Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.77821

Certain Subclasses of Bi-Univalent Functions Defined by q-Analogue of Ruscheweyh Differential Operator

N. Ravikumar, M. Madhushree and P. Siva Kota Reddy*

ABSTRACT: In this paper, we find a new subclasses of the function class $\sum$ of bi-univalent functions defined in the open unit disk, which are associated with the q-analogue of Ruscheweyh differential operator and satisfy some subordination conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|v_2|$ and $|v_3|$ for functions in the new subclasses introduced here.

Key Words: Analytic functions, univalent functions, bi-univalent, Starlike and convex functions, q-Ruscheweyh differential operator.

Contents

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$\psi(u) = u + \sum_{j=2}^{\infty} v_j u^j \tag{1.1}$$

normalized by the conditions $\psi(0) = 0 = \psi'(0) - 1$, defined in the open unit disk

$$\mathrm{U} = \{u \in \mathcal{C} \ : \ |u| < 1\}.$$

Let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of function of the form (1) which are also univalent in U. Consider an analytic function ξ with positive real part in the unit disk U, $\xi(0) = 1, \xi'(0) > 0$ and ξ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. In the sequel, it is assumed that such a function has a series expansion of the form

$$\zeta(u) = 1 + B_1 u + B_2 u^2 + B_3 u^3 + ..., (B_1 > 0). \tag{1.2}$$

In particular, for the class of strongly starlike functions of order $\alpha(0 < \alpha \leq 1)$, the function ζ is given by

$$\zeta(u) = \left[\frac{1+u}{1-u}\right]^{\alpha} = 1 + 2\alpha u + 2\alpha^2 u^2 + \cdots \qquad (0 < \alpha \leq 1), \tag{1.3}$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$ and on the other hand, for the class of starlike functions of order $\beta(0 \leq \beta < 1)$,

$$\zeta(u) = \frac{1 + (1-2\beta)u}{1-u} = 1 + 2(1-\beta)u + 2(1-\beta)u^2 + \cdots \qquad (0 \leq \beta < 1), \tag{1.4}$$

* Corresponding author.

Submitted July 12, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* 30C45, 30C50.

we have $B_1 = B_2 = 2(1-\beta)$.

A function $\psi \in \mathcal{A}$ is said to be bi-univalent in U if both ψ and ψ^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U. Since $\psi \in \Sigma$ has the Maclaurian series given by (1), a computation shows that its inverse $\phi = \psi^{-1}$ has the expansion

$$\phi(w) = \psi^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots \tag{1.5}$$

Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (See [2,3,4,8,11,12,13]).

For $0 < q < 1$, the Jackson's q-derivative of a function $\psi(u) \in \mathcal{A}$ is given by ([5]):

$$D_q\psi(u) = \begin{cases} \dfrac{\psi(u) - \psi(qu)}{(1-q)u} & for \quad u \neq 0; \\ \psi'(0) & for \quad u = 0. \end{cases} \tag{1.6}$$

For $\psi(u)$ of the form (1), we have

$$D_q\psi(u) = 1 + \sum_{j=2}^{\infty} [j]_q v_j u^{j-1}, \tag{1.7}$$

where

$$[j]_q = \frac{1-q^j}{1-q} \quad (0 < q < 1; j \in \mathbb{N} = \{1, 2, ...\}). \tag{1.8}$$

Kanas and Raducanu [7] (See also Aldweby and Darus [1]) defined the q-analogue of Ruscheweyh operator by:

$$R_q^{\lambda}\psi(u) = u + \sum_{j=2}^{\infty} \frac{[j+\lambda-1]_q!}{[\lambda]_q![j-1]_q!} v_j u^j \quad (0 < q < 1; \lambda \geq 0), \tag{1.9}$$

where

$$[j]_q! = \begin{cases} [j]_q \ [j-1]_q \ldots \ [1]_q, & j \in \mathbb{N}; \\ 1, & j = 0. \end{cases} \tag{1.10}$$

From (9), we obtain that

$$R_q^0\psi(u) = \psi(u) \text{ and } R_q^1\psi(u) = uD_q\psi(u),$$

and

$$\lim_{q \to 1^-} R_q^{\lambda}\psi(u) = u + \sum_{j=2}^{\infty} \frac{[j+\lambda-1]_q!}{[\lambda]_q![j-1]_q!} v_j u^j = R^{\lambda}\psi(u), \tag{1.11}$$

where R^{λ} is the Ruscheweyh differential operator [10].

