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# Energy decay result in a flexible structure system of thermoelasticity of type III with distributed delay

Younes Bidi and Madani Douib*

ABSTRACT: In this paper, we consider a flexible structure system of thermoelasticity of type III with distributed delay term. Under suitable assumptions on the weight of the delay, we establish a decay result by introducing a suitable Lyaponov functional.

Key Words: Flexible structure, thermoelasticity of type III, distributed delay term, energy decay.

## Contents



## 1. Introduction

One of the primary challenges related to vibrations in flexible structural system models is ensuring the stability of the structure. It's essential to avoid resonance effects in the system and to aim for a reduction in total energy, preferably at a polynomial rate, and ideally at an exponential rate. Therefore, it is crucial to explore the theory behind stabilization processes in flexible structural systems to effectively manage their vibrations. To achieve energy dissipation and reduce system energy, damping forces can be introduced. Various types of damping, including boundary, internal, and localized damping can be utilized to accomplish this (for instance [1,2,11]). In [5], Gorain studied the uniform exponential stability of the problem,

$$m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x = f(x), \text{ on } (0, L) \times \mathbb{R}^+,$$

where $u = u(x,t)$ is the displacement of a particle at position $x \in (0, L)$ and time $t > 0$. The parameters $m(x)$, $\delta(x)$ and $p(x)$ is responsible for the non-uniform structure of the body, where $m(x)$ denote mass per unit length of structure, $\delta(x)$ coefficient of internal material damping and $p(x)$ a positive function related to the stress acting on the body at a point $x$. We recall the assumptions of the functions $m(x), \delta(x)$ and $p(x)$ in [2] such that

$$m, \delta, p \in W^{1,\infty}(0, L), \quad m(x), \delta(x), p(x) > 0, \quad \forall x \in [0, L].$$

The distributed force $f : (0, L) \times \mathbb{R}^+ \to \mathbb{R}$ is the uncertain disturbance appearing in the model, which is assumed to be continuously differentiable for all $t \geq 0$.

In 2014, Misra et al. [12] showed the exponential stability of the vibrations of an inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + k\theta_x = f(x), \\ \theta_t - \theta_{xx} + ku_{xt} = 0. \end{cases}$$

Considering thermal effects in flexible structures is of great physical importance. However, in the above model, the temperature propagates instantaneously (according to the heat equation). This characteristic is not consistent with reality, where heating or cooling a flexible structure typically takes some time (see also [4]). Consequently, numerous studies have been conducted to modify the thermal effect model to

* Corresponding author.
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reflect this. In [2], the authors consider the vibrations of an inhomogeneous flexible structure system under Cattaneo's law of heat condition

$$\begin{cases} m(x)\,u_{tt} - (p(x)\,u_x + 2\delta(x)\,u_{xt})_x + \eta\theta_x = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \theta_t + \kappa q_x + \eta u_{tx} = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \tau q_t + \beta q + \kappa\theta_x = 0, & (x,t) \in (0,L)\times(0,+\infty), \end{cases}$$

with boundary condition and initial condition

$$\begin{cases} u(0,t) = u(L,t) = 0,\ \theta(0,t) = \theta(L,t) = 0,\ t \geq 0, \\ u(x,0) = u_0(x)\,, u_t(x,0) = u_1(x)\,, \theta(x,0) = \theta_0(x)\,, \\ q(x,0) = q_0(x)\,,\ x \in [0,L]\,, \end{cases}$$

they proved the well-posedness and obtained an exponential stability result for one set of boundary conditions, and at least polynomial for another set of boundary conditions. In 1990s, Green and Naghdi proposed three new thermoelastic theories based on an entropy equality rather than the conventional entropy inequality (see [6,7,8,9]). The constitutive assumptions for the heat flux vector differ in any theory. Thus, they obtained three theories that they called thermoelasticity of type I, II and III. When the theory of type is linearized, we obtain the classical system thermoelasticity given by Fourier's law. The theory of type II does not allow the dissipation of the energy, and it's usually known as thermoelasticity without energy dissipation. But in thermoelasticity of type III, we suggest for heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed.

Time delays occur in many applications because most phenomena depend not only on the present state but also past events, Therefore, the stability of system with time delay effects has emerged as a significant area of research over the past few years. That is to say, since the 1970s of the last century. See for example [3,10]. In recent years, numerous mathematicians have investigated the stability of delay systems due to their extensive application in various sciences. Apalara in [3] studied this model in his article entitled " Well posedness and exponential stability for a linear damped Timoshenko system with second sound and internal distributed delay". And Kafinia et al. in [10] studied the model with distributed delay, entitled "Energy decay result in a Timoshenko-type system of thermoelasticity of type III with distributive delay".

