Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.77819

On a Class of Anisotropic Elliptic Equations with Hardy Potential and Homogeneous Neumann Boundary Conditions

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ABSTRACT: We study a class of anisotropic nonlinear elliptic problems involving a Hardy potential and subject to homogeneous Neumann boundary conditions. The differential operator under consideration is of the Leray-Lions anisotropic type. By employing Berkovits' topological degree theory, combined with the properties of anisotropic Sobolev spaces and the abstract Hammerstein equation, we establish the existence of weak solutions to the problem under investigation.

Key Words: Hardy potential, topological degree theory, anisotropic nonlinear elliptic problems, abstract Hammerstein equation.

Contents

1. Introduction and assumptions

In this paper, we consider $\mathcal{D} \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset with a Lipschitz boundary, and $p_i := \{p_1, p_2, \dots p_N\}$ such that $p_i \in]2, \infty[$. Using the topological degree theory introduced by Berkovits in [8], we establish, under some suitable assumptions, the existence of weak solutions to the following nonlinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i \kappa_i(\tau, w, \nabla w) - \Theta \frac{|w|^{p_0-2}w}{|\tau|^{p_0}} + \Gamma |w|^{p_0-2} w = \lambda \mathcal{A}(\tau, w, \nabla w) & \text{in } \mathcal{D}, \\ \displaystyle\sum_{i=1}^{N} \kappa_i(\tau, w, \nabla w)\eta_i = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where λ is a non-negative real parameter, η_i denotes the outer unit normal derivative, and $p_0 = \max\{p_i : i = 1, \dots, N\}$. The functions κ_i are Carathéodory functions respecting these conditions

(κ_1) $\displaystyle\sum_{i=1}^{N} \kappa_i(\tau, s, \zeta)\, \zeta_i \geq \varrho \sum_{i=1}^{N} |\zeta_i|^{p_i}.$

(κ_2) $|\kappa_i(\tau, s, \zeta)| \leq \mu \left(|\zeta_i|^{p_i-1} + |s|^{p_i-1} + m_i(\tau) \right),$ for each $i = 1, \dots, N$.

(κ_3) $(\kappa_i(\tau, s, \zeta) - \kappa_i(\tau, s, \zeta'))(\zeta_i - \zeta_i') > 0$ for each $i = 1, \dots, N$.

where $\zeta, \zeta' \in \mathbb{R}^N$, $(\zeta \neq \zeta')$, $s \in \mathbb{R}$, ϱ, μ are positive constants and $m_i(\tau) \in L^{p_i'}(\mathcal{D})$ is a positive function such that $p_i' = \dfrac{p_i}{p_i - 1}$. In addition, $\mathcal{A}$ is the Carathéodory function satisfies a growth condition for all $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{R}^N$, for all $s \in \mathbb{R}$, and a.e $\tau \in \mathcal{D}$. Specifically:

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2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted July 12, 2025. Published September 25, 2025

$(\mathcal{A}_1)$ there exists a positive real number $\mathcal{C}$ such that

$$|\mathcal{A}(\tau,s,\zeta)|\leq \mathcal{C}\Big(\sum_{i=1}^{N}|\zeta_i|^{\frac{p_i}{p_0'}}+|\rho(\tau)|+|s|^{p_0-1}\Big),$$

with $\rho(\tau)\in L^{p_i'}(\mathcal{D})$.

By utilizing Berkovits' topological degree for $(\mathcal{S}_+)$-type operators as presented in [8], along with the theory of anisotropic Sobolev spaces, we establish the existence of a weak solution for the given problem. The construction of a topological degree for continuous mappings on a bounded domain of $\mathbb{R}^n$ was initiated by Brouwer [10]. Among various examples, we refer the reader to classical works for further insights. [6,16,2,3,14,22,23,24] For additional details. Anisotropic problems, which can become critical in certain regions of the domain, have recently been studied in [7,19]. In these cases, the right-hand side has low regularity, belonging, for example, to L^1. Additionally, Lions' concentration-compactness principle for critical problems has been extended to the anisotropic setting in [12]. Lastly, the connection between different notions of weak solutions and their regularity has been recently analyzed in [18] for non-homogeneous equations involving the $p(\tau)$-Laplacian operator. In recent years, several authors have investigated problems involving the so-called Hardy potential. For instance, Porzio [20] has studied the following quasilinear elliptic problem.In [1], the influence of the Hardy potential on the existence and nonexistence of solutions has been emphasized. Specifically, Abdellaoui et al. [1] studied the following nonlinear elliptic equation:

$$\begin{cases} -\Delta u\pm|\nabla u|^2=\lambda\dfrac{u}{|x|^2}+f & \text{in } \mathcal{D},\\ u=0 & \text{on } \partial\mathcal{D},\end{cases}$$

The paper is structured as follows. In Section 2, we present the basic notation, definitions, key properties, and the functional framework. Section 3 introduces Berkovits' degree theory and its generalization from [8]. Finally, in the last section, we establish preliminary lemmas and prove the main result of the paper.

2. Preliminaries

In this part, we first recall some necessary results which will be used in the next part, we present some properties on the theory of topological degree and anisotropic space.
Consider $\mathcal{D}\subset\mathbb{R}^N$ $(N\in[2;+\infty[)$ bounded open subset in with smooth boundary and let $p_i\in]2;+\infty[$ for any $i=1,\dots,N$, $p_-=\min\{p_1,\dots,p_N\}$. We denote $\partial_i=\frac{\partial}{\partial\tau_i}$.
The anisotropic Sobolev space is given by

$$W^{1,\vec{p}}(\mathcal{D})=\{w\in L^{p_0}(\mathcal{D}),\ \partial_i w\in L^{p_i}(\mathcal{D}) \text{ for } i=1,2,\dots,N\}$$

is a Banach space we associate it with the norm

$$\|w\|=\|w\|_{L^{p_0}(\mathcal{D})}+\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}(\mathcal{D})}\,.$$

In what follows, introducing certain categories of mappings. Subsequently, consider $\mathcal{X}$ real, separable, reflexive Banach space, and $\mathcal{X}^*$ be its dual space with dual pairing $\langle\cdot,\cdot\rangle$ and given a nonempty set $\mathcal{D}$ of $\mathcal{X}$, $\overline{\mathcal{D}}$ and $\partial\mathcal{D}$ will denote the closure and the boundary of $\mathcal{D}$ in $\mathcal{X}$, strong (weak) convergence is represented by the notation $\to$ $(\rightharpoonup)$ respectively.
Let us recall the Hardy inequality

$$\int_{\mathcal{D}}\frac{|\ u(x)\ |^p}{|\ x\ |^p}\ \mathrm{d}x\leq\frac{1}{H}\int_{\mathcal{D}}|\ \nabla u(x)\ |^p\ \ \mathrm{d}x$$

where $\mathcal{D}$ is open set in $\mathbb{R}^n$ containing the origin and H is the best constant in the inclusion of $W_0^{1,p}(\mathcal{D})$ in $L^p(\mathcal{D})$ with weight $|\ x\ |^{-p}$ In particular, when $\mathcal{D}$ is a ball,$H=(\frac{n-p}{p})^p$

