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Fixed Point Theorems Using Kannan Contraction on Suprametric Space

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ABSTRACT: in this article,the fixed point theorems are explored using Kannan type contraction in the setting of suprametric space. An example is given in support of the results obtained. Solution of the Fredholm type integral equation is given as an application of the results proved.

Keywords: Control function, Fixed Point Theorem, Kannan Contraction mapping, suprametric space.

Contents

1. Introduction and Preliminaries

Fixed point theory has been a crucial and captivating subject of research for scholars over the past century. Due to its wide range of applications across various fields of mathematics, extensive research has been dedicated to its extensions and generalizations [9,10,13,14].
In some research works, the metric space has been generalized to more general spaces such as partial metric space, b-metric space [1,2], quasi-partial b-metric space [7,8], and some work was carried out by relaxing the contraction [4,11,12,13,16,17,18,19]. In 1922, Banach established a significant result relevant to a metric fixed point theory, known as the Banach Contraction Principal (BCP). BCP laid the foundation for research in the field of fixed point theorem during last century. Kannan enhanced the Banach contraction mapping concept by introducing a new type of contraction, known as Kannan contraction [5,6,15]. In 2022, Maher Berzig [2,3] introduced the concept of a suprametric space by replacing the triangular inequality condition in a metric space and demonstrated that a Banach contraction has a unique fixed point in a complete suprametric space. The applications of fixed point ranges across approximation theory, optimization, techniques numerical analysis and many more.
The objective of this work is to investigate Kannan-type contraction in the context of suprametric space. An example is constructed to validate the result. A generalization of Kannan contraction with the constant replaced by a control function, is also explored in suprametric space.As an application, the existence of a solution of the Fredholm type integral equation is presented. Throughout this paper, $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ denote the set of real numbers, the set of non-negative real numbers and the set of natural numbers, respectively. We begin with the following:

Definition 1.1 [3]Let Y be a non-empty set. A function $\mathfrak{d} : Y \times Y \to \mathbb{R}^+$ is called suprametric on Y if for all $v, w, z \in Y$, the following hold:

1. $\mathfrak{d}(v, w) = 0$ if and only if $v = w$.
2. $\mathfrak{d}(v, w) = \mathfrak{d}(w, v)$.
3. $\mathfrak{d}(v, w) \leq \mathfrak{d}(v, z) + \mathfrak{d}(z, w) + \gamma\, \mathfrak{d}(v, z)\mathfrak{d}(z, w)$, for some constant $\gamma \in \mathbb{R}^+$.

The pair $(Y, \mathfrak{d})$ is called suprametric space with $\mathfrak{d}$ as suprametric on Y.

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Example 1.1 [3] Let δ be a metric on Y and let α and β be two positive real numbers. It is straightforward to observe that

$$\mathfrak{d}_1^{\alpha}(v,w) = \delta(v,w)(\delta(v,w)+\alpha)$$

and

$$\mathfrak{d}_2^{\beta}(v,w) = \beta(e^{\delta(v,w)} - 1) \tag{1}$$

both are suprametrics with constants γ equal to $\frac{2}{\alpha}$ and $\frac{1}{\beta}$, respectively. However, $\mathfrak{d}_1^{\alpha}$ and $\mathfrak{d}_2^{\beta}$ are not necessarily metrics. For example, if $\mathfrak{d} = \mathfrak{d}_1^1$ (or $\mathfrak{d} = \mathfrak{d}_2^1$) is defined on $\mathbb{R}$ and $\delta(v,w) = |v-w|$, then we have $\mathfrak{d}(0,1) + \mathfrak{d}(1,2) < \mathfrak{d}(0,2)$.

Definition 1.2 [3] A sequence $\{v_m\}_{m\in\mathbb{N}}$ in a suprametric space $(Y,\mathfrak{d})$ is said to converge to $v_0 \in Y$,if for every $\varepsilon > 0$,there is a natural number i such that $\mathfrak{d}(v_m, v_0) < \varepsilon$ for all $m \geq i$.

Definition 1.3 [3] Let $(Y,\mathfrak{d})$ be a supermetric space. A sequence $\{v_m\}_{m\in\mathbb{N}}$ is called a Cauchy sequence if, for every $\varepsilon > 0$, there exists a natural number i such that $\mathfrak{d}(v_m, v_p) < \varepsilon$ for all $m, p \geq i$.

