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Existence of Solutions of an infinite Mixed Volterra-Fredholm integral system in ℓ_1 space

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ABSTRACT: Applying the FPT, we offer an existence result for an infinite mixed Volterra-Fredholm type nonlinear integral system in the sequence space ℓ_1. Here Meir-Keeler FPT is used and we use the concept of measure of noncompactness. To further demonstrate the given existence result, we provide some examples.

Key Words: Measure of noncompactness (MNC), Volterra Fredholm Integral equation, Fixed point theorem (FPT).

Contents

1. Introduction

Mixed Volterra-Fredholm integral equations is broad field and it contains various sectors, with benefaction from sectors of mathematicians, physicists, engineers, and researchers from other fields. This mixed integral equations are typically occur across a wide range of fields and sectors including modeling, mathematical and physical dynamics, electrodynamics and biology. The term Mixed Volterra-Fredholm integral equations uses in the study of parabolic boundary value problems and is popularly used in mathematical models that describes the spatio-temporal dynamics of epidemics, as well as in various physical and biological systems. There are two versions of the mixed Volterra-Fredholm integral equations, namely:

$$w(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)w(t)dt + \lambda_2 \int_a^b k_2(x,t)w(t)dt \tag{1.1}$$

and the mixed form

$$w(x) = f(x) + \lambda \int_a^x \int_a^b k(r,t)w(t)dtdr. \tag{1.2}$$

Equation (1.1) is of the form of disjoint Volterra and Fredholm integrals, whereas equation (1.2) is of the form of mixed Volterra and Fredholm integrals. Detailed discussions of these types of integral equations can be found in [20]. In this article, we will discuss (1.2) type of equation. In recent times, many research papers have been published to study these type of equations and understand their properties. Researchers are showing interest in exploring how these equations work and what characteristics they have. This implies the fact that these equations are topic of active study. Many generalizations of the same have been given by many researchers, which includes the collocation method in [9], Taylor expansion methods [6], block-pulse functions [18], spectral methods [8] etc. Numerous techniques have been used for computation of nonlinear two dimensional Volterra-Fredholm integral equations, we have given following reference for better understanding [11,19]. Solvability of linear mixed Volterra- Fredholm

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integral equations by applying Banach fixed point theorem discussed in the article [10]. In [18] authors studied nonlinear mixed Volterra-Fredholm type integral equation by using generalized Banach FPT.

In 1930 Kuratowski [16] first proposed the idea of MNC. An entirely new FPT was created by Darbo [8] connecting the concept of MNC. Later it referred by Darbo FPT. By generalization of Banach principle of contraction [4], Meir and Keeler [15] proved a new FPT, which is very interesting. There are lots of works have been done by using fixed point theorems in the field of integral equations. We have referred some recent works on application of fixed point theorems in integral equations for better understanding [12,13,14].

We found that there are lots of works have been done in sequence spaces by many authors. M.Mursaleen and S.A. Mohiuddine [17] established results in l_p space for infinite differential system of equations by using Meir Keeler FPT. Later, A.R. Arab et al. [3] proved results regarding system of integral equations containing two variables. The result of an infinite integral system in sequence spaces c_0 and ℓ_1 discussed by A. Das et al. [7].

Solution existence of infinite mixed Volterra–Fredholm integral system have already done in sequence space. But study of infinite mixed Volterra–Fredholm integral system in sequence space is still an innovative research area. This fact motivate us to approach this study to fill this research gap. In this manuscript, we try to illustrate the existence of solutions of an infinite mixed Volterra-Fredholm integral system by using Meir Keeler FPT in the sequence space ℓ_1 and clarified our result with some examples.

The manuscript is systematically structured as: In Section 2 we have stated some notations and supporting facts which will be helpful for our main finding. The Section 3 is devoted to prove our main result. Next in the section 4 we have given some examples to support of our main result and also verified the examples.

2. Definitions and results

Before going next part, Consider $\mathfrak{E}$ is a Banach space whose norm is $||.||$ and a closed ball $\mathfrak{D}(y_0, d)$ is defined in the Banach space $\mathfrak{E}$ whose centre y_0,radius d.
If χ is a nonempty subset of $\mathfrak{E}$, then:-
$\overline{\chi}$ denotes Closure of the set χ,
$Conv\chi$ denotes Convex closure of χ,
$\mathcal{M}_{\mathfrak{E}}$ represents collection of all bounded subset of $\mathfrak{E}$ which are nonempty,
$\mathcal{N}_{\mathfrak{E}}$ represents sub-collection of $\mathfrak{E}$ containing all relatively compact sets.