Remark 1 *[13,23] Thanks to the Sobolev embedding theorem and the continuous embedding $W^{1,\vec{p}}(\mathcal{D}) \hookrightarrow W^{1,p^-}(\mathcal{D})$, it follows that the embedding $W^{1,\vec{p}}(\mathcal{D}) \hookrightarrow L^{p^-}(\mathcal{D})$ is compact.*

Definition 1 *Let $\mathcal{Y}$ is a real Banach space and $\mathcal{L}$ is an operator defined : $\mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ where.*

i) if $\mathcal{L}$ takes any bounded set into a bounded set, then $\mathcal{L}$ is bounded.

ii) $\mathcal{L}$ is compact, if it is continuous and the image of any bounded set is relatively compact.

iii) demicontinuous, if for any sequence $(\vartheta_k)_{k\in\mathbb{N}} \subset \mathcal{D}$, ϑ_k converge strongly to ϑ implies $\mathcal{L}(\vartheta_k)$ converge weakly to $\mathcal{L}(\vartheta)$.

Definition 2 *Let $\mathcal{L} : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}^*$ be a mapping.*

i) quasimonotone, if for each $(\vartheta_k)_{k\in\mathbb{N}} \subset \mathcal{D}$ with ϑ_k converge weakly to ϑ, we have $\limsup_{n\to\infty}\langle \mathcal{L}\vartheta_k,\ \vartheta_k - \vartheta\rangle \geq 0$.

ii) If for each $(\vartheta_k) \subset \mathcal{D}$ with ϑ_k converge weakly to ϑ and $\limsup_{n\to\infty}\langle \mathcal{L}\vartheta_k, \vartheta_k - \vartheta\rangle \leq 0$, we have ϑ_k converge strongly to ϑ, then $\mathcal{L}$ is of condition $(\mathcal{S}_+)$.

Definition 3 *Consider $T : \mathcal{D}_1 \subset \mathcal{X}$ into $\mathcal{X}^*$ be a bounded operator such that $\mathcal{D} \subset \mathcal{D}_1$. For each $\mathcal{L} : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}$, we say that*

i) If for any $(\vartheta_k)_{k\in\mathbb{N}} \subset \mathcal{D}$ with ϑ_k converge weakly to ϑ, $\sigma_k := T\vartheta_k \rightharpoonup \sigma$ and $\limsup_{k\to\infty}\langle \mathcal{L}\vartheta_k,\ \sigma_k - \sigma\rangle \leq 0$, we have ϑ_k converge strongly to ϑ, then $\mathcal{L}$ satisfies condition $(\mathcal{S}_+)_{\mathcal{T}}$.

ii) $\mathcal{L}$ has the property $(QM)_T$ If for any sequence $(\mathcal{V}_k) \subset \mathcal{D}$ with ϑ_k converge weakly to ϑ, $\sigma_k := T\vartheta_k \rightharpoonup \sigma$, we have $\limsup_{k\to\infty}\langle \mathcal{L}\vartheta_k, \sigma - \sigma_k\rangle \geq 0$.

Now, we consider Θ be the union of all open bounded sets in $\mathcal{X}$. For each $\mathcal{D} \subset \mathcal{X}$, we take the following classes of operators:

$$
\begin{aligned}
&\Re_1(\mathcal{D}) := \{\mathcal{L} : \mathcal{D} \to \mathcal{X}^* : \mathcal{L} \text{ is of condition } (\mathcal{S}_+), \text{ bounded and demicontinuous } \},\\
&\Re_{T,B}(\mathcal{D}) := \{\mathcal{L} : \mathcal{D} \to \mathcal{X} : \mathcal{L} \text{ is of condition } (\mathcal{S}_+)_T \text{ demicontinuous and bounded } \},\\
&\Re_T(\mathcal{D}) := \{\mathcal{L} : \mathcal{D} \to \mathcal{X} : \mathcal{L} \text{ is of condition } (\mathcal{S}_+)_T \text{ and demicontinuous}\},\\
&\Re_B(\mathcal{X}) := \{\mathcal{L} \in \Re_{T,B}(\overline{E}) :\ \mathrm{T} \in \Re_1(\overline{\mathrm{E}}), E \in \Theta\ \}.
\end{aligned}
$$

during this document. $T \in \Re_1(\overline{E})$ is said an essential inner map to $\mathcal{L}$.

Lemma 1 *[8, Lemmas 2.2 and 2.4], Let $T \in \Re_1(\overline{E})$ be continuous, where E is a bounded open set in a real reflexive Banach space $\mathcal{X}$, $\mathcal{I}$ denotes the identity operator and $\mathcal{S} : D_{\mathcal{S}} \subset \mathcal{X}^* \to \mathcal{X}$ be demicontinuous such that $T(\overline{E}) \subset D_{\mathcal{S}}$. Then the following assertions are completely correct:*

i) $\mathcal{S}$ is quasimonotone then $\mathcal{I} + \mathcal{S} \circ T \in \Re_T(\overline{E})$,

ii) $\mathcal{S}$ is of condition $(\mathcal{S}_+)$ then $\mathcal{S} \circ T \in \Re_T(\overline{E})$.

Definition 4 *Suppose that E is bounded open subset of a real reflexive Banach space, consider $\mathcal{S}, \mathcal{L} \in \Re_T(\overline{E})$ and $\mathcal{X}$,$T \in \Re_1(\overline{E})$ be continuous. The affine homotopy $A : [0,1] \times \overline{E} \to \mathcal{X}$ defined by*

$$A(t,\ w) := tSw + (1-t)\mathcal{L}w \qquad \text{for} \qquad (t,\ w) \in [0,1] \times \overline{E}$$

Remark 2 *[8] The previously mentioned affine homotopy conforms to the class $(\mathcal{S}_+)_T$.*

We are now presenting the Berkovits topological degree for $\Re_B(\mathcal{X})$ for further information see [8,22,14].