Definition 1.4 [3] A suprametric space $(Y,\mathfrak{d})$ is called a complete suprametric space if every cauchy sequence in it is convergent in $(Y,\mathfrak{d})$.

Proposition 1.1 *[3] If Y is a non-empty set and $\mathfrak{d}$ is one of the suprametric given by (1) on Y such that (Y,δ) is a complete metric space, then the pair $(Y,\mathfrak{d})$ is a complete suprametric space.*

Proposition 1.2 *[3]Let $(Y,\mathfrak{d})$ be a suprametric space. If a sequence $\{v_m\}_{m\in\mathbb{N}} \subset Y$ has a limit, then it is unique.*

Proposition 1.3 *[3] Every suprametric is continuous.*

Proposition 1.4 *[3] Let $(Y,\mathfrak{d})$ be a suprametric space and let $T : Y \to Y$ be a Lipschitz mapping, that is, there is a constant $\lambda \in [0,\infty)$ such that $\mathfrak{d}(Tv,Tw) \leq \lambda\, \mathfrak{d}(v,w)$ for all $v,w \in Y$. Then T is continuous.*

Theorem 1.1 *[3] Let $(Y,\mathfrak{d})$ be a complete suprametric space and $T : Y \to Y$ be a mapping. Assume that there exists $\lambda \in [0,1)$ such that $\mathfrak{d}(Tv,Tw) \leq \lambda\mathfrak{d}(v,w)$ for all $v,w \in Y$. Then T has a unique fixed point, and for every $v_\circ \in Y$, the iterative sequence defined by $v_m = Tv_{m-1}, m \in \mathbb{N}$ converges to this fixed point.*

In 1968, Kannan proved the following.

Definition 1.5 [9] Let (Y,δ) be a metric space and $T : Y \to Y$ be a mapping,then T is said to be *Kannan contraction* if there exists $\beta \in [0, \frac{1}{2})$ such that $\delta(Tv,Tw) \leq \beta\ (\delta(v,Tv) + \delta(w,Tw))$ for all $v,w \in Y$.

Theorem 1.2 *[9] Let (Y,δ) be a complete metric space and T be a Kannan contraction on Y. Then T has a unique fixed point $\mathtt{v} \in Y$, and for every $\mathtt{v}_0 \in Y$, the sequence of iterations $\{T^m\mathtt{v}_0\}$ converges to $\mathtt{v}$.*

2. Fixed Point Theorem in Suprametric Space

Definition 2.1 Let $(Y,\mathfrak{d})$ be a suprametric space with constant $\gamma \geq 0$ and $T : Y \to Y$ be a mapping. Then T is said to be Kannan contraction if there exists $\beta \in [0, \frac{1}{2})$ such that

$$\mathfrak{d}(Tv,Tw) \leq \beta\left(\mathfrak{d}(v,Tv) + \mathfrak{d}(w,Tw)\right), \text{ for all } v,w \in Y.$$

Theorem 2.1 *Every function on a complete suprametric space which is a Kannan contraction admits a unique fixed point, more precisely, if $(Y,\mathfrak{d})$ be a complete suprametric space with constant $\gamma \geq 0$ and $T : Y \to Y$ is such that there exists $\beta \in [0, \frac{1}{2})$ such that*

$$\mathfrak{d}(Tv,Tw) \leq \beta\left(\mathfrak{d}(v,Tv) + \mathfrak{d}(w,Tw)\right), \text{ for all } v,w \in Y, \tag{2}$$

then T has a unique fixed point in Y.

Proof:

In the case $\beta = 0$, T becomes a constant map and thus has a unique fixed point. Now, let us consider the complete suprametric space $(Y, \mathfrak{d})$ with constant $\gamma > 0$ and $\beta \in (0, \frac{1}{2})$. Define the sequence $\{v_m\}$ with $v_m = Tv_{m-1}$ for all $m \in \mathbb{N}$ for some arbitrary $v_0 \in Y$. Let us denote $\mathfrak{d}_{m,n} = \mathfrak{d}(v_m, v_n)$ for all $m, n \in \mathbb{N} \cup \{0\}$. Assume that $v_m \neq v_{m+1}$ for all m, otherwise v_m is a fixed point of T. Therefore $\mathfrak{d}_{m,m+1} > 0$ for all $m \in \mathbb{N}$. Now, using equation (2) we have

$$\begin{aligned} \mathfrak{d}_{m,m+1} = \mathfrak{d}(v_m, v_{m+1}) &= \mathfrak{d}(Tv_{m-1}, Tv_m) \\ &\leq \beta \left(\mathfrak{d}(v_{m-1}, Tv_{m-1}) + \mathfrak{d}(v_m, Tv_m) \right) \\ &= \beta \left(\mathfrak{d}_{m-1,m} + \mathfrak{d}_{m,m+1} \right). \end{aligned}$$