Definition 2.1 [5]MNC is a function $\nu : \mathcal{M}_{\mathfrak{E}} \to [0, \infty)$ that satisfies following properties:

(i) $\ker \nu = \{\chi \in \mathcal{M}_{\mathfrak{E}} : \nu(\chi) = 0\} \neq \emptyset$ and $\ker \nu$ is a subset of $\mathcal{N}_{\mathfrak{E}}$.

(ii) If χ is a subset of $\mathfrak{Y}$ then it implies $\nu(\chi) \leq \nu(\mathfrak{Y})$.

(iii) $\nu(\overline{\chi}) = \nu(\chi)$.

(iv) $\nu(\text{Conv}\, \chi) = \nu(\chi)$.

(v) $\nu(\lambda\chi + (1-\lambda)\mathfrak{Y}) \leq \lambda\nu(\chi) + (1-\lambda)\nu(\mathfrak{Y})$ for $\lambda \in [0, 1]$.

(vi) if $\mathcal{X}_i$ is a sequence of closed sets from $\mathcal{M}_{\mathfrak{E}}$, such that $\chi_{i+1} \subset \chi_i$ for $i = 1, 2, 3,$...and if $\lim_{i\to\infty} \nu(\chi_i) = 0$, then $\chi_\infty := \bigcap_{i=1}^{\infty} \chi_i$ is nonempty.

The family $\ker \nu$ is known as kernel of measure ν.
Measures ν to be sublinear, if it holds the conditions

(i) $\nu_n(\lambda\chi) = |\lambda|\nu_n(\chi)$, for $\lambda \in \mathcal{R}, n \in \mathcal{N}$.

(ii) $\nu_n(\chi + \mathfrak{Y}) \leq \nu_n(\chi) + \nu_n(\mathfrak{Y})$.

A sublinear MNC ν holds the following property

$$\nu(\chi \cup \mathfrak{Y}) = \max\{\nu(\chi), \nu(\mathfrak{Y})\}$$

following the condition $\ker \nu = \mathcal{N}_{\mathfrak{E}}$ is known as regular. If $\mathfrak{S} \subset \mathcal{X}$, where $\mathfrak{S}$ is bounded subset and $\mathcal{X}$ is a metric space then we define Kuratowski MNC as :

$$\alpha(\mathfrak{S}) = \inf\{\varrho > 0 : \mathfrak{S} = \bigcup_{i=1}^{n} \mathfrak{S}_i, \text{diam}\mathfrak{S}_i \leq \varrho \text{ for } 1 \leq i \leq n \leq \infty\},$$

Here diam$\mathfrak{S}$ = diameter of $\mathfrak{S}_i$, means

$$\text{diam}\mathfrak{S}_i = \sup\{d(x, y) : x, y \in \mathfrak{S}_i\}.$$

Taking bounded set, Hausdorff MNC $\mathfrak{S}$ is denoted as

$$\psi(\mathfrak{S}) = \inf\{\varepsilon > 0 : \text{ There exist a finite } \varepsilon - \text{net for } \mathfrak{S} \text{ in } \chi\}.$$

Let us recollect some basic assets of Hausdorff MNC. For a metric space (χ, d), we take $\mathfrak{F}$, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ as bounded subsets. Then we have

(i) $\psi(\mathfrak{F}) = 0$ iff $\mathfrak{F}$ is totally bounded;

(ii) $\psi(\mathfrak{F}) = \psi(\overline{\mathfrak{F}})$, here $\overline{\mathfrak{F}}$ is the closure of $\mathfrak{F}$;

(iii) If $\mathfrak{F}_1$ is a subset of $\mathfrak{F}_2 \implies \psi(\mathfrak{F}_1) \leq \psi(\mathfrak{F}_2)$;

(iv) $\psi(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \max\{\psi(\mathfrak{F}_1), \psi(\mathfrak{F}_2)$;

(v) $\psi(\mathfrak{F}_1 \cap \mathfrak{F}_2) \leq \min\{\psi(\mathfrak{F}_1), \psi(\mathfrak{F}_2)\}$;

By taking normed space $(X, ||.||)$, we have some additional properties for the function ψ which is linked with the linearity. Thus, we have

(i) $\psi(\mathfrak{F}_1 + \mathfrak{F}_2) \leq \psi(\mathfrak{F}_1) + \psi(\mathfrak{F}_2)$;

(ii) $\psi(\mathfrak{F} + x) = \psi(\mathfrak{F}), \forall\, x \in \chi$;

(iii) $\psi(\alpha\mathfrak{F}) = |\alpha|\psi(\mathfrak{F}), \forall\, \alpha \in \mathbb{C}$.