Theorem 1 *A unique function degree exists*

$$\delta : \big\{(\mathcal{H}, \mathcal{E}, k) \mid \mathcal{E} \in \Theta,\ T \in \Re_1(\overline{\mathcal{E}}),\ \mathcal{H} \in \Re_{T,B}(\overline{\mathcal{E}}),\ k \notin F(\partial\mathcal{E})\big\} \longrightarrow \mathbb{Z}$$

which verifies the properties listed below

1. *For all k in $\mathcal{E}$, we have $\delta(\mathcal{I}, \mathcal{E}, k) = 1$ (**Normalization property**).*
2. *If $\delta(\mathcal{H}, \mathcal{E}, k) \neq 0$, then the equation $\mathcal{H}w = k$ has a solution in $\mathcal{E}$, (**Existence property**).*
3. *If h: $[0,1] \to \mathcal{X}$ is a continuous map in $\mathcal{X}$ such that $k(t) \notin A(t, \partial\mathcal{E})$ for all $0 \leq t \leq 1$ and $A : [0,1] \times \overline{\mathcal{E}} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map, then the value of $\delta(A(t,\cdot), \mathcal{E}, k(t))$ is constant for all $0 \leq t \leq 1$ (**Homotopy invariance property**).*

3. Notions of solution

In this part, we define what constitutes a weak solution for the Problem (1.1) and present a few helpful lemmas and techniques to support proofs of existence findings.

Definition 5 *w is a weak solution of (1.1) (it is a measurable function), if $w \in W^{1,\vec{p}}(\mathcal{D})$. Then, using integration by parts, then we have*

$$\sum_{i=1}^{N} \int_{\mathcal{D}} \kappa_i(\tau, w, \nabla w)\partial_i\vartheta d\tau - \Theta \int_{\mathcal{D}} \frac{|w|^{p_0-2}w}{|\tau|^{p_0}}\vartheta\, d\tau + \Gamma \int_{\mathcal{D}} |w|^{p_0-2}w\vartheta d\tau = \lambda \int_{\mathcal{D}} \mathcal{A}(\tau, w, \nabla w)\vartheta d\tau,$$

for all $\vartheta \in W^{1,\vec{p}}(\mathcal{D})$.

Lemma 2 *[5] Let $h \in L^{r_i}(\mathcal{D})$ and let a sequence $(h_n)_{n\in\mathbb{N}} \subset L^{r_i}(\mathcal{D})$ such that*

$$\|h_n\|_{r_i} \leq C, \quad 1 < r_i < \infty,$$

for all $i = 1, \ldots, N$. If $h_n(\tau) \to h(\tau)$ almost everywhere in $\mathcal{D}$, then

$$h_n \rightharpoonup h \quad \text{weakly in } L^{r_i}(\mathcal{D}) \text{ for any } i = 1, \ldots, N.$$

We will defined the nonlinear operator B in $W^{1,\vec{p}}(\mathcal{D})$ into its dual by

$$\langle Bw, \vartheta\rangle = \sum_{i=1}^{N} \int_{\mathcal{D}} \kappa_i(\tau, w, \nabla w)\partial_i\vartheta\, d\tau - \Theta \int_{\mathcal{D}} \frac{|w|^{p_0-2}w}{|\tau|^{p_0}}\vartheta\, d\tau + \Gamma \int_{\mathcal{D}} |w|^{p_0-2}w\vartheta d\tau, \text{ for all } \vartheta \text{ in } W^{1,\vec{p}}(\mathcal{D}).$$

Lemma 3 *Under the conditions $(\kappa_1) - (\kappa_3)$. Then,*

(1) B is coercive, continuous and bounded,

(2) B is of class $(\mathcal{S}_+)$.

Proof 1 *We start by demonstrate that B is an operator bounded.*
We consider $w, v \in W^{1,\vec{p}}(\mathcal{D})$, We decompose B into three components:

$$\langle Bw, \vartheta\rangle = \langle B_1w, \vartheta\rangle + \langle B_2w, \vartheta\rangle + \langle B_3w, \vartheta\rangle,$$

where

$$\langle B_1w, \vartheta\rangle = \sum_{i=1}^{N} \int_{\mathcal{D}} \kappa_i(\tau, w, \nabla w)\partial_i\vartheta\, d\tau,$$

$$\langle B_2w, \vartheta\rangle = -\Theta \int_{\mathcal{D}} \frac{|w|^{p_0-2}w}{|\tau|^{p_0}}\vartheta\, d\tau,$$

$$\langle B_3 w, \vartheta\rangle = \Gamma \int_{\mathcal{D}} |w|^{p_0-2} w\vartheta \, d\tau.$$

with the help of inequality the Hölder's and we have

$$\langle Bw, \vartheta\rangle \leq C \sum_{i=1}^{N} \Big(\int_{\mathcal{D}} |\kappa_i(\tau, w, \nabla w)|^{p_i'} d\tau \Big)^{\frac{1}{p_i'}} \Big(\int_{\mathcal{D}} |\partial_i \vartheta|^{p_i} d\tau \Big)^{\frac{1}{p_i}} + \Theta \int_{\mathcal{D}} \frac{|w|^{p_0-1}}{|\tau|^{p_0}} \vartheta \, d\tau + \Gamma \int_{\mathcal{D}} |w|^{p_0-2} w\vartheta d\tau.$$

Step 1: Bound for B_1.

$$\begin{aligned} \langle B_1 w, \vartheta\rangle &\leq C_1 \sum_{i=1}^{N} \Big(\int_{\mathcal{D}} |\partial_i \vartheta|^{p_i} d\tau \Big)^{1/p_i} \\ &\leq C_1 \sum_{i=1}^{N} \|\partial_i \vartheta\|_{L^{p_i}} \\ &\leq C_1 \|\vartheta\| . \end{aligned}$$

Thus B_1 *is bounded.*
Step 2: Bound for B_2.
For the Hardy term,

$$\Theta \int_{\mathcal{D}} \frac{|w|^{p_0-1}}{|\tau|^{p_0}} |\vartheta| \, d\tau \leq \Theta \left\| \frac{|w|^{p_0-1}}{|\tau|^{p_0}} \right\|_{L^{p_0/(p_0-1)}} \|\vartheta\|_{L^{p_0}}.$$

Since $\|w\|_{L^{p_0}} \leq C_1 M$, *we have*

$$\|w\|_{L^{p_0}}^{p_0-1} \leq (C_1 M)^{p_0-1}.$$

The function $\frac{1}{|\tau|^{p_0}}$ *is locally bounded, and*

$$\begin{aligned} \|\vartheta\|_{L^{p_0}} &\leq C_1 \|\vartheta\| \\ &\leq C_1. \end{aligned}$$