Hence, we have

$$\mathfrak{d}_{m,m+1} \leq \frac{\beta}{1-\beta} \mathfrak{d}_{m-1,m} < \mathfrak{d}_{m-1,m}. \tag{3}$$

Hence, the sequence $\{\mathfrak{d}_{m,m+1}\}$ is a non-negative decreasing sequence of real numbers and so there exists a non-negative constant k such that $\lim_{m\to\infty} \mathfrak{d}_{m,m+1} = \mathrm{k}$. We claim that $\mathrm{k} = 0$. Using equation (3), we derive the following:

$$\mathfrak{d}_{m,m+1} \leq \left(\frac{\beta}{1-\beta} \right)^m \mathfrak{d}_{0,1}. \tag{4}$$

Since $\beta \in \left(0, \frac{1}{2}\right)$, therefore we have the following:

$$\lim_{m\to\infty} \mathfrak{d}_{m,m+1} = 0. \tag{5}$$

This implies that $k = 0$.
Using Definition 1.1(iii), for all positive integers $m, r \in \mathbb{N}$ yields:

$$\begin{aligned} \mathfrak{d}_{m,m+r} &\leq \mathfrak{d}_{m,m+1} + \mathfrak{d}_{m+1,m+r} + \gamma\ \mathfrak{d}_{m,m+1}\ \mathfrak{d}_{m+1,m+r} \\ &\leq \left(\frac{\beta}{1-\beta} \right)^m \mathfrak{d}_{0,1} + \mathfrak{d}_{m+1,m+r} + \gamma \left(\frac{\beta}{1-\beta} \right)^m \mathfrak{d}_{0,1}\ \mathfrak{d}_{m+1,m+r} \\ &\leq \left(\frac{\beta}{1-\beta} \right)^m \mathfrak{d}_{0,1} + \mathfrak{d}_{m+1,m+r} \left(1 + \gamma (\frac{\beta}{1-\beta})^m \mathfrak{d}_{0,1} \right), \end{aligned}$$

and

$$\mathfrak{d}_{m+1,m+r} \leq \left(\frac{\beta}{1-\beta} \right)^{m+1} \mathfrak{d}_{0,1} + \mathfrak{d}_{m+2,m+r} \left(1 + \gamma \left(\frac{\beta}{1-\beta} \right)^{m+1} \mathfrak{d}_{0,1} \right). \tag{6}$$

Thus, using equation (6), we get that

$$\begin{aligned} \mathfrak{d}_{m,m+r} \leq & \left(\frac{\beta}{1-\beta} \right)^m \mathfrak{d}_{0,1} + \left(\frac{\beta}{1-\beta} \right)^{m+1} \mathfrak{d}_{0,1} \left(1 + \gamma (\frac{\beta}{1-\beta})^m \mathfrak{d}_{0,1} \right) \\ & + \left(1 + \gamma (\frac{\beta}{1-\beta})^m \mathfrak{d}_{0,1} \right) \left(1 + \gamma (\frac{\beta}{1-\beta})^{m+1} \mathfrak{d}_{0,1} \right) \mathfrak{d}_{m+2,m+r}. \end{aligned}$$

Iterating the above equation, we get the following:

$$\mathfrak{d}_{m,m+r} \leq \left(\frac{\beta}{1-\beta} \right)^m \mathfrak{d}_{0,1} \sum_{i=0}^{r-1} \left((\frac{\beta}{1-\beta})^i \prod_{j=0}^{i-1} \left(1 + \gamma (\frac{\beta}{1-\beta})^{m+j} \mathfrak{d}_{0,1} \right) \right). \tag{7}$$

Since $\frac{\beta}{1-\beta} < 1$, therefore, $\left(\frac{\beta}{1-\beta} \right)^m < 1 \quad$ for all $m \in \mathbb{N}$. Thus,

$$1 + \gamma \left(\frac{\beta}{1-\beta} \right)^{m+j} \mathfrak{d}_{0,1} < 1 + \gamma \left(\frac{\beta}{1-\beta} \right)^j \mathfrak{d}_{0,1}.$$