2. Bi-Univalent Function Class $\mathcal{N}\Sigma_q^{\lambda}(\delta, \zeta)$

In this section, we introduce a subclass $\mathcal{N}\Sigma_q^{\lambda}(\delta, \zeta)$ of Σ and find the estimate on the coefficients $|v_2|$ and $|v_3|$ for the functions in this new subclass, by subordination. Throughout our study, unless otherwise stated, we let

$$0 \leq \delta \leq 1 \text{ and } 0 < q < 1.$$

Definition 2.1 *For $0 \leq \delta \leq 1$, a function $\psi \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{N}\Sigma_q^{\lambda}(\delta, \zeta)$, if the following subordination hold:*

$$(1-\delta)\frac{uD_qR_q^{\lambda}\psi(u)}{R_q^{\lambda}\psi(u)} + \delta\frac{D_q(uD_qR_q^{\lambda}\psi(u))}{D_q(R_q^{\lambda}\psi(u))} \prec \zeta(u) \tag{2.1}$$

and

$$(1-\delta)\frac{wD_qR_q^{\lambda}\phi(w)}{R_q^{\lambda}\phi(w)}+\delta\frac{D_q(wD_qR_q^{\lambda}\phi(w))}{D_q(R_q^{\lambda}\phi(w))}\prec\zeta(w), \tag{2.2}$$

where $u, w \in \mathrm{U}$ and ϕ is given by (5).

Note that if $\lambda = 0$ and $q \to 1^-$ the class $\mathcal{N}\Sigma_q^{\lambda}(\delta,\zeta)$ reduces to class $M_{\Sigma}(\alpha,\lambda)$, $0<\alpha\leq 1$ and $\lambda\geq 0$ studied by Xiao-Fei Li and Au-Ping Wang [14].

If $\lambda = 1$ and $q \to 1^-$ the class $\mathcal{N}\Sigma_q^{\lambda}(\delta,\zeta)$ satisfying the subordination $\frac{u\psi'(u)}{\psi(u)}\prec\zeta(u)$ and $1+\frac{u\psi''(u)}{\psi'(u)}\prec\zeta(u)$ studied by the class of Ma and Minda [9] starlike and convex function respectively.

If $\lambda = 1$ and $q \to 1^-$ the class $\mathcal{N}\Sigma_q^{\lambda}(\delta,\zeta)$ reduces to class $M_{\sigma}(\alpha,\varphi)$ studied by Jothi Latha and Cynthiya Margaret Indrani [6].

Lemma 2.1 *If a function $p \in \mathcal{P}$ is given by*

$$p(u)=1+p_1u+p_2u^2+\cdots \quad (u\in\mathrm{U}),$$

then

$$|p_i|\leq 2 \quad (i\in\mathbb{N}),$$

where is $\mathcal{P}$ is the family of all functions p, analytic in U, for which

$$p(0)=1 \quad \textit{and} \quad \Re(p(u))>0 \quad (u\in\mathrm{U}).$$

Theorem 2.1 *If ψ given by (1) is in the class $\mathcal{N}\Sigma_q^{\lambda}(\delta,\zeta)$, then*

$$|v_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\begin{array}{r}|([1+\delta([3]_q)-1]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)-[1+\delta([2]_q^2-1)](\lambda+1)_q^2\\ ([2]_q-1))B_1^2+[1+\delta([2]_q)-1](B_1-B_2)(\lambda+1)_q^2([2]_q-1)^2|\end{array}}} \tag{2.3}$$

and

$$|v_3|\leq\frac{B_1}{[1+\delta([3]_q)-1]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)}+\left(\frac{B_1}{[1+\delta([2]_q)-1](\lambda+1)_q([2]_q-1)}\right)^2, \tag{2.4}$$

where $0\leq\delta\leq 1$.