Thus,

$$\Theta \int_{\mathcal{D}} \frac{|w|^{p_0-1}}{|\tau|^{p_0}} |\vartheta| \, d\tau \leq \Theta (C_1 M)^{p_0-1} C_1 \left\| \frac{1}{|\tau|^{p_0}} \right\|_{L^{p_0/(p_0-1)}}.$$

This is bounded by a constant depending on M.
Step 3: Bound for B_3.
For the new term,

$$\begin{aligned} \Gamma \int_{\mathcal{D}} |w|^{p_0-1} |\vartheta| \, d\tau &\leq \Gamma \|w\|_{L^{p_0}}^{p_0-1} \|\vartheta\|_{L^{p_0}} \\ &\leq \Gamma (C_1 M)^{p_0-1} C_1 \|\vartheta\| \\ &\leq \Gamma C_1^2 M^{p_0-1}. \end{aligned}$$

This is also bounded by a constant depending on M.
We obtain that B *is bounded.*

Second, we demonstrate B is coercive.
To prove coercivity, we show that

$$\frac{\langle Bw, w\rangle}{\|w\|} \to +\infty \quad \text{as} \quad \|w\| \to +\infty.$$

Compute

$$\langle Bw, w\rangle = \sum_{i=1}^{N} \int_{\mathcal{D}} \kappa_i(\tau, w, \nabla w)\partial_i w \, d\tau - \Theta \int_{\mathcal{D}} \frac{|w|^{p_0}}{|\tau|^{p_0}} \, d\tau + \Gamma \int_{\mathcal{D}} |w|^{p_0} \, d\tau.$$

By condition (κ_1),

$$\sum_{i=1}^{N} \kappa_i(\tau, w, \nabla w)\partial_i w \geq \varrho \sum_{i=1}^{N} |\partial_i w|^{p_i},$$

thus

$$\begin{aligned}\sum_{i=1}^{N} \int_{\mathcal{D}} \kappa_i(\tau, w, \nabla w)\partial_i w \, d\tau &\geq \varrho \sum_{i=1}^{N} \int_{\mathcal{D}} |\partial_i w|^{p_i} \, d\tau \\ &= \varrho \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}^{p_i}.\end{aligned}$$

By the Hardy inequality,

$$\int_{\mathcal{D}} \frac{|w|^{p_0}}{|\tau|^{p_0}} \, d\tau \leq \frac{1}{H} \int_{\mathcal{D}} |\nabla w|^{p_0} \, d\tau,$$

There exists a constant $C > 0$ such that

$$\int_{\mathcal{D}} |\nabla w|^{p_0} \, d\tau \leq C \sum_{i=1}^{N} \int_{\mathcal{D}} |\partial_i w|^{p_i} \, d\tau,$$

due to the embedding $W^{1,\vec{p}}(\mathcal{D}) \hookrightarrow W^{1,p_0}(\mathcal{D})$. Thus,

$$\begin{aligned}\Theta \int_{\mathcal{D}} \frac{|w|^{p_0}}{|\tau|^{p_0}} \, d\tau &\leq \frac{\Theta C}{H} \sum_{i=1}^{N} \int_{\mathcal{D}} |\partial_i w|^{p_i} \, d\tau \\ &= \frac{\Theta C}{H} \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}^{p_i}.\end{aligned}$$

Such that

$$\Gamma \int_{\mathcal{D}} |w|^{p_0} \, d\tau = \Gamma \|w\|_{L^{p_0}}^{p_0}.$$

Combine terms

$$\begin{aligned}\langle Bw, w\rangle &\geq \varrho \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}^{p_i} - \frac{\Theta C}{H} \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}^{p_i} + \Gamma \|w\|_{L^{p_0}}^{p_0}. \\ &= \left(\varrho - \frac{\Theta C}{H}\right) \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}^{p_i} + \Gamma \|w\|_{L^{p_0}}^{p_0}.\end{aligned}$$

Then

$$\frac{\langle Bw, w\rangle}{\|w\|} \geq \frac{\left(\varrho - \frac{\Theta C}{H}\right) \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}^{p_i} + \Gamma \|w\|_{L^{p_0}}^{p_0}}{\|w\|_{L^{p_0}} + \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}}}.$$

We consider two cases, as $\|w\| = \|w\|_{L^{p_0}} + \sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}} \to +\infty$.

- Case 1: $\|w\|_{L^{p_0}} \to +\infty$.

Suppose $\|w\|_{L^{p_0}} \to +\infty$. The numerator includes $\Gamma\|w\|_{L^{p_0}}^{p_0}$, where $p_0 > 2$, by the embedding $W^{1,\vec{p}}(\mathcal{D}) \hookrightarrow L^{p_0}(\mathcal{D})$,

$$\|w\|_{L^{p_0}} \le C_1\|w\|,$$

and

$$\sum_{i=1}^{N} \|\partial_i w\|_{L^{p_i}} \le \|w\|.$$

Thus, the denominator is bounded by $(C_1+N)\|w\|_{L^{p_0}}$.

$$\begin{aligned}\frac{\Gamma\|w\|_{L^{p_0}}^{p_0}}{\|w\|_{L^{p_0}} + \sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}} &\ge \frac{\Gamma\|w\|_{L^{p_0}}^{p_0}}{(C_1+N)\|w\|_{L^{p_0}}}\\ &= \frac{\Gamma}{C_1+N}\|w\|_{L^{p_0}}^{p_0-1}.\end{aligned}$$

Since $p_0 - 1 > 1$, this term tends to $+\infty$ as $\|w\|_{L^{p_0}} \to +\infty$.

- Case 2: $\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}} \to +\infty$.

Suppose $\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}} \to +\infty$. The numerator includes $\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}^{p_i}$, where $p_i > 2$.

Let $\beta = \varrho - \frac{\Theta C}{H}$. Even if $\beta \le 0$, the term $\Gamma\|w\|_{L^{p_0}}^{p_0}$ is non-negative. Since $\|w\|_{L^{p_0}} \le C_1\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}$ (by Sobolev embeddings). Thus

$$\begin{aligned}\frac{\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}^{p_i}}{\|w\|_{L^{p_0}} + \sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}} &\ge \frac{\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}^{p_i}}{(C_1+1)\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}}\\ &\ge \frac{1}{C_1+1}\sum_{i=1}^{N}\|\partial_i w\|_{L^{p_i}}^{p_i-1}.\end{aligned}$$

Since $p_i - 1 > 1$, this tends to $+\infty$.

In both cases, the ratio $\dfrac{\langle Bw, w\rangle}{\|w\|} \to +\infty$, we deduce that B is coercive.

Then B is an operator coercive.