So, equation (7) yields:

$$\mathfrak{d}_{m,m+r} \leq \left(\frac{\beta}{1-\beta}\right)^m \mathfrak{d}_{0,1} \sum_{i=0}^{r-1} \left((\frac{\beta}{1-\beta})^i \prod_{j=0}^{i-1} \left(1+\gamma(\frac{\beta}{1-\beta})^j \mathfrak{d}_{0,1}\right)\right).$$

Therefore, if the series $\sum_{i=0}^{\infty} u_i$, where $u_i = (\frac{\beta}{1-\beta})^i \prod_{j=0}^{i-1} \left(1+\gamma(\frac{\beta}{1-\beta})^j \mathfrak{d}_{0,1}\right)$ is convergent,then so is the sequence $\mathfrak{d}_{m,m+r}$. Now,

$$\lim_{i\to\infty} \left|\frac{u_i+1}{u_i}\right| = \lim_{i\to\infty} \frac{\beta}{1-\beta} \left(1+\gamma \left(\frac{\beta}{1-\beta}\right)^i \mathfrak{d}_{0,1}\right) = \frac{\beta}{1-\beta} < 1.$$

Hence, the series $\sum_{i=0}^{\infty} u_i$ converges by ratio test. Therefore, $\mathfrak{d}_{m,m+r}$ tends to zero as m, r approach infinity. Hence, the sequence $\{v_m\}$ is a Cauchy sequence and by the completeness of $(Y, \mathfrak{d})$, it is concluded that $\{v_m\}$ converges for some $v \in Y$. Next step is to establish that v is a fixed point of T. Subsequently, employing Definition 1.1(iii) and equation (2) we get the following:

$$\begin{aligned}
\mathfrak{d}(v, Tv) &\leq \mathfrak{d}(v, v_{m+1}) + \mathfrak{d}(v_{m+1}, Tv) + \gamma\, \mathfrak{d}(v, v_{m+1})\, \mathfrak{d}(v_{m+1}, Tv) \\
&\leq \mathfrak{d}(v, v_{m+1}) + \mathfrak{d}(Tv_m, Tv) + \gamma\, \mathfrak{d}(v, v_{m+1})\, \mathfrak{d}(Tv_m, Tv) \\
&= \mathfrak{d}(v, v_{m+1}) + \mathfrak{d}(Tv_m, Tv)(1 + \gamma\mathfrak{d}(v, v_{m+1})) \\
&\leq \mathfrak{d}(v, v_{m+1}) + \beta(\mathfrak{d}(v_m, v_{m+1}) + \mathfrak{d}(v, Tv))(1 + \gamma\mathfrak{d}(v, v_{m+1})).
\end{aligned}$$

Considering the limit as m tends to infinity, we get that $\mathfrak{d}(v, Tv) \leq \beta\, \mathfrak{d}(v, Tv)$, which is possible only if $\mathfrak{d}(v, Tv) = 0$, that is, $v = Tv$. Therefore, v is a fixed point of T. Let us assume that T has two fixed points, namely v and w. Then equation (2) yields

$$\begin{aligned}
0 \leq \mathfrak{d}(v, w) &= \mathfrak{d}(Tv, Tw) \\
&\leq \beta\, (\mathfrak{d}(v, Tv) + \mathfrak{d}(w, Tw)) \\
&= 0.
\end{aligned}$$

Therefore, $\mathfrak{d}(v, w) = 0$ and hence $v = w$ implying that T has unique fixed point. □

Example 2.1 Let $Y = [0, 2]$ then using Example 1.2 and Proposition 1.3, $\mathfrak{d}(v, w) = |v-w|(|v-w|+1)$ is a complete suprametric space with $\gamma = 2$. Now, define $T : Y \to Y$ such that

$$Tv = \begin{cases} 0 & \text{if } v \in [0, 1) \\ \frac{1}{3} & \text{if } v \in [1, 2]. \end{cases}$$

Then, T is a Kannan contraction on the suprametric space $(Y, \mathfrak{d})$ with $\beta = \frac{9}{20} < \frac{1}{2}$, that is, $\mathfrak{d}(Tv, Tw) \leq \frac{9}{20}\,(\mathfrak{d}(v, Tv) + \mathfrak{d}(w, Tw))$ for all $v, w \in Y$. Consequently, we have the following:

$$|Tv-Tw|^2 + |Tv-Tw| \leq \frac{9}{20}\left(|Tv-v|^2 + |Tv-v| + |Tw-w|^2 + |Tw-w|\right), \text{ for all } v, w \in Y.$$