Definition 2.2 [2] Consider two Banach spaces $\mathfrak{E}_1$ and $\mathfrak{E}_2$ and two arbitrary MNC ν_1 and ν_2 on $\mathfrak{E}_1$ and $\mathfrak{E}_2$ accordingly. $\mathcal{T} : \mathfrak{E}_1 \to \mathfrak{E}_2$ is an operator known as (ν_1, ν_2)-condensing operator if satisfies continuity and the property $\nu_2(\mathcal{T}(\mathcal{D}_1)) < \nu_1(\mathcal{D}_1)$ for each set $\mathcal{D}_1 \subset \mathfrak{E}_1$ with compact closure.

Remark 2.1 An operator $\mathcal{T}$ is known as ν-condensing operator if it satisfies $\mathfrak{E}_1 = \mathfrak{E}_2$ and $\nu_1 = \nu_2 = \nu$.

Theorem 2.1 *[3] Let $\mathfrak{E}$ be a Banach space and ω be a nonempty, closed, bounded and convex subset of $\mathfrak{E}$. For the continuous mapping $\mathcal{T} : \omega \to \omega$, a constant $k \in [0, 1)$ such that $\nu_2(\mathcal{T}(\omega)) < k\nu_1(\mathcal{T}(\omega))$.Then $\mathcal{T}$ has a fixed point in ω.*

Definition 2.3 [15] For a metric space "(χ, d)", the mapping $\mathcal{T}$ is on χ is called Meir-Keeler contraction if for each $\varepsilon > 0$, $\exists\ \varrho > 0$ which satisfies,

$$\varepsilon \leq d(x, y) < \varepsilon + \varrho \implies d(\mathcal{T}x, \mathcal{T}y) < \varepsilon,\ \forall\ x, y \in \chi.$$

Theorem 2.2 *[15] For a complete metric space (χ, d), if the mapping $\mathcal{T} : \chi \to \chi$ is a Meir-Keeler contraction; then $\mathcal{T}$ has a unique fixed point.*

Definition 2.4 [1] Let ν be an arbitrary MNC on the Banach space $\mathfrak{E}$ and $\mathcal{C}$ be a nonempty subset of $\mathfrak{E}$. The operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called Meir-Keeler condensing operator if for each $\varepsilon > 0$, $\exists\ \varrho > 0$ satisfies the condition

$$\varepsilon \leq \nu(\chi) < \varepsilon + \varrho \implies \nu(\mathcal{T}(\chi)) < \varepsilon$$

holds for any bounded subset χ of the set $\mathcal{C}$.

Theorem 2.3 *[1] Consider $\mathfrak{E}$ is a Banach space and consider it's closed convex bounded subset $\mathcal{C}$ which is nonempty. Let ν be an arbitrary MNC on $\mathfrak{E}$. If the operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ satisfies continuity along with properties of Meir-Keeler condensing operator, then there exist at least one fixed point of $\mathcal{T}$ and the family of all fixed points in $\mathcal{C}$ is compact.*

3. Main Finding

In this part, we are discussed the existence of solutions of infinite mixed Volterra-Fredholm integral system using the Meir-Keeler condensing operators in the sequence space ℓ_1.

In the Banach space $(\ell_1, ||.||)$, the Hausdorff MNC ψ can be expressed as

$$\psi(\mathcal{D}_1) = \lim_{n\to\infty}\left[\sup_{w(t)\in\mathcal{D}_1}\left(\sum_{l=n}^{\infty}|w_l(t)|\right)\right],$$

here we define $w(t) = (w_i(t))_i^{\infty} \in \ell_1$ for all t is in the positive real number $\mathcal{R}_+$ and $\mathcal{D}_1 \in \mathcal{M}_{\ell_1}$.
Let us assume the infinite mixed Volterra-Fredholm type integral system

$$w_n(t) = f_n\bigg(t, \int_0^x\int_0^a b_n(t, v, w(v))dvdt, w(t)\bigg), \tag{3.1}$$

where $w(t) = (w_i(t))_i^{\infty}$,$t$ is in the positive real number $\mathcal{R}_+$, n is in the natural number $\mathcal{N}$ and $(w_i(t)) \in C(\mathcal{R}_+, \mathcal{R})$, $\forall i \in \mathcal{N}$.