Next, we prove that B is continuous, we consider $w_n \to w$ in $W^{1,\vec{p}}(\mathcal{D})$. Then $w_n \to w$ in $L^{p_0}(\mathcal{D})$ and $\partial_i w_n \to \partial_i w$ in $L^{p_i}(\mathcal{D})$.

This implies the existence of a subsequence (w_k) of (w_n) and $g_i, h \in L^{p_i}(\mathcal{D})$ (two measurable functions) with the condition that

$$\begin{aligned}&w_k(\tau) \to w(\tau) \quad \text{in } L^{p_0}(\mathcal{D}) \quad \text{and} \quad |w_k(\tau)| \le h(\tau),\\ &\partial_i w_k(\tau) \to \partial_i w(\tau) \text{ in } L^{p_i}(\mathcal{D}) \quad \text{and} \quad |\partial_i w_k(\tau)| \le |g_i(\tau)| \text{ for any } i = 1,\dots,N,\end{aligned}$$

for almost everywhere $\tau \in \mathcal{D}$ and all $k \in \mathbb{N}$, we infer

$$\kappa_i(\tau, w_k, \nabla w_k) \to \kappa_i(\tau, w, \nabla w) \qquad \text{a. e. } \tau \in \mathcal{D} \tag{3.1}$$

According to (κ_2), we have

$$\Big|\kappa_i(\tau, w_k, \nabla w_k)\Big| \le \mu(|h(\tau)|^{p_0-1} + m_i(\tau) + |g_i(\tau)|^{p_i-1}), \qquad \text{for each } i = 1,\dots,N.$$

Given that $\left(|g_i(\tau)|^{p_i-1}+|h(\tau)|^{p_0-1}+m_i(\tau)\right)\in L^{p_i'}(\mathcal{D})$, for each $i=1,\ldots,N$.
Using the (3.1), for all $i=1,\ldots,N$, we have

$$\int_{\mathcal{D}}|\kappa_i(\tau,w_k,\nabla w_k)-\kappa_i(\tau,w,\nabla w)|^{p_i'}d\tau\longrightarrow 0.$$

According to the dominated convergence theorem we have, for all $i=1,\ldots,N$, then

$$\kappa_i(\tau,w_k,\nabla w_k)\to\kappa_i(\tau,w,\nabla w)\qquad\text{in}\quad L^{p_i'}(\mathcal{D}).$$

$$\frac{|w_k|^{p_0-2}w_k}{|\tau|^{p_0}}\to\frac{|w|^{p_0-2}w}{|\tau|^{p_0}}\qquad\text{in}\quad L^{p_0'}(\mathcal{D}).$$

Consider the term $\Gamma\int_{\mathcal{D}}|w_j|^{p_0-2}w_j\vartheta\,d\tau$.
As before, $w_j\to w$ in $L^{p_0}(\mathcal{D})$ implies $|w_j|^{p_0-2}w_j\to|w|^{p_0-2}w$ in $L^{p_0/(p_0-1)}(\mathcal{D})$. Since $\vartheta\in L^{p_0}(\mathcal{D})$, we have

$$|w_j|^{p_0-2}w_j\vartheta\to|w|^{p_0-2}w\vartheta\quad\text{in }L^1(\mathcal{D}),$$

by Hölder's inequality. Thus,

$$\int_{\mathcal{D}}|w_j|^{p_0-2}w_j\vartheta\,d\tau\to\int_{\mathcal{D}}|w|^{p_0-2}w\vartheta\,d\tau,$$

Consequently, we have $\langle Bw_n,\vartheta\rangle$ converges to $\langle Bw,\vartheta\rangle$, for any $\vartheta\in W^{1,\vec{p}}(\mathcal{D})$ then B is continuous.

To prove that B satisfies condition $(\mathcal{S}_+)$, assume that $\{w_j\}\subset W^{1,\vec{p}}(\mathcal{D})$ is a sequence such that

$$w_j\rightharpoonup w\quad\text{weakly in }W^{1,\vec{p}}(\mathcal{D}),$$

and

$$\limsup_{j\to\infty}\langle Bw_j,w_j-w\rangle\le 0.$$

We need to show that $w_j\to w$ strongly in $W^{1,\vec{p}}(\mathcal{D})$.

$$\langle Bw_j,w_j-w\rangle=\sum_{i=1}^{N}\int_{\mathcal{D}}\kappa_i(\tau,w_j,\nabla w_j)\partial_i(w_j-w)\,d\tau-\Theta\int_{\mathcal{D}}\frac{|w_j|^{p_0-2}w_j}{|\tau|^{p_0}}(w_j-w)\,d\tau+\Gamma\int_{\mathcal{D}}|w_j|^{p_0-2}w_j(w_j-w)\,d\tau.$$

Since $w_j\rightharpoonup w$ weakly in $W^{1,\vec{p}}(\mathcal{D})$, the compact embedding $W^{1,\vec{p}}(\mathcal{D})\hookrightarrow L^{p_0}(\mathcal{D})$ (see Remark 1) implies

$$w_j\to w\quad\text{strongly in }L^{p_0}(\mathcal{D}).$$

For the Hardy term:

$$\Theta\int_{\mathcal{D}}\frac{|w_j|^{p_0-2}w_j}{|\tau|^{p_0}}(w_j-w)\,d\tau.$$

Since $w_j\to w$ in $L^{p_0}(\mathcal{D})$, the map $w\mapsto|w|^{p_0-2}w$ is continuous, so

$$|w_j|^{p_0-2}w_j\to|w|^{p_0-2}w\quad\text{in }L^{p_0/(p_0-1)}(\mathcal{D}).$$

Thus, $\frac{|w_j|^{p_0-2}w_j}{|\tau|^{p_0}}\to\frac{|w|^{p_0-2}w}{|\tau|^{p_0}}$ in $L^{p_0/(p_0-1)}(\mathcal{D})$, and since $w_j-w\to 0$ in $L^{p_0}(\mathcal{D})$,

$$\Theta\int_{\mathcal{D}}\frac{|w_j|^{p_0-2}w_j}{|\tau|^{p_0}}(w_j-w)\,d\tau\to 0.$$

For the term:

$$\Gamma\int_{\mathcal{D}}|w_j|^{p_0-2}w_j(w_j-w)\,d\tau.$$

Similarly, $|w_j|^{p_0-2}w_j \to |w|^{p_0-2}w$ in $L^{p_0/(p_0-1)}(\mathcal{D})$, and

$$\Gamma\int_{\mathcal{D}} |w_j|^{p_0-2}w_j(w_j-w)\,d\tau \to 0.$$

From the assumption $\limsup\limits_{j\to\infty}\langle Bw_j, w_j-w\rangle \le 0$, and since the second and third terms converge to zero, we have

$$\limsup_{j\to\infty}\sum_{i=1}^{N}\int_{\mathcal{D}} \kappa_i(\tau, w_j, \nabla w_j)\partial_i(w_j-w)\,d\tau \le 0.$$