If $v \in [0, 1)$ and $w \in [1, 2]$ then

$$\frac{4}{9} \leq \frac{9}{20}\left(v^2 + v + \left|w-\frac{1}{3}\right|^2 + \left|w-\frac{1}{3}\right|\right)$$

or equivalently,

$$1 \leq \frac{81}{80}\left(v^2 + v + \left|w-\frac{1}{3}\right|^2 + \left|w-\frac{1}{3}\right|\right).$$

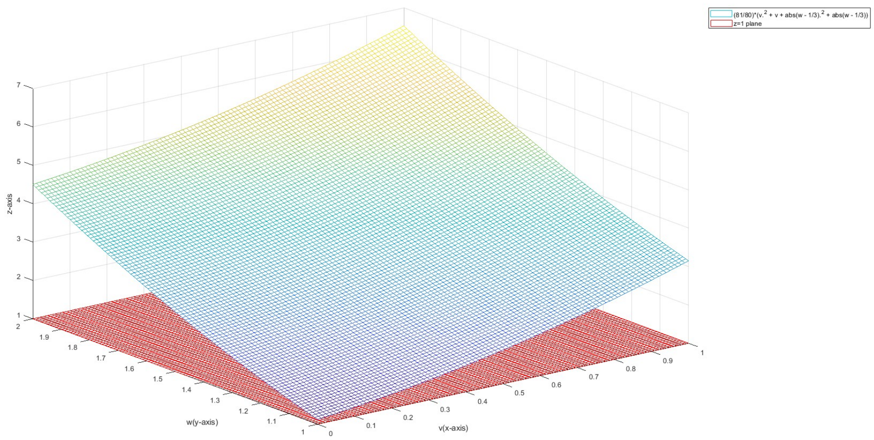

Figure 1: Graphical representation.

In particular, for $v = 0, w = 1$,the above inequality gives, $1 < \frac{9}{8}$ which is true. Now, for $v = 0, w = \frac{3}{2}$,it becomes, $1 < 2.559$, which is also true, and for $v = \frac{1}{3}, w = 1, 1 < 1.575$, which again satisfy the above inequality. Figure 1,shows the graphical representation of the plane $z = 1$ and that of $\frac{81}{80}\left(v^2 + v + \left|w - \frac{1}{3}\right|^2 + \left|w - \frac{1}{3}\right|\right)$.

Hence, T is a Kannan contraction on the complete suprametric space Y and thus by Theorem 2.1, T must have a unique fixed point, which is 0 in this case.

3. Generalized Kannan Contraction with Control Function

The following idea is due to Geraghty [4] and Górnicki [5]:
Let Λ denote the class of functions which satisfy the condition

$$\Lambda = \left\{\psi : (0, \infty) \to \left[0, \frac{1}{2}\right) \text{ such that } \psi(t_n) \to \frac{1}{2} \text{ implies } t_n \to 0 \text{ as } n \to \infty\right\}. \tag{8}$$

Here, ψ may not be continuous.

Theorem 3.1 *Let $(Y, \mathfrak{d})$ be a complete metric space with constant $\gamma \geq 0$ and let $T : Y \to Y$ be a mapping. Suppose that there exists $\psi \in \Lambda$ such that for each $v, w \in Y$ with $v \neq w$,*

$$\mathfrak{d}(Tv, Tw) \leq \psi(\mathfrak{d}(v, w))(\mathfrak{d}(v, Tv) + \mathfrak{d}(w, Tw)). \tag{9}$$

Then T has a unique fixed point in Y.

Proof: Define the sequence $\{v_m\}$ with $v_m = Tv_{m-1}$ for all $m \in \mathbb{N}$ for some arbitrary $v_0 \in Y$. let us denote $\mathfrak{d}_{m,n} = \mathfrak{d}(v_m, v_n)$ for all $m, n \in \mathbb{N} \cup \{0\}$. Assume that $v_m \neq v_{m+1}$ for all m, otherwise v_m is a fixed point of T. Therefore, $\mathfrak{d}_{m,m+1} > 0$ for all $m \in \mathbb{N}$. Now using equation (9), we have

$$\begin{aligned}
\mathfrak{d}_{m,m+1} &= \mathfrak{d}(v_m, v_{m+1}) = \mathfrak{d}(Tv_{m-1}, Tv_m) \\
&\leq \psi(\mathfrak{d}_{m-1,m})(\mathfrak{d}(v_{m-1}, Tv_{m-1}) + \mathfrak{d}(v_m, Tv_m)) \\
&= \psi(\mathfrak{d}_{m-1,m})(\mathfrak{d}_{m-1,m} + \mathfrak{d}_{m,m+1}) \qquad (10) \\
&< \frac{1}{2}(\mathfrak{d}_{m-1,m} + \mathfrak{d}_{m,m+1}).
\end{aligned}$$