3.1. Result of existence of the system (3.1):

Consider the assumptions

1. The variable, $x : \mathcal{R}_+ \to [0, \infty)$ is continuous.
2. Infinite system of functions $f_n : \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R}^{\infty} \to \mathcal{R}$ $(n \in \mathcal{N})$ are continuous with
$$\left\{\sum_{n\geq 1}|f_n(t, 0, w^0(t)| : t \in \mathcal{R}_+\right\}$$
converges to zero, where $w^0(t) = (w_n^0(t))_{n=1}^{\infty} \in \mathbb{R}^{\infty}$ and $w_n^0(t) = 0$, $\forall\ n \in \mathcal{N}, t \in \mathbb{R}_+$. Also, there exist $r_n, g_n : \mathcal{R}_+ \to \mathcal{R}_+ (n \in \mathcal{N})$ are continuous functions such that
$$|f_n(t, p(t), w(t)) - f_n(t, q(t), \overline{w}(t))| \leq r_n(t)|w_i(t) - \overline{w_i}(t)| + g_n(t)|p(t) - q(t)|,$$
where $w(t) = (w_i(t))_{i=1}^{\infty}, \overline{w}(t) = (\overline{w_i}(t))_{i=1}^{\infty} \in \mathcal{R}^{\infty}$
3. $b_n : \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R}^{\infty} \to \mathcal{R}$ $(n \in \mathcal{N})$ are continuous. Here we define
$$b_n = \sup\left\{\sum_{n\geq k}[g_n(t)|\int_0^x\int_0^a b_n(t, v, w(v))dvdt|] : t, v \in \mathcal{R}_+\right\}.$$
Also
$$\lim_{t\to\infty}\sum_n\left|g_n(t)\int_0^x\int_0^a[b_n(t, v, w(v)) - b_n(t, v, \overline{w}(v))]dvdt\right| = 0.$$
4. Let us assume an operator W from $\mathbb{R}_+ \times \ell_1$ to ℓ_1 as follows
$(t, w(t)) \to (Ww)(t)$, where
$$(Ww)(t) = (f_1(t, v_1, w(t)), f_2(t, v_2, w(t)), f_3(t, v_3, w(t)), ...),$$
and
$$v_n(w) = \int_0^x\int_0^a b_n(t, v, w(v))dvdt.$$
5. $\sup b_n = G$; $b_n \to 0$ when $n \to \infty$; $\sup\{r_n(t) : t \in \mathcal{R}_+\} = U < \infty$ such that $0 < U < 1$. Also, $M = \sup_n\{\sum_n g_n(t)\}\ \forall\ t \in \mathcal{R}_+$.

Theorem 3.1 *By considering conditions* (1)−(5) *we can say at least one solution* $w(t) = (w_i(t))_{i=0}^{\infty} \in \ell_1$, $t \in \mathcal{R}_+$, $n \in \mathcal{N}$ *of the system* (3.1) *exists where* $(w_i(t)) \in \mathcal{C}(\mathcal{R}_+, \mathcal{R})$ $\forall\ i \in \mathcal{N}$.

Proof: We have

$$\begin{aligned}
||w(t)||_{\ell_1} &= \sum_{n\geq 1} |f_n(t, \int_0^x \int_0^a b_n(t,v,w(v))dvdt, w(t))| \\
&= \sum_{n\geq 1} |f_n(t, \int_0^x \int_0^a b_n(t,v,w(v))dvdt, w(t)) \\
&- f_n(t,0,w^0(t)) + f_n(t,0,w^0(t))| \\
&\leq \sum_{n\geq 1} |f_n(t, \int_0^x \int_0^a b_n(t,v,w(v))dvdt, w(t)) \\
&- f_n(t,0,w^0(t))| + \sum_{n\geq} |f_n(t,0,w^0(t))| \\
&\leq \sum_{n\geq 1} \{r_n(t)|w(t) - w^0(t)|\} \\
&+ g_n(t)| \int_0^x \int_0^a b_n(t,v,w(v))dvdt| \\
&\leq U||w(t)||_{\ell_1} + G
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\Longrightarrow ||w(t)||_{\ell_1} \leq U||w(t)||_{\ell_1} + G \\
&\Longrightarrow (1-U)||w(t)||_{\ell_1} \leq G \\
&\Longrightarrow ||w(t)||_{\ell_1} \leq \frac{G}{1-U} = \mathfrak{d}(say).
\end{aligned}$$

Assume that $\overline{\mathfrak{D}} = \overline{\mathfrak{D}}(w^0(t), \mathfrak{d})$ is a radius $\mathfrak{d}$ closed ball with center $w^0(t)$, which concludes that $\overline{\mathfrak{D}}$ is a convex subset of ℓ_1 along with closed and nonempty. Let us take the operator $W = (W_i)$ on $C(\mathcal{R}_+, \overline{\mathfrak{D}})$ stated as bellow. For each $t \in \mathcal{R}_+$

$$(Ww)(t) = (W_i w)(t) = \{f_i(t, v_i(w), w(t)\},$$

here $w(t) = (w_i(t)) \in \overline{\mathfrak{D}}$ and $w_i(t)$ is in $C(\mathcal{R}_+ \times \mathcal{R}_+, \mathcal{R})$, $\forall i \in \mathcal{N}$.
Since all $t \in \mathcal{R}_+$,Then by the assumption (4) we have