We need to show that $\nabla w_j \to \nabla w$ strongly in $L^{p_i}(\mathcal{D})$. Assume for contradiction, that ∇w_j does not converge to ∇w. By condition (κ_3), the operator defined by κ_i is strictly monotone. Since $w_j \to w$ in $L^{p_0}(\mathcal{D})$. For the term

$$\sum_{i=1}^{N}\int_{\mathcal{D}} \left(\kappa_i(\tau, w_j, \nabla w_j) - \kappa_i(\tau, w_j, \nabla w)\right)\partial_i(w_j-w)\,d\tau$$

is positive unless $\nabla w_j = \nabla w$. Since $w_j \rightharpoonup w$, we use the monotonicity to deduce that the limsup condition forces $\nabla w_j \to \nabla w$ in measure, and by the boundedness of $\{w_j\}$, we conclude $\partial_i w_j \to \partial_i w$ in $L^{p_i}(\mathcal{D})$.

Since $w_j \to w$ in $L^{p_0}(\mathcal{D})$ and $\partial_i w_j \to \partial_i w$ in $L^{p_i}(\mathcal{D})$, we have

$$\|w_j - w\| = \|w_j - w\|_{L^{p_0}} + \sum_{i=1}^{N}\|\partial_i w_j - \partial_i w\|_{L^{p_i}} \to 0.$$

Thus, $w_j \to w$ strongly in $W^{1,\vec{p}}(\mathcal{D})$, and B satisfies condition $(\mathcal{S}_+)$.

Lemma 4 *Under the condition $(\mathcal{A}_1)$ holds. Then $\mathcal{M} : W^{1,\vec{p}}(\mathcal{D}) \to \left(W^{1,\vec{p}}(\mathcal{D})\right)^*$ is an operator given by the expression*

$$\langle \mathcal{M}w,\ v\rangle = -\int_{\mathcal{D}} \lambda\mathcal{A}(\tau, w, \nabla w)v\,d\tau, \qquad \text{for any } w,\ v \in W^{1,\vec{p}}(\mathcal{D})$$

is compact.

Proof 2 *The proof is structured in two parts.*

First step:

Let $\psi : W^{1,\vec{p}}(\mathcal{D}) \to L^{p_0'}(\mathcal{D})$ be the operator established by

$$\psi w(\tau) := -\mathcal{A}(\tau, w(\tau), \nabla w(\tau)), \quad \text{for all } w \in W^{1,\vec{p}}(\mathcal{D}).$$

We next show that ψ is well-defined, sends bounded set into bounded sets (that is, ψ is a bounded operator), and it is continuous.

$$\begin{aligned}
\|\psi w\|_{p'}^{p_0'} &\le \int_{\mathcal{D}} \lambda^{p_0'}|\mathcal{A}(\tau, w, \nabla w)|^{p_0'}d\tau\\
&\le (C\lambda)^{p_0'}\Big(\int_{\mathcal{D}} |\rho(\tau)|^{p_0'} + |w|^{(p_0-1)p_0'} + \sum_{i=1}^{N}|\partial_i w|^{p_i}d\tau\Big)\\
&\le (C\lambda)^{p_0'}\int_{\mathcal{D}} |\rho(\tau)|^{p_0'}d\tau + (C\lambda)^{p_0'}\Big(\int_{\mathcal{D}} |w|^{p_0}d\tau + \sum_{i=1}^{N}\int_{\mathcal{D}} |\partial_i w|^{p_i}d\tau\Big)\\
&\le (C\lambda)^{p_0'}\int_{\mathcal{D}} |\rho(\tau)|^{p_0'}d\tau + (C\lambda)^{p_0'}\|w\|^{p_-^+}\\
&\le C_{mx}\|w\|^{p_m},
\end{aligned}$$

where

$$p_m = \begin{cases} p^- & \text{if } \| \partial_i v \| \leq 1, \\ p_0 & \text{if } \| \partial_i v \| > 1, \end{cases}$$

and $C_{mx} = max((C\lambda)^{p_0'} \|\rho(\tau)\|^{p_0'}, (C\lambda)^{p_0'})$, *($\rho(\tau)$ is positive function in $L^{p_i'}(\mathcal{D})$).*
Hence, ψ is an operator bounded.
To prove that ψ is an operator continuous, let $w_k \to w$ in $W^{1,\vec{p}}(\mathcal{D})$. Consequently, we have $w_k \to w$ in $L^{p_0}(\mathcal{D})$ and $\partial_i w_k \to \partial_i w$ in $L^{p_i}(\mathcal{D})$ for all $i = 1, \ldots, N$.

Consider any subsequence (w_{k_l}) of $(w_k)_{k\in\mathbb{N}}$, which we will still denote by (w_k) for simplicity. By standard arguments, there exist functions $v \in L^{p_0}(\mathcal{D})$ and $\varpi_i \in L^{p_i}(\mathcal{D})$ for $i = 1, \ldots, N$, such that for a further subsequence, also denoted by (w_k), the following inequalities hold

$$|w_k(\tau)| \leq v(\tau) \quad \text{and} \quad |\partial_i w_k(\tau)| \leq \varpi_i(\tau), \quad \text{for almost everywhere } \tau \in \mathcal{D},\ i = 1, \ldots, N.$$

By the assumption $(\mathcal{A}_1)$, we then have

$$\mathcal{A}(\tau, w_k, \nabla w_k) \leq C \left(|\rho(\tau)| + |w(\tau)|^{p_0 - 1} + \sum_{i=1}^{n} |\partial_i w(\tau)|^{p_i} \right) \in L^{p_0}(\mathcal{D}).$$

Additionally, the sequence satisfies

$$|\mathcal{A}(\tau, w_k, \nabla w_k) - \mathcal{A}(\tau, w, \nabla w)|^{p_0'} \to 0 \quad \text{a.e. in } \mathcal{D}.$$

Thus, we obtain by Lebesgue Theorem

$$\int_{\mathcal{D}} |\mathcal{A}(\tau, w_k, \partial_i w_k) - \mathcal{A}(\tau, w, \partial_i w)|^{p_0'} \, d\tau \to 0, \qquad \text{as } k \to \infty.$$

Since this holds for any subsequence of (w_k), we prove that ψ is continuous.
Second step:
$\mathcal{M}$ is compact, observe that $\mathcal{M} = \mathcal{I}^* \circ \psi + \mathcal{I}^* \circ \phi$ and $\mathcal{I} : W^{1,\vec{p}}(\mathcal{D}) \to L^{p_0}(\mathcal{D})$ is compact, where $\mathcal{I}^*$ is the adjoint operator $\mathcal{I}^* : L^{p_0'}(\mathcal{D}) \to \left(W^{1,\vec{p}}(\mathcal{D})\right)^*$ is additionally compact. Thus, the compositions $\mathcal{I}^* \circ \phi$ and $\mathcal{I}^* \circ \psi$ from $W^{1,\vec{p}}(\mathcal{D}) \to \left(W^{1,\vec{p}}(\mathcal{D})\right)^*$ are compact. Hence, we conclude that $\mathcal{M}$ is compact.