In this work, we aimed to combine two important concepts, which are flexible structural systems of thermoelasticity of type III and delay terms. We consider the following flexible structure system of thermoelasticity of type III with distributed delay, which has the form

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \beta\theta_{tx} = 0, \\ \rho\theta_{tt} - \sigma\theta_{xx} + \gamma u_{tx} - k\theta_{txx} - \displaystyle\int_{\tau_1}^{\tau_2} g(s)\,\theta_{txx}(x,t-s)\,ds = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,L)\times(0,+\infty)$, with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x)\,,\ u_t(x,0) = u_1(x)\,, & x \in [0,L], \\ \theta(x,0) = \theta_0(x)\,,\ \theta(x,0) = \theta_1(x)\,, & x \in [0,L], \\ u(0,t) = u(L,t) = 0, & t \in [0,+\infty), \\ \theta_x(0,t) = \theta_x(L,t) = 0, & t \in [0,+\infty), \\ \theta_{tx}(x,-t) = f_0(x,t)\,, & (x,t) \in (0,L)\times(0,\tau_2). \end{cases} \tag{1.2}$$

The paper is organized as follows.

In Section 2, we present some assumptions and preliminary works.

In Section 3, we use the perturbed energy method and construct some Lyapunov functionals to prove our stability result.

## 2. Preliminaries

In this section, we present some materials needed in the proof of our result. Firstly, to deal with the delay term, we introduce the new variable

$$\varpi(x,\xi,s,t) = \theta_{tx}(x,t-s\xi),\ x \in (0,L), \xi \in (0,1), s \in (\tau_1,\tau_2)\,,\ t > 0.$$

A simple differentiation shows that $\varpi$ satisfies,

$$\xi\varpi_t(x,\xi,s,t)+\varpi_\xi(x,\xi,s,t)=0,\ x\in(0,L),\xi\in(0,1),s\in(\tau_1,\tau_2)\,,\ t>0.$$

Then the problem (1.1) is equivalent to

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\beta\theta_{tx}=0,\\ \rho\theta_{tt}-\sigma\theta_{xx}+\gamma u_{tx}-k\theta_{txx}-\int_{\tau_1}^{\tau_2}g\left(s\right)\varpi_x(x,1,s,t)ds=0,\\ \xi\varpi_t(x,\xi,s,t)+\varpi_\xi(x,\xi,s,t)=0, \end{cases}\tag{2.1}$$

with the following initial and boundary conditions

$$\begin{cases} u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right), & x\in[0,L],\\ \theta\left(x,0\right)=\theta_0\left(x\right),\ \theta\left(x,0\right)=\theta_1\left(x\right), & x\in[0,L],\\ u\left(0,t\right)=u\left(L,t\right)=0, & t\in[0,+\infty),\\ \theta_x\left(0,t\right)=\theta_x\left(L,t\right)=0, & t\in[0,+\infty),\\ \theta_{tx}\left(x,-t\right)=f_0\left(x,t\right), & (x,t)\in(0,L)\times(0,\tau_2). \end{cases}\tag{2.2}$$

In order to be able to use Poincaré's inequality for $\theta$ , we introduce

$$\bar{\theta}\left(x,t\right)=\theta\left(x,t\right)-t\int_0^L\theta_1\left(x\right)dx-\int_0^L\theta_0\left(x\right)dx.$$

Then by $(2.1)_2$ we have

$$\int_0^L\bar{\theta}\left(x,t\right)dx=0.\ \ \forall t\geq 0.$$

In this case, Poincaré's inequality is applicable for $\bar{\theta}$. On the other hand, it is easy to check that $\left(u,\bar{\theta},\varpi\right)$ satisfies the same equations and boundary conditions of (2.2). In what follows, we will work with $\bar{\theta}$ but for convenience, we write $\theta$ instead of $\bar{\theta}$. In the sequel we consider $(u,\theta,\varpi)$ to be a solution of system (2.1)-(2.2) with the regularity needed to justify the calculations in this paper.
We shall use the following hypotheses, $g:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function and

$$k-\int_{\tau_1}^{\tau_2}|g(s)|ds>0.\tag{2.3}$$

The associated energy is give by

$$\mathcal{E}(t)=\frac{\beta}{2}\int_0^L\left(\rho\theta_t^2+\delta\theta_x^2\right)dx+\frac{\beta}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s|g(s)|\varpi(x,\xi,s,t)d\xi dsdx+\frac{\gamma}{2}\int_0^L m(x)u_t^2dx+\frac{\gamma}{2}\int_0^L p(x)u_x^2dx.\tag{2.4}$$

The following lemma shows that the system is dissipative.