$$\sum_{i\geq 1} |(W_i w)(t)| = \sum_{i\geq 1} f_i(t, v_i(w), w(t))| < \infty.$$

Therefore $(Ww)(t) \in \ell_1$. It follows from the fact $||(Ww)(t) - w^0(t)||_{\ell_1} \leq \mathfrak{d}$, w is a self mapping on $\overline{\mathfrak{D}}$. Next we check the continuity of w on $C(\mathcal{R}_+, \overline{\mathfrak{D}})$.
Consider arbitrary $c(t) = (c_j(t))_{j=1}^{\infty}$, $e(t) = (e_j(t))_{j=1}^{\infty} \in \ell_1$ and $\varepsilon > 0$ which fulfills the inequality

$$||c - e||_{\ell_1} < \frac{\varepsilon}{2U}.$$

Now $\forall\ t \in \mathcal{R}_+$, we have

$$\begin{aligned}
&|(w_n c(t)) - (w_n e(t))| \\
&= |f_n(t, v_n(c), c(t)) - f_n(t, v_n(e), e(t))| \\
&\leq r_n(t)|c_i(t) - e_i(t)| + g_n(t)|v_n(c) - v_n(e)| \\
&\leq U|c_i(t) - e_i(t)| + g_n(t)|v_n(c) - v_n(e)| \\
&< \frac{\varepsilon}{2} + g_n(t)| \int_0^x \int_0^a [b_n(t,v,c(v)) - b_n(t,v,e(v))]dvdt|.
\end{aligned}$$

By using pre-defined condition (3), we can choose $T_1 > 0$ for which $\max(t) > T_1$

$$\sum_n |g_n(t) \int_0^x \int_0^a [b_n(t, v, c(v)) - b_n(t, v, e(v))]dvdt| < \frac{\varepsilon}{2}.$$

Hence $|(W_n c(t)) - (W_n e(t))| < \varepsilon$.
For $t \in [0, T_1]$ let $X = \sup\{x(t) : t \in [0, T_1]\}$; $M = \sup\{g_n(t) : t \in [0, T_1]\}$ and $g = \sup_n\{|b_n(t, v, c(v)) - b_n(t, v, e(v))| : t \in [0, T_1], v \in [0, X]\}$.
Then $\sum_n |(W_n c(t)) - (W_n e(t))| < \frac{\varepsilon}{2} + MgXa$.
Since b_n is continuous on $[0, T_1] \times [0, X] \times \ell_1$, when $\varepsilon \to 0$ we have $b_n \to 0$. Therefore, $\sum_n |(W_n c(t)) - (W_n e(t))| \to 0$ as $||c(t) - e(t)||_{\ell_1} \to 0$. Which defines continuity of W on $\overline{\mathfrak{D}} \subset \ell_1$.
Next we show that W is a condensing operator of Meir-Keeler. Given any $\varepsilon > 0$ we can choose $\varrho > 0$ satisfying $\varepsilon \le \psi(\overline{\mathfrak{D}}) < \varepsilon + \varrho \implies \psi(W(\overline{\mathfrak{D}})) < \varepsilon$.
We have,

$$\begin{aligned}
\psi(W(\overline{\mathfrak{D}})) &= \lim_{n\to\infty}[\sup_{w(t)\in\overline{\mathfrak{D}}}\{\sum_{k\ge n}|f_n(t, v_n(w), w(t))|\}] \\
&= \lim_{n\to\infty}[\sup_{w(t)\in\overline{\mathfrak{D}}}\{\sum_{k\ge n}|f_n(t, v_n(w), w(t)) + f_n(t, 0, w^0) - f_n(t, 0, w^0)|\}] \\
&= \lim_{n\to\infty}[\sup_{w(t)\in\overline{\mathfrak{D}}}\{\sum_{k\ge n}(r_n(t)|w_i(t)| + g_n(t)|\int_0^x \int_0^a b_n(t, v, w(v))dvdt|)\} \\
&\le \lim_{n\to\infty}\sup_{w(t)\in\overline{\mathfrak{D}}}[U\sum_{k\ge n}|w_k(t)| + b_n] \\
&\le U\psi(\overline{\mathfrak{D}}).
\end{aligned}$$

It can be seen that

$$\psi(W(\overline{\mathfrak{D}})) \le U\psi(\overline{\mathfrak{D}}) < \varepsilon \implies \psi(\overline{\mathfrak{D}}) < \frac{\varepsilon}{U}.$$

Now taking $\varrho = \frac{\varepsilon(1-U)}{U}$, we get $\varepsilon \le \psi(\overline{\mathfrak{D}}) \le \frac{\varepsilon}{U} = \varepsilon + \varrho$.
Thus we can say the operator W is a condensing Meir-Keeler operator on the set $\overline{\mathfrak{D}} \subset \ell_1$. Consequently, the operator W satisfies all the assumptions of the Theorem 2.3 implying the fact that there is a fixed point of W in $\overline{\mathfrak{D}}$. Thus in ℓ_1 space there exist a solution of the system (3.1) . □

4. Problems

Problem 1 *For mixed integral system*

$$w_n(t) = \sum_{i=n}^{\infty} \frac{|w_i(t)|}{3i^2} + \frac{1}{n^2 e^t} \int_0^x \int_0^\pi \frac{\sin(w_i(v))}{2 + \cos(\sum_{i=1}^{\infty} w_i(v))} dvdt. \tag{4.1}$$

Examine the solution existency of the system (4.1) in ℓ_1 space.