4. Principal results

We investigate the existence of weak solution to the problem (1.1), relying on the recent Berkovits topological degree theory in anisotropic spaces.

Theorem 2 *Under the conditions $(\kappa_1) - (\kappa_3)$ and $(\mathcal{A}_1)$. Then, our problem has a weak solution w in $W^{1,\vec{p}}(\mathcal{D})$, according to definition 5*

Proof 3 *Consider $w \in W^{1,\vec{p}}(\mathcal{D})$, the problem (1.1) has a weak solution w if and only if*

$$Bw = -\mathcal{M}w. \tag{4.1}$$

Following Lemma 4, $\mathcal{M}$ is an operator bounded, continuous and quasimonotone. According to Lemma 3 B is an operator strictly monotone. Additionally, observe by Applying the Minty-Browder Theorem 2.6 (see [24]), it exists an operator $G := B^{-1} : \left(W^{1,\vec{p}}(\mathcal{D})\right)^* \to W^{1,\vec{p}}(\mathcal{D})$ is satisfies condition (S_+), continuous and bounded.
Therefore, (4.1) is equivalent to solve the following Hammerstein equation :

$$G\vartheta = w \quad \text{and} \quad \mathcal{M} \circ G\vartheta + \vartheta = 0. \tag{4.2}$$

To determine solutions to equations (4.2), we will apply the Berkovits topological degree presented in the second section. To achieve this, we start by demonstrating that the set

$$\Upsilon := \left\{ \vartheta \in \left(W^{1,\vec{p}}(\mathcal{D})\right)^* : \quad \vartheta + t\mathcal{M} \circ G\vartheta = 0 \quad \text{for any} \quad 0 \leq t \leq 1 \right\}$$

is bounded. Indeed, let $\vartheta \in \Upsilon$ and take $w := G\vartheta$. Thanks to (κ_1), $(\mathcal{A}_1)$, Jensen's inequality, the compact embedding $(W^{1,\vec{p}}(\mathcal{D}) \hookrightarrow\hookrightarrow L^{p_0}(\mathcal{D}))$, the continuous imbedding$(W^{1,\vec{p}} \hookrightarrow W^{1,p_0})$ and the Young's inequality, then

$$
\begin{aligned}
\|G\vartheta\|^{p_-^+} = \Big(\|w\|_{p_0} + \sum_{i=1}^N \|\partial_i w\|_{p_i}\Big)^{p_-^+} &\leq 2^{p_-^+-1}\Big[\Big(\int_{\mathcal{D}} |w|^{p_0}\,d\tau\Big)^{\frac{p_-^+}{p_0}} + \Big(\sum_{i=1}^N \Big(\int_{\mathcal{D}} |\,\partial_i w\,|^{p_i}\,d\tau\Big)^{\frac{1}{p_i}}\Big)^{p_-^+}\Big] \\
&\leq 2^{p_-^+-1}\Big[\int_{\mathcal{D}} |w|^{p_0}\,d\tau + C\sum_{i=1}^N \int_{\mathcal{D}} |\,\partial_i w\,|^{p_i}\,d\tau\Big] \\
&\leq C_m\Big[\int_{\mathcal{D}} |w|^{p_0}\,d\tau + \frac{1}{\varrho}\sum_{i=1}^N \int_{\mathcal{D}} \kappa_i(\tau, w, \nabla w)\partial_i w d\tau\Big] \\
&\leq C_m\Big[\int_{\mathcal{D}} |w|^{p_0} d\tau + \frac{1}{\varrho}\big(\langle Bw, w\rangle + \Theta\int_{\mathcal{D}} \frac{|w|^{p_0}}{|\tau|^{p_0}} d\tau + \Gamma\int_{\mathcal{D}} |w|^{p_0} d\tau\big)\Big] \\
&\leq C_m\Big((\Gamma+1)\int_{\mathcal{D}} |w|^{p_0}\,d\tau + \frac{1}{\varrho}\big(\langle\vartheta, Gv\rangle + \Theta\int_{\mathcal{D}} \frac{|w|^{p_0}}{|\tau|^{p_0}} d\tau\big)\Big) \\
&\leq C_m\Big((\Gamma+1)\int_{\mathcal{D}} |w|^{p_0} d\tau + C_h\int_{\mathcal{D}} |\nabla w|^{p_0} d\tau + \frac{t}{\varrho}\big(|\langle\mathcal{M}\circ Gv, Gv\rangle|\big)\Big) \\
&\leq C_m\Big(C_h'\|w\|^{p_0}_{W^{1,p_0}} + \frac{t}{\varrho}\int_{\mathcal{D}} |\mathcal{A}(\tau, w, \nabla w)| w d\tau\Big) \\
&\leq C_m\Big(C_h'\|w\|^{p_-^+} + \frac{t\Gamma}{\varrho}\int_{\mathcal{D}} |w|^{p_i} d\tau + \frac{t}{\varrho}C_{p'}\int_{\mathcal{D}} |\mathcal{A}(\tau, w, \nabla w)|^{p_i'} d\tau + \frac{t}{\varrho}C_p\int_{\mathcal{D}} |w|^{p_i} d\tau \\
&\leq C_2\|w\|^{p_-^+} + C_2\|w\|^{p_i/p_i'} \\
&\leq C_s\big(\|Gv\|^{p_-^+} + \|Gv\|^{p_-^+-1}\big),
\end{aligned}
$$

where $C_m = max(2^{p_-^+-1}, C)$.
Hence, we have that $\{G\vartheta \mid \vartheta \in \Upsilon\}$ is bounded. Given that the operator $\mathcal{M}$ is bounded, from (4.2), it is obvious that the set G in $\big(W^{1,\vec{p}}(\mathcal{D})\big)^*$ is bounded. Therefore, there exists a positive constant $\mathcal{K}$ knowing that

$$\|\vartheta\|_{\left(W^{1,\vec{p}}(\mathcal{D})\right)^*} < \mathcal{K} \qquad \text{for all } \vartheta \in \Upsilon.$$