Proof: Let $\psi\in\mathcal{N}\Sigma_q^{\lambda}(\delta,\zeta)$ and $\phi\in\psi^{-1}$. Then there are analytic functions $a, b:\mathrm{U}\to\mathrm{U}$, with $a(0)=0=b(0)$, satisfying

$$(1-\delta)\frac{uD_qR_q^{\lambda}\psi(u)}{R_q^{\lambda}\psi(u)}+\delta\frac{D_q(uD_qR_q^{\lambda}\psi(u))}{D_q(R_q^{\lambda}\psi(u))}=\zeta(a(u)) \tag{2.5}$$

and

$$(1-\delta)\frac{wD_qR_q^{\lambda}\phi(w)}{R_q^{\lambda}\phi(w)}+\delta\frac{D_q(wD_qR_q^{\lambda}\phi(w))}{D_q(R_q^{\lambda}\phi(w))}=\zeta(b(w)). \tag{2.6}$$

Define the functions $p(u)$ and $q(u)$ by

$$p(u):=\frac{1+a(u)}{1-a(u)}=1+p_1u+p_2u^2+\cdots$$

and

$$q(u) := \frac{1+b(u)}{1-b(u)} = 1 + q_1 u + q_2 u^2 + \cdots$$

or, equivalently,

$$a(u) := \frac{p(u)-1}{p(u)+1} = \frac{1}{2}\left[p_1 u + \left(p_2 - \frac{p_1^2}{2}\right)u^2 + \cdots\right] \tag{2.7}$$

and

$$b(u) := \frac{q(u)-1}{q(u)+1} = \frac{1}{2}\left[q_1 u + \left(q_2 - \frac{q_1^2}{2}\right)u^2 + \cdots\right]. \tag{2.8}$$

Then $p(u)$ and $q(u)$ are analytic in U with $p(0) = 1 = q(0)$. Since $a, b : \mathrm{U} \to \mathrm{U}$, the functions $p(u)$ and $q(u)$ have a positive real part in $\mathrm{U}, |p_i| \leq 2$ and $|q_i| \leq 2$.

Using (18) and (19) in (16) and (17) respectively, we have

$$(1-\delta)\frac{uD_qR_q^\lambda\psi(u)}{R_q^\lambda\psi(u)} + \delta\frac{D_q(uD_qR_q^\lambda\psi(u))}{D_q(R_q^\lambda\psi(u))} = \zeta\left(\frac{1}{2}\left[p_1 u + \left(p_2 - \frac{p_1^2}{2}\right)u^2 + \cdots\right]\right) \tag{2.9}$$

and

$$(1-\delta)\frac{wD_qR_q^\lambda\phi(w)}{R_q^\lambda\phi(w)} + \delta\frac{D_q(wD_qR_q^\lambda\phi(w))}{D_q(R_q^\lambda\phi(w))} = \zeta\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \cdots\right]\right). \tag{2.10}$$

In light of (1)-(5), and from (20) and (21), we have

$$\begin{aligned} &1 + [1+\delta([2]_q-1)](\lambda+1)_q([2]_q-1)v_2u + \Bigg\{\Bigg([1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)v_3\Bigg) - \\ &\qquad [1+\delta([2]_q^2-1)](\lambda+1)_q^2([2]_q-1)v_2^2\Bigg\}u^2 + \cdots \\ &= 1 + \frac{1}{2}B_1p_1u + \left[\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2p_1^2\right]u^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} &1 - [1+\delta([2]_q-1)](\lambda+1)_q([2]_q-1)v_2w + \Bigg\{\Bigg(2[1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1) - \\ &[1+\delta([2]_q^2-1)](\lambda+1)_q^2([2]_q-1)\Bigg)v_2^2 - [1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)v_3\Bigg\}w^2 + \cdots \\ &= 1 + \frac{1}{2}B_1q_1w + \left[\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2q_1^2\right]w^2 + \cdots, \end{aligned}$$

which yields the following relations:

$$[1+\delta([2]_q-1)](\lambda+1)_q([2]_q-1)v_2 = \frac{1}{2}B_1p_1 \tag{2.11}$$

$$\begin{aligned} &-[1+\delta([2]_q^2-1)](\lambda+1)_q^2([2]_q-1)v_2^2 \;+\; [1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)v_3 \\ &= \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2p_1^2 \end{aligned} \tag{2.12}$$

$$-[1+\delta([2]_q-1)](\lambda+1)_q([2]_q-1)v_2 = \frac{1}{2}B_1q_1 \tag{2.13}$$

$$\left(2[1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)-[1+\delta([2]_q^2-1)](\lambda+1)_q^2([2]_q-1)\right)v_2^2-$$

$$[1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)v_3=\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2. \tag{2.14}$$