Solution: Here $f_n(t, v_n(w(t)), w(t)) = \sum_{i=n}^{\infty} \frac{|w_i(t)|}{3i^2} + \frac{1}{n^2 e^t} v_n(w(t))$,
where $v_n(w(t)) = \int_0^x \int_0^\pi b_n(t, v, w(v))dvdt$ and $b_n = \frac{\sin(w_i(v))}{2 + \cos(\sum_{i=1}^{\infty} w_i(v))}$.

By checking $|v_n(w(t))|$

$$|\int_0^x \int_0^\pi \frac{\sin(w_i(v))}{2 + \cos(\sum_{i=1}^{\infty} w_i(v))} dvdt| \le 2|\int_0^x \int_0^\pi dvdw| = 2\pi x. \tag{4.2}$$

If $w(t) \in \ell_1$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} |f_n(t, v_n(w(t)), w(t))| \\
&\leq \sum_{i=n}^{\infty}\sum_{n=1}^{\infty} \frac{|w_i(t)|}{3i^2} + \frac{1}{e^t}\sum_{i=n}^{\infty}\frac{|v_n(w(t))|}{n^2} \\
&\leq (\sum_{i=n}^{\infty}\sum_{i=n}^{\infty} \frac{|w_i(t)|}{3i^2} + \frac{1}{e}\sum_{i=n}^{\infty}\frac{|v_n(w(t))|}{n^2} \\
&\leq \frac{\pi^2}{18}|w_i(t)|_{\ell_1} + \frac{x\pi^3}{3e} \\
&< \infty.
\end{aligned}$$

Therefore $(f_n(t, v_n(w(t)), w(t))) \in \ell_1$. Considering $\gamma(t) = (\gamma_i(t)) \in \ell_1$ we get

$$\begin{aligned}
&|f_n(t, v_n(w(t)), w(t)) - f_n(t, v_n(\gamma(t)), \gamma(t))| \\
&\leq \sum_{i=n}^{\infty} \frac{1}{3i^2}|w_i(t) - \gamma_i(t)| + \frac{1}{n^2e^t}|v_n(w(t)) - v_n(\gamma(t))| \\
&\leq (\sum_{i=n}^{\infty} \frac{1}{3i^2})|w_i(t) - \gamma_i(t)| + \frac{1}{n^2e^t}|v_n(w(t)) - v_n(\gamma(t))| \\
&\leq \frac{\pi^2}{18}|w_i(t) - \gamma_i(t)| + \frac{1}{n^2e^t}|v_n(w(t)) - v_n(\gamma(t))|.
\end{aligned}$$

Here $r_n(t) = \frac{\pi^2}{18}$, $g_n(t) = \frac{1}{n^2e^t}$.
Here we find $0 < U < 1$ and $\sum_{n\geq 1} |f_n(t, 0, w^0(t))| \to 0$ for each $t \in \mathcal{R}_+$. And

$$\sum_{n\geq k} g_n(t)|v_n w(t)| \leq \frac{2\pi x}{e^t}\sum_{n\geq k}\frac{1}{n^2}$$

As well as,

$$b_n \leq \sup\{\frac{2\pi x}{e^t}\sum_{n\geq k}\frac{1}{n^2} : t, v \in \mathcal{R}_+\}.$$

Therefore $n \to \infty$ we get $\sum_{n\geq k}\frac{1}{n^2} \to 0$. Thus $b_n \to 0$ as $n \to \infty$ and $G = \frac{x\pi^3}{3e}$.
Now,

$$\begin{aligned}
&\sum_n \left|\frac{1}{n^2e^t}\int_0^x\int_0^a [b_n(t, v, w(v)) - b_n(t, v, \overline{w}(v))]dvdt\right| \\
&\leq \sum_n \left(\frac{2\pi x}{n^2e^t}\right) \\
&= \frac{x\pi^3}{3e^t}.
\end{aligned}$$

We have as $t \to \infty$, $\frac{x}{e^t} \to 0$. Therefore,

$$\lim_{t\to\infty}\sum_n \left|g_n(t)\int_0^x\int_0^a [b_n(t, v, w(v)) - b_n(t, v, \overline{w}(v))]dvdt\right| = 0.$$

We can show that $M = \sup_t \left\{ \sum_n g_n(t) \right\} = \frac{\pi^2}{6}$. Moreover f_n and b_n are continuous functions. Hence the equation (4.1) satisfies all the assumptions (1)-(5). Hence the system (4.1) has a solution in ℓ_1.