Hence, we have $\mathfrak{d}_{m,m+1} < \mathfrak{d}_{m-1,m}$. Therefore, the sequence $\{\mathfrak{d}_{m,m+1}\}$ is a non-negative decreasing sequence of real numbers and thus there exists a non-negative constant k such that $\lim_{m\to\infty} \mathfrak{d}_{m,m+1} = k$. Subsequently, we claim that $k = 0$, for if it is not so, then replacing $m + 1$ by m+2 in (10), we get

$$\mathfrak{d}_{m+1,m+2} \leq \psi(\mathfrak{d}_{m,m+1})(\mathfrak{d}_{m,m+1} + \mathfrak{d}_{m+1,m+2}).$$

Equivalently,

$$\frac{\mathfrak{d}_{m+1,m+2}}{\mathfrak{d}_{m,m+1} + \mathfrak{d}_{m+1,m+2}} \leq \psi(\mathfrak{d}_{m,m+1}), \text{ for all } m \in \mathbb{N}.$$

As $m \to \infty$, we see that

$$\frac{k}{2k} \leq \lim_{m\to\infty} \psi(\mathfrak{d}_{m,m+1}).$$

Hence,

$$\frac{1}{2} \leq \lim_{m\to\infty} \psi(\mathfrak{d}_{m,m+1}).$$

Since, $\psi \in \Lambda$, we have

$$\frac{1}{2} \leq \lim_{m\to\infty} \psi(\mathfrak{d}_{m,m+1}) \leq \frac{1}{2}.$$

Therefore, equation (8) yields

$$\lim_{m\to\infty} \mathfrak{d}_{m,m+1} = 0, \tag{11}$$

which gives a contradiction and then $k = 0$. Now, we shall prove that $\{v_m\}$ is a Cauchy sequence in Y. On the contrary, if $\{v_m\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence of integers $\{M_\ell\}, \{m_\ell\}$ and $\{n_\ell\}$ such that $M_\ell \to \infty,\ m_\ell \geq n_\ell \geq M_\ell$

$$\varepsilon \leq \mathfrak{d}(v_{n_\ell}, v_{m_\ell}) = \mathfrak{d}(Tv_{n_\ell-1}, Tv_{m_\ell-1}).$$

The above inequality implies that $v_{n_\ell-1} \neq v_{m_\ell-1}$. Therefore, by using equation (9)

$$\begin{aligned}
\varepsilon &\leq \mathfrak{d}(v_{n_\ell}, v_{m_\ell}) \\
&\leq \psi(\mathfrak{d}(v_{n_\ell-1}, v_{m_\ell-1}))(\mathfrak{d}(v_{n_\ell-1}, Tv_{n_\ell-1}) + \mathfrak{d}(v_{m_\ell-1}, Tv_{m_\ell-1})) \\
&= \psi(\mathfrak{d}_{n_\ell-1,m_\ell-1})(\mathfrak{d}_{n_\ell-1,n_\ell} + \mathfrak{d}_{m_\ell-1,m_\ell}) \\
&< \frac{1}{2}(\mathfrak{d}_{n_\ell-1,n_\ell} + \mathfrak{d}_{m_\ell-1,m_\ell}).
\end{aligned}$$

Letting $\ell \to \infty$, it follows from equation (11) that $\varepsilon \leq 0$, which is a contradiction. Thus, $\{v_m\}$ is a Cauchy sequence in Y. Since Y is a complete suprametric space, therefore, $\{v_m\}$ converges to $v \in Y$. Next step is to establish that v is a fixed point of T. Subsequently, employing definition 1.1(iii) and equation (9), we get the following:

$$\begin{aligned}
\mathfrak{d}(v, Tv) &\leq \mathfrak{d}(v, v_{m+1}) + \mathfrak{d}(v_{m+1}, Tv) + \gamma\mathfrak{d}(v, v_{m+1})\mathfrak{d}(v_{m+1}, Tv) \\
&\leq \mathfrak{d}(v, v_{m+1}) + \mathfrak{d}(Tv_m, Tv) + \gamma\mathfrak{d}(v, v_{m+1})\mathfrak{d}(Tv_m, Tv) \\
&= \mathfrak{d}(v, v_{m+1}) + \mathfrak{d}(Tv_m, Tv)(1 + \gamma\mathfrak{d}(v, v_{m+1})) \\
&\leq \mathfrak{d}(v, v_{m+1}) + \psi(\mathfrak{d}(v_m, v))(\mathfrak{d}(v_m, v_{m+1}) + \mathfrak{d}(v, Tv))(1 + \gamma\mathfrak{d}(v, v_{m+1})) \\
&\leq \mathfrak{d}(v, v_{m+1}) + \frac{1}{2}(\mathfrak{d}(v_m, v_{m+1}) + \mathfrak{d}(v, Tv))(1 + \gamma\mathfrak{d}(v, v_{m+1})).
\end{aligned}$$

Considering the limit as $m \to \infty$, we get that

$$\mathfrak{d}(v, Tv) \leq \lim_{m\to\infty} \mathfrak{d}(v, v_{m+1}) + \frac{1}{2}(\lim_{m\to\infty} \mathfrak{d}(v_m, v_{m+1}) + \mathfrak{d}(v, Tv)).$$

Therefore, $\mathfrak{d}(v,Tv) \leq \frac{1}{2}\mathfrak{d}(v,Tv)$, which is possible only if $\mathfrak{d}(v,Tv)=0$. that is, $v=Tv$. Consequently, v is a fixed point of T.
Let us assume that T has two fixed points, namely v and w. Then equation (9) yields

$$\begin{aligned} 0 \leq \mathfrak{d}(v,w) &= \mathfrak{d}(Tv,Tw) \\ &\leq \psi(\mathfrak{d}(v,w))(\mathfrak{d}(v,Tv)+\mathfrak{d}(w,Tw)) \\ &< \frac{1}{2}(\mathfrak{d}(v,v)+\mathfrak{d}(w,w)) \\ &= 0. \end{aligned}$$

Therefore, $\mathfrak{d}(v,w)=0$ and hence $v=w$ implying that T has unique fixed point. □

4. Application

Consider the Fredholm integral equation

$$v(t) = \int_0^1 \mathcal{K}(t,\zeta,v(\zeta))d\zeta + \mathrm{f}(t), \quad \text{for all } t \in [0,1], \tag{12}$$

where $\mathcal{K} : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ and $\mathrm{f} : [0,1] \to \mathbb{R}$ is continuous.
Let $Y = C[0,1]$ be the set of all continuous functions with real values in $[0,1]$ and α be a positive real number and let $\mathfrak{d}(v,w) = \left(\sup_{t\in[0,1]} \{|v(t)-w(t)|\}\right)^2 + \alpha \sup_{t\in[0,1]} \{|v(t)-w(t)|\}$. Then, by Example 1.2, $(Y,\mathfrak{d})$ is a complete suprametric space with constant γ equal to $\frac{2}{\alpha}$. Now, define $T : Y \to Y$ as

$$Tv(t) = \int_0^1 \mathcal{K}(t,\zeta,v(\zeta))d\zeta + \mathrm{f}(t), \quad \text{for all } t \in [0,1]. \tag{13}$$

Next, assume that for $v,w \in Y$

$$|\mathcal{K}(t,\zeta,v(\zeta)) - \mathcal{K}(t,\zeta,w(\zeta))| \leq \beta\ \left(\ |v(\zeta)-Tv(\zeta)+w(\zeta)-Tw(\zeta)|\ \right), \text{for all} t,\zeta \in [0,1], \beta \in \left[0,\frac{1}{2}\right). \tag{14}$$

Then the integral equation (12) has a unique solution.

Solution: To prove the claim, it suffices to prove that the given map T on the suprametric space $(Y,\mathfrak{d})$ satisfies the hypothesis of Theorem 2.2. For any $v,w \in Y$

$$\mathfrak{d}(Tv,Tw) = \left(\sup_{t\in[0,1]} |Tv(t)-Tw(t)|\right)^2 + \alpha \sup_{t\in[0,1]} \{|Tv(t)-Tw(t)|\},$$

and for each $t \in [0,1]$

$$\begin{aligned} |Tv(t)-Tw(t)| &= \left|\int_0^1 \mathcal{K}(t,\zeta,v(\zeta))d\zeta - \int_0^1 \mathcal{K}(t,\zeta,w(\zeta))d\zeta\right| \\ &= \left|\int_0^1 \left(\mathcal{K}(t,\zeta,v(\zeta)) - \mathcal{K}(t,\zeta,w(\zeta))\right) d\zeta\right| \\ &\leq \int_0^1 |\mathcal{K}(t,\zeta,v(\zeta)) - \mathcal{K}(t,\zeta,w(\zeta))|\,|d\zeta|. \end{aligned}$$

