} < 1$, we can choose $\epsilon > 0$ such that $2\epsilon + \frac{\mu}{\sqrt{\gamma}} < 1$. Substituting this ϵ in above three inequalities and combining them with (6.3) , we have

$$|\nabla \Delta v_n^m|^2_{L^2(\Omega)} \leqslant C \left(|h_n|_{L^\infty(\Omega)} + |g_n|_{L^\infty(\Omega)} \right).$$

This proves the claim. As a consequence there exists $v_n \in \mathcal{H}$ such that up to a subsequence $u_n^m \rightharpoonup v_n$ in $\mathcal{H}$ as $m \to \infty$ and $v_n^m \to v_n$ in $L^2(\Omega)$. Therefore we can conclude the theorem as we did in Lemma 3.5.

Since v_n is a nonnegative supersolution to (5.4), the theorem results from Lemma 3.5.

□

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