Problem 2 *For mixed integral system*

$$w_n(t) = \sum_{i=n}^{\infty} \frac{\sin t \cos t w_i(t)}{2i^2} + \frac{1}{n^4 e^t} \int_0^x \int_0^{\pi} \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dvdt. \tag{4.3}$$

examine the solution existency in ℓ_1 space.
Solution: From (4.3), we have

$$\begin{aligned} w_n(t) &= \sum_{i=n}^{\infty} \frac{\sin t \cos t w_i(t)}{2i^2} + \frac{1}{n^4 e^t} \int_0^x \int_0^{\pi} \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dvdt \\ &= \sum_{i=n}^{\infty} \frac{\sin(2t) w_i(t)}{4i^2} + \frac{1}{n^4 e^t} \int_0^x \int_0^{\pi} \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dvdt. \end{aligned}$$

Here $f_n(t, v_n(w(t)), w(t)) = \sum_{i=n}^{\infty} \frac{\sin(2t) w_i(t)}{4i^2} + \frac{1}{n^4 e^t} v_n(w(t))$, where $v_n(w(t)) = \int_0^x \int_0^{\pi} b_n(t, v, w(v)) dvdt$ and $b_n(t) = \dfrac{\cos(w_i(v)) + \sin\left(\sum_{i=1}^{\infty} w_i(v)\right)}{3 + \sin(w_i(v))}$.

By checking $|v_n(w(t))|$,

$$\left| \int_0^x \int_0^{\pi} \frac{\cos(w_i(v)) + \sin\left(\sum_{i=1}^{\infty} w_i(v)\right)}{3 + \sin(w_i(v))} dvdt \right| \leq 2 \left| \int_0^x \int_0^{\pi} dvdw \right| = 2\pi x. \tag{4.4}$$

If $w(t) \in \ell_1$ then we have

$$\begin{aligned} &\sum_{n=1}^{\infty} |f_n(t, v_n(w(t)), w(t))| \\ &\leq \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \frac{|\sin 2t|}{4i^2} |w_i(t)| + \frac{1}{e^t} \sum_{n=1}^{\infty} \frac{|v_n(w(t))|}{n^4} \\ &\leq \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \frac{1}{4i^2} |w_i(t)| + \frac{1}{e} \sum_{n=1}^{\infty} \frac{|v_n(w(t))|}{n^4} \\ &\leq \frac{\pi^2}{24} |w_i(t)|_{\ell_1} + \frac{x\pi^5}{45e} \\ &< \infty. \end{aligned}$$

Therefore, $(f_n(t, v_n(w(t)), w(t))) \in \ell_1$. By taking $\gamma(t) = (\gamma_i(t)) \in \ell_1$ we get

$$\begin{aligned} &|f_n(t, v_n(w(t)), w(t)) - f_n(t, v_n(\gamma(t)), \gamma(t))| \\ &\leq \sum_{i=n}^{\infty} \frac{|\sin 2t|}{4i^2} |w_i(t) - \gamma_i(t)| + \frac{1}{n^4 e^t} |v_n(w(t)) - v_n(\gamma(t))| \\ &\leq (\sum_{i=n}^{\infty} \frac{1}{4i^2}) |w_i(t) - \gamma_i(t)| + \frac{1}{n^4 e^t} |v_n(w(t)) - v_n(\gamma(t))| \\ &\leq \frac{\pi^2}{24} |w_i(t) - \gamma_i(t)| + \frac{1}{n^4 e^t} |v_n(w(t)) - v_n(\gamma(t))|. \end{aligned}$$

Here $r_n(t) = \frac{\pi^2}{24}$, $g_n(t) = \frac{1}{n^4 e^t}$. Here we get $0 < U < 1$ and $\sum_{n\geq 1} |f_n(t, 0, w^0(t))|$ converging to zero $\forall$ $t \in \mathcal{R}_+$. And

$$\sum_{n\geq k} g_n(t)|v_n(w(t))| \leq \frac{2\pi x}{e^t} \sum_{n\geq k} \frac{1}{n^4}$$

As well as,

$$b_n \leq \sup\left\{\frac{2\pi x}{e^t} \sum_{n\geq k} \frac{1}{n^4} : t, x \in \mathcal{R}_+\right\}.$$

For, $n \to \infty$ we get $\sum_{n\geq k} \frac{1}{n^4} \to 0$. Thus $b_n \to 0$ as $n \to \infty$ and $G = \frac{x\pi^5}{45e}$.