From (22) and (24), it follows that

$$p_1=-q_1 \tag{2.15}$$

and

$$8[1+\delta([2]_q-1)]^2(\lambda+1)_q^2([2]_q-1)^2v_2^2=B_1^2(p_1^2+q_1^2). \tag{2.16}$$

From (23), (25) and (27), we obtain

$$v_2^2=\frac{B_1^3(p_2+q_2)}{4\Big\{\Big([1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)-[1+\delta([2]_q^2-1)](\lambda+1)_q^2([2]_q-1)\Big)B_1^2+[1+\delta([2]_q-1)](B_1-B_2)(\lambda+1)_q^2([2]_q-1)^2\Big\}}. \tag{2.17}$$

Applying Lemma 2.1 to the coefficients p_2 and q_2, we have

$$|v_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{|([1+\delta([3]_q)-1]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)-[1+\delta([2]_q^2-1)](\lambda+1)_q^2([2]_q-1))B_1^2+[1+\delta([2]_q)-1](B_1-B_2)(\lambda+1)_q^2([2]_q-1)^2|}}. \tag{2.18}$$

By substracting (25) from (23) and using (26) and (27), we get

$$v_3=\frac{B_1^2(p_1^2+q_1^2)}{8[1+\delta([2]_q)-1]^2(\lambda+1)_q^2([2]_q-1)^2}+\frac{B_1(p_2-q_2)}{4[1+\delta([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)}. \tag{2.19}$$

Applying Lemma 2.1 once again to the coefficients p_1,p_2,q_1 and q_2, we get

$$|v_3|\le\frac{B_1}{[1+\delta([3]_q)-1]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}([3]_q-1)}+\left(\frac{B_1}{[1+\delta([2]_q)-1](\lambda+1)_q([2]_q-1)}\right)^2. \tag{2.20}$$

□

3. Bi-Univalent Function Class $\mathcal{F}\Sigma_q^\lambda(\mu,\zeta)$

Definition 3.1 *For $0\le\mu\le1$, a function $\psi\in\Sigma$ of the form (1) is said to be in the class $\mathcal{F}\Sigma_q^\lambda(\mu,\zeta)$, if the following subordination hold:*

$$(1-\mu)\frac{R_q^\lambda\psi(u)}{u}+\mu D_qR_q^\lambda\psi(u)\prec\zeta(u) \tag{3.1}$$

and

$$(1-\mu)\frac{R_q^\lambda\phi(w)}{w}+\mu D_qR_q^\lambda\phi(w)\prec\zeta(w), \tag{3.2}$$

where $u,w\in\mathrm{U},\phi$ is given by (5) and $R_q^\lambda\psi(u)$ is given by (9).

Theorem 3.1 *Let ψ given by (1) be in the class $\mathcal{F}\Sigma_q^\lambda(\mu,\zeta)$. Then*

$$|v_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|[1+\mu([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}B_1^2+[1+\mu([2]_q-1)]^2(\lambda+1)_q^2(B_1-B_2)\right|}} \tag{3.3}$$

and

$$|v_3| \leq \frac{B_1}{[1+\mu([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}} + \left(\frac{B_1}{[1+\mu([2]_q-1)](\lambda+1)_q}\right)^2. \tag{3.4}$$

Proof: Proceeding as in the proof of Theorem 2.1, we can arrive the following relations:

$$[1+\mu([2]_q-1)](\lambda+1)_q v_2 = \frac{1}{2}B_1 p_1, \tag{3.5}$$

$$[1+\mu([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}v_3 = \frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2, \tag{3.6}$$

$$-[1+\mu([2]_q-1)](\lambda+1)_q v_2 = \frac{1}{2}B_1 q_1, \tag{3.7}$$

$$\begin{aligned}2[1+\mu([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}v_2^2 \;-\; [1+\mu([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}v_3 \\ = \frac{1}{2}B_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}B_2q_1^2\end{aligned} \tag{3.8}$$

From (36) and (38) it follows that

$$p_1 = -q_1 \tag{3.9}$$

and

$$8[1+\mu([2]_q-1)]^2(\lambda+1)_q^2v_2^2 = B_1^2(p_1^2+q_1^2). \tag{3.10}$$

From (37), (39) and (41), we obtain

$$v_2^2 = \frac{B_1^3(p_2+q_2)}{4\left\{[1+\mu([3]_q-1)]\frac{(\lambda+1)_q(\lambda+2)_q}{[2]_q!}B_1^2 \;+\; (B_1-B_2)[1+\mu([2]_q-1)]^2(\lambda+1)_q^2\right\}}. \tag{3.11}$$