Next, equation (14) gives us the following:

$$\begin{aligned}|Tv(t)-Tw(t)| &\leq \int_0^1 \beta|v(\zeta)-Tv(\zeta)+w(\zeta)-Tw(\zeta)|d\zeta \\ &\leq \beta\int_0^1 \left(|v(\zeta)-Tv(\zeta)|+|w(\zeta)-Tw(\zeta)|\right)d\zeta \\ &\leq \beta\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|+\sup_{t\in[0,1]}|w(t)-Tw(t)|\right).\end{aligned}$$

Hence,

$$\sup_{t\in[0,1]}|Tv(t)-Tw(t)| \leq \beta\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|+\sup_{t\in[0,1]}|w(t)-Tw(t)|\right). \tag{15}$$

Further applying equation (15), we derive the following:

$$\begin{aligned}\left(\sup_{t\in[0,1]}|Tv(t)-Tw(t)|\right)^2 &\leq \beta^2\left(\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|\right)^2+\left(\sup_{t\in[0,1]}|w(t)-Tw(t)|\right)^2\right) \\ &\quad + 2\beta^2\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|\right)\left(\sup_{t\in[0,1]}|w(t)-Tw(t)|\right) \\ &\leq 2\beta^2\left(\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|\right)^2+\left(\sup_{t\in[0,1]}|w(t)-Tw(t)|\right)^2\right).\end{aligned}$$

Since $\beta \in [0, \frac{1}{2})$,therefore,

$$\left(\sup_{t\in[0,1]}|Tv(t)-Tw(t)|\right)^2 \leq \beta\left(\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|\right)^2+\left(\sup_{t\in[0,1]}|w(t)-Tw(t)|\right)^2\right). \tag{16}$$

Therefore, equations (15) and (16) yield

$$\begin{aligned}\mathfrak{d}(Tv,Tw) &\leq \beta\left(\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|\right)^2+\left(\sup_{t\in[0,1]}|w(t)-Tw(t)|\right)^2\right) \\ &\quad + \alpha\beta\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|+\sup_{t\in[0,1]}|w(t)-Tw(t)|\right) \\ &= \beta\left(\left(\sup_{t\in[0,1]}|v(t)-Tv(t)|\right)^2+\alpha\sup_{t\in[0,1]}|v(t)-Tv(t)|\right) \\ &\quad + \beta\left(\left(\sup_{t\in[0,1]}|w(t)-Tw(t)|\right)^2+\alpha\sup_{t\in[0,1]}|w(t)-Tw(t)|\right) \\ &= \beta\left(\mathfrak{d}(v,Tv)+\mathfrak{d}(w,Tw)\right).\end{aligned}$$

Therefore, $\mathfrak{d}(Tv,Tw) \leq \beta\left(\mathfrak{d}(v,Tv)+\mathfrak{d}(w,Tw)\right)$ for all $v, w \in Y$. Thus, by Theorem (2.2), the operator T has a unique fixed point. Thus,the Fredholm integral equations have unique solutions.

Conclusion and future work

In this paper, we conducted a comprehensive study of Kannan-type contractions within the setting of suprametric spaces with an appropriate control function. This generalization broadens the class of admissible mappings and provides a more flexible framework for investigating fixed-point problems.To emphasize the practical significance of our findings, we applied the developed theory to demonstrate the existence of a solution for a Fredholm-type integral equation. This example highlights not only the relevance of our results but also their potential applications in addressing nonlinear problems across both pure and applied mathematics.

The results proved in this article can be extended to more generalized structures such as partial suprametric spaces, b-suprametric spaces, or complex suprametric spaces. Another interesting direction would be to analyze the stability and uniqueness of fixed points under weaker contraction conditions. Furthermore, the generalized Kannan-type mappings introduced in this work may be explored in greater depth, particularly related to integral and differential equations encountered in applied scientific contexts.

Statements and Declarations

Conflict of interest

The authors have no financial or proprietary interests in any material discussed in this article.

Availability of data and material

Data sharing is not applicable to this article as no datasets were generated during the current study.

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