Now,

$$\sum_n \left|\frac{1}{n^4 e^t} \int_0^x \int_0^a [b_n(t, v, w(v)) - b_n(t, v, \overline{w}(v))]dvdt\right|$$
$$\leq \sum_n (\frac{2\pi x}{n^4 e^t})$$
$$= \frac{x\pi^5}{45e^t}.$$

As, $t \to \infty$, We have $\frac{x}{e^t} \to 0$. Therefore,

$$\lim_{t\to\infty} \sum_n \left|g_n(t) \int_0^x \int_0^a [b_n(t, v, w(v)) - b_n(t, v, \overline{w}(v))]dvdt\right| = 0.$$

We can show that $M = \sup_t \sum_n g_n(t) = \frac{\pi^4}{90}$. Moreover f_n and b_n are continuous functions. Hence, the equation (4.3) satisfies all the assumptions (1)-(5). Thus the system (4.3) has a solution in ℓ_1.

5. Conclusion

There are lots of works have been done in sequence spaces, but study of an infinite mixed Volterra–Fredholm integral system in sequence spaces is still a research area where we can explore more results. In our present work, we have solved theoretically an infinite mixed Volterra–Fredholm integral system in the sequence space ℓ_1. In our future work we can explore numerical methods and it's application by taking this type of an infinite mixed Volterra–Fredholm integral system. Another future work of our paper is that we can try to solved mixed integral equations in other sequence spaces.

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References

1. A. Aghajani, M. Mursaleen, A.S. Haghighi, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci. 35,552–566,(2015).

2. R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, *Measures of noncompactness and condensing operators*, J. Operator Theory, 55, 1–244,(1992).

3. A.R. Arab, R. Allahyari, A.S. Haghighi, *Existence of solutions of infinite systems of integral equations in two variables via measure of noncompactness*, J. Appl. Math. Comput. 246, 283–291,(2014).

4. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3, 133–181,(1922).

5. J. Banaś, M. Lecko, *Solvability of infinite systems of differential equations in Banach sequence spaces*, J. Comput. Appl. Math. 137, 363–375, (2001).

6. Z. Chen, W. Jiang, *An approximate solution for a mixed linear Volterra–Fredholm integral equation*, Appl. Math. Lett. 25, 1131–1134, (2012).
7. A. Das, B. Hazarika, R. Arab, M. Mursaleen, *Solvability of the infinite system of integral equations in two variables in the sequence spaces c_0 and ℓ_1*, J. Comput. Appl. Math. 326, 183–192, (2017).
8. G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Semin. Mat. Univ. Padova, 24, 84–92, (1955).
9. R. Ezzati, S. Najafalizadeh, *Numerical methods for solving linear and nonlinear Volterra-Fredholm integral equations by using CAS wavelets*, World Appl. Sci. J. 18, 1847–1854, (2012).
10. P.M. A. Hasan, N.A. Sulaiman, F. Soleymani, A. Akgül, *The existence and uniqueness of solution for linear system of mixed Volterra-Fredholm integral equations in Banach space*, AIMS Math. 5, 226–235,(2020).
11. S.A. Hosseini, S. Shahmorad, F. Talati, *A matrix based method for two dimensional nonlinear Volterra-Fredholm integral equations*, Numer. Algorithms, 68,511–529,(2015).
12. M. Kazemi, H. Chaudhary, A. Deep, *Existence and approximate solutions for Hadamard fractional integral equations in a Banach space*, J. Integral Equations Appl. 35, 27–40,(2023).
13. M. Kazemi, R. Ezzati, A. Deep, *On the solvability of non-linear fractional integral equations of product type*, J. Pseudo-Differ. Oper. Appl. 14, 39,(2023).
14. M. Kazemi, *On existence of solutions for some functional integral equations in Banach algebra by fixed point theorem*, Int. J. Nonlinear Anal. Appl. 13, 451–466,(2022).
15. E. Keeler, A. Meir, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28, 326–329,(1969).
16. K. Kuratowski, *Sur les espaces complets*, Fund. Math. 15, 301–309,(1930).
17. M. Mursaleen, S.A. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in l_p spaces*, Nonlinear Anal. 75, 2111–2115,(2012).
18. C. Nwaigwe, D.N. Benedict, *Generalized Banach fixed-point theorem and numerical discretization for nonlinear Volterra–Fredholm equations*, J. Comput. Appl. Math. 425, 115019,(2023).
19. K. Wang, Q. Wang, K. Guan, *Iterative method and convergence analysis for a kind of mixed nonlinear Volterra–Fredholm integral equation*, J. Appl. Math. Comput. Mech.225, 631–637,(2013).
20. A.M. Wazwaz, *Linear and nonlinear integral equations: Methods and applications*, Springer Berlin, Heidelberg (2011).

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