Applying Lemma 2.1 to the coefficient p_2 and q_2, we immediately get the desired estimate on $|v_2|$ as asserted in (34). By subtracting (39) from (37) and using (40) and (41), we get

$$v_3 = \frac{B_1(p_2-q_2)}{4[1+\mu([3]_q-1)]\frac{(\lambda+1)(\lambda+2)_q}{[2]_q!}} + \frac{B_1^2(p_1^2+q_1^2)}{8[1+\mu([2]_q-1)]^2(\lambda+1)_q^2}. \tag{3.12}$$

Applying Lemma 2.1 to the coefficients p_1, p_2, q_1 and q_2, we get the desired estimate on $|v_3|$ as asserted in (35). □

4. Conclusion

We considered the q-Analogue of Ruscheweyh differential operator and defined a new subclasses of the bi-univalent functions in open unit disk. We investigated Taylor-Maclaurin coefficients $|v_2|$ and $|v_3|$ for functions belonging to this new subclasses and its subclasses and discussed some geometric properties of these subclasses.

Acknowledgments

The authors thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

References

1. Aldweby, H. and Darus, M., *Some subordination results on q-analogue of Ruscheweyh differential operator*, Abstr. Appl. Anal., Volume 2014, Article Id: 958563, 6 Pages, (2014).
2. Brannan, D. A. and Clunie, J. and Kirwan, W. E., *Coefficient estimates for a class of star-like functions*, Can. J. Math., 22, 476–485, (1970).
3. Brannan, D. A. and Taha, T. S., *On some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai, Math., 31(2), 70–77, (1986).
4. Deniz, E., *Certain subclasess of bi-univalent functions satisfying subordinate conditions*, Journal of Classical Analysis, 2(1), 49–60, (2013).
5. Jackson, F. H., *XI.—On q-Functions and a certain Difference Operator*, Transactions of the Royal Society of Edinburgh, 46(2), 253–281, (1909).
6. Latha, T. J. and Indrani, S. C. M., *Coefficient Estimates for Bi-univalent Ma-Minda Starlike and Convex Functions*, Journal of Emerging Technologies and Innovative Research, 6(6), 68–74, (2019).
7. Kanas, S. and Răducanu, D., *Some class of analytic functions related to conic domains*, Math. Slovaca, 64(5), 1183–1196, (2014).
8. Lewin, M., *On a coefficient problem for bi-univalent functions*, Proc. Am. Math. Soc., 18, 63–68, (1967).
9. Ma, W. and Minda, D., *A unified treatment of some special classes of univalent functions*, Proceedings of the conference on complex analysis, held June 19-23, 1992 at the Nankai Institute of Mathematics, Tianjin, China, Cambridge, MA: International Press, Pages 157–169, (1994).
10. Ruscheweyh, S., *New criteria for univalent functions*, Proc. Am. Math. Soc., 49, 109–115, (1975).
11. Srivastava, H. M., Mishra, A. K. and Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., 23(10), 1188–1192, (2010).
12. Hayami, T. and Owa, S., *Coefficient bounds for bi-univalent functions*, Panam. Math. J., 22(4), 15–26, (2012).
13. Xu. Q-H., Srivastava, H. M. and Li, Z., *A certain subclass of analytic and close-to-convex functions*, Appl. Math. Lett., 24(3), 396–401, (2011).
14. Li, X-F. and Wang, A-P., *Two new subclasses of bi-univalent functions*, Int. Math. Forum, 7, 1495–1504, (2012).

N. Ravikumar
PG Department of Mathematics
JSS College of Arts, Commerce and Science
Mysuru-570 025, India.
E-mail address: ravisn.kumar@gmail.com

and

M. Madhushree
PG Department of Mathematics
JSS College of Arts, Commerce and Science
Mysuru-570 025, India.
E-mail address: madhumaths89@gmail.com

and

P. Siva Kota Reddy (Corresponding author)
Department of Mathematics
JSS Science and Technology University
Mysuru-570 006, India.

and

Universidad Bernardo O'Higgins
Facultad de Ingeniería, Ciencia y Tecnología
Departamento de Formación y Desarrollo Científico en Ingeniería
Av. Viel 1497, Santiago, Chile
E-mail address: pskreddy@jssstuniv.in