As a result, we find that

$$t\mathcal{M}\circ G\vartheta + \vartheta \neq 0 \qquad \text{for each } \vartheta \in \partial B_{\mathcal{K}}(0) \qquad \text{and every} \qquad 0 \leq t \leq 1.$$

By Lemma 1, we deduct

$$\mathcal{M}\circ G + \mathcal{I} \in \Re_T(\overline{B_{\mathcal{K}}(0)}) \qquad \text{and} \quad \mathcal{I} = B\circ G \in \Re_T(\overline{B_{\mathcal{K}}(0)}).$$

Then G, $\mathcal{M}$ and $\mathcal{I}$ tow operators bounded, $\mathcal{M}\circ G + \mathcal{I}$ is bounded. It follows that

$$\mathcal{I} + \mathcal{M}\circ G \in \Re_{T,B}(\overline{B_{\mathcal{K}}(0)}) \qquad \text{and} \quad \mathcal{I} \in \Re_{T,B}(\overline{B_{\mathcal{K}}(0)}).$$

Let an affine homotopy $A : [0,1] \times \overline{B_{\mathcal{K}}(0)} \to (W^{1,\vec{p}}(\mathcal{D}))^*$ defined by

$$A(t,\vartheta) := \vartheta + t\mathcal{M}\circ Gv \qquad \text{for} \qquad (t,\vartheta) \in [0,1] \times \overline{B_{\mathcal{K}}(0)}.$$

according Theorem 1, we have

$$\delta(\mathcal{I} + \mathcal{M}\circ G, B_{\mathcal{K}}(0), 0) = \delta(\mathcal{I}, B_{\mathcal{K}}(0), 0) = 1.$$

As a result, there is a point $\vartheta \in B_{\mathcal{K}}(0)$ Knowing that

$$\vartheta + \mathcal{M}\circ Gv = 0.$$

which affirms that $w = G\vartheta$ is a weak solution of (1.1).

References

1. Abdellaoui, B., Peral, I., Primo, A.: Elliptic problems with a Hardy potential and critical growth in the gradient: non-resonance and blow-up results. J. Differ. Equations 239, 386–416 (2007)

2. A. Abbassi, C. Allalou and A. Kassidi, Existence of weak solutions for nonlinear p-elliptic problem by topological degree, Nonlinear Dyn. Syst. Theory, 20(3) (2020) 229-241.

3. A. Abbassi, C. Allalou and A. Kassidi, Existence results for some nonlinear elliptic equations via topological degree methods. Journal of Elliptic and Parabolic Equations, 7, 121–136 (2021).

4. A. Abbassi, C. Allalou and A. Kassidi, Topological degree methods for a Neumann problem governed by nonlinear elliptic equation. Moroccan Journal of Pure and Applied Analysis, 6(2), 231-242(2020).

5. Y. Akdim, E. Azroul and A. Benkirane, Existence of solutions for quasilinear degenerate elliptic equations. Electronic Journal of Differential Equations (EJDE), 2001, vol. 2001, p. Paper No. 71, 19.

6. J. P. García Azorero, and I. Peral Alonso, Existence and nonuniqueness for the p-Laplacian, Communications in Partial Differential Equations 12.12 (1987): 126-202.

7. H. Ayadi and F. Mokhtari, Anisotropic elliptic equations with variable exponents and degenerate coercivity. Electronic Journal of Differential Equations 2018 43(2018), 1-23.

8. J. Berkovits, Extension of the Leray–Schauder degree for abstract Hammerstein type mappings. Journal of Differential Equations, 2007, vol. 234, no 1, p. 289-310.

9. L. Boccardo, F. Murat, J.-P. Puel, Existence of bounded solution for nonlinear elliptic unilateral problems. Ann. Mat. Pura Appl. 152(1988); 183-196.

10. F. E. Browder, Fixed point theory and nonlinear problems, Bull. Amer. Math. Soc. (N.S.), 9 (1983), 1-39.

11. Carl S, Le VK, Motreanu D. Nonsmooth variational problems and their inequalities: comparison principles and applications. Springer monographs in mathematics. New York: Springer; 2007.

12. N. Eddine, M. Ragusa and D. Repovs, On the concentration compactness principle for anisotropic variable exponent Sobolev space and applications. Fractional Calculus and Applied Analysis 27 (2024), 725-756.

13. Edmunds, D. E., Evans, W. D. (1987). Spectral theory and differential operators. Oxford.

14. H. El Hammar, C. Allalou, S. Melliani and A. Kassidi, Existence results for a parabolic problem of Kirchhoff type via topological degree. Advanced Mathematical Models and Applications, 8(2023).

15. Hammou MA, Azroul E, Lahmi B. Topological degree methods for a strongly nonlinear p(x)-elliptic problem. Rev Colomb Mat. 2019;53(1):27–39.

16. I.S. Kim, A topological degree and applications to elliptic problems with discontinuous nonlinearity, J. Nonlinear Sci. Appl. 10, 612-624 (2017).

17. D. Motreanu, V. V. Motreanu and N. Papageorgiou, Topological and variational methods with applications to nonlinear boundary value problems. Vol. 1. No. 7. New York: Springer, 2014.

18. P. Ochoa and F. Valverde, Equivalence and regularity of weak and viscosity solutions for the anisotropic $p(x)-$Laplacian. To appear in Nonlinear Differential Equations and Applications.

19. A. Ourraoui, and M. Ragusa, An Existence Result for a Class of p(x)-Anisotropic Type Equations. Symmetry 13, (2021), 1-12.

20. M.M. Porzio : On some quasilinear elliptic equations involving Hardy potential. Rend. di Mat. Ser. VII, Roma 27, 277–297 (2007)

21. G.G. dos Santos, G.M. Figueiredo, and L.S. Tavares, Sub super solution method for nonlocal systems involving the $p(x)-$Laplacian operator, Electron. J. Differ. Equ. 2020 (2020), no. 25, 1–19.

22. G. Zineddaine, A. Sabiry, S. Melliani and A. Kassidi, On a discontinuous nonlinear elliptic problem of nonlocal-type with Neumann boundary condition. J Elliptic Parabol Equ 10, 19–38.

23. G. Zineddaine, A. Sabiry, S. Melliani and A. Kassidi, Anisotropic obstacle Neumann problems in weighted Sobolev spaces and variable exponent, Journal of Applied Analysis, 2024.

24. E. Zeider, Nonlinear Functional Analysis and its Applications, II\B : Nonlinear Monotone Operators, Springer, New York, 1990.

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