**Lemma 2.1** *Let $(u,\theta,\varpi)$ be the solution of the system (2.1)-(2.2) and assume (2.3) holds, then energy functional satisfies*

$$\mathcal{E}'(t)\leq-2\gamma\int_0^L\delta(x)u_{tx}^2dx-\beta\left(k-\int_{\tau_1}^{\tau_2}|g(s)|ds\right)\int_0^L\theta_{tx}^2dx\leq 0,\tag{2.5}$$

*for all $t\geq 0$.*

**Proof:** A simple multiplication of $(2.1)_1$ and $(2.1)_2$ by $\gamma u_t$ and $\beta\theta_t$ , respectively, and integrating over $(0, L)$, on the other hand, multiplying $(2.1)_3$ by $\beta|g(s)|\varpi$, integrating the product over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$. Using integration by parts and the boundary conditions in (2.2) and summing up to obtain

$$\begin{aligned}\mathcal{E}'(t) = -k\beta \int_0^L \theta_{tx}^2 dx - 2\gamma \int_0^L \delta(x) u_{tx}^2 dx + \beta \int_0^L \int_{\tau_1}^{\tau_2} g(s)\varpi_x(x,1,s,t)\theta_t ds dx \\ -\beta \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |g(s)|\varpi_\xi(x,\xi,s,t)\varpi(x,\xi,s,t) ds d\xi dx.\end{aligned} \tag{2.6}$$

Now, by using integration by parts and Young's inequality and recall that $\varpi(x, 0, s, t) = \theta_{tx}$, we get

$$\begin{aligned}\beta \int_0^L \int_{\tau_1}^{\tau_2} g(s)\varpi_x(x,1,s,t)\theta_t ds dx &= -\beta \int_0^L \int_{\tau_1}^{\tau_2} g(s)\varpi(x,1,s,t)\theta_{tx} ds dx \\ &\le \frac{\beta}{2} \int_0^L \int_{\tau_1}^{\tau_2} |g(s)|\varpi^2(x,1,s,t) ds dx + \frac{\beta}{2} \int_{\tau_1}^{\tau_2} |g(s)| ds \int_0^L \theta_{tx}^2 dx.\end{aligned} \tag{2.7}$$

And by a simple calculation, we have

$$\begin{aligned}&-\beta \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |g(s)|\varpi_\xi(x,\xi,s,t)\varpi(x,\xi,s,t) ds d\xi dx \\ &= -\frac{\beta}{2} \int_0^L \int_{\tau_1}^{\tau_2} |g(s)| \int_0^1 \frac{d}{d\xi}\left(\varpi^2(x,\xi,s,t)\right) d\xi ds dx \\ &= -\frac{\beta}{2} \int_0^L \int_{\tau_1}^{\tau_2} |g(s)|\varpi^2(x,1,s,t) ds dx + \frac{\beta}{2} \int_{\tau_1}^{\tau_2} |g(s)| ds \int_0^L \theta_{tx}(x,t) dx.\end{aligned} \tag{2.8}$$

The substitution of (2.7) and (2.8) into (2.6) and using (2.3) gives (2.5), which concludes the proof. □

To prove the decay result, we need the following lemmas.

**Lemma 2.2 (Poincaré-type scheeffer's inquality)** *Let $\varphi$ in $H_0^1(0, L)$. Then we have,*

$$\int_0^L \varphi^2 ds \le \frac{L^2}{\pi^2} \int_0^L \varphi_x^2 ds. \tag{2.9}$$

**Lemma 2.3 (mean value theorem)** *Let $h$ be a function define on $(0, L)$ and $p, m, \delta$ function given above. Then there exists $(\varsigma_i)_{i=1}^3 \in [0, L]$, such that,*

$$\begin{aligned}\int_0^L p(x)h^2 dx = p(\varsigma_1) \int_0^L h^2 dx, \\ \int_0^L m(x)h^2 dx = m(\varsigma_2) \int_0^L h^2 dx, \\ \int_0^L \delta(x)h^2 dx = \delta(\varsigma_3) \int_0^L h^2 dx.\end{aligned} \tag{2.10}$$

## 3. Decay result

In this section, we are ready to prove this important result, it's called stability results of the system (2.1)-(2.2), by using the Lyapunov function. Let's prove the following lemmas:

**Lemma 3.1** *Let $(u, \theta, \varpi)$ be a solution of the system (2.1)-(2.2), we consider the functional*

$$\mathcal{I}_1(t) := \int_0^L m(x) u_t u dx + \int_0^L \delta(x) u_x^2 dx.$$

*For any positive constant $\epsilon_1$ and $\iota_1$, $\iota_2$ in $[0, L]$, we have the following estimate*

$$\mathcal{I}_1'(t) \leq -\left(p(\iota_2) - \epsilon_1\right) \int_0^L u_x^2 dx + m(\iota_1) \int_0^L u_t^2 dx + \frac{\beta^2}{4\epsilon_1} \int_0^L \theta_t^2 dx. \tag{3.1}$$

**Proof:** By differentiating $\mathcal{I}_1(t)$ and using $(2.1)_1$, we obtain

$$\begin{aligned} \mathcal{I}_1'(t) &= \int_0^L m(x) u_{tt} u dx + \int_0^L m(x) u_t^2 dx + 2 \int_0^L \delta(x) u_{tx} u_x dx \\ &= \int_0^L m(x) u_t^2 dx + \int_0^L \left(p(x) u_x + 2\delta(x) u_{tx}\right)_x u dx - \int_0^L \theta_{tx} u dx + 2 \int_0^L \delta(x) u_{tx} u_x dx \\ &= \int_0^L m(x) u_t^2 dx - \int_0^L p(x) u_x^2 dx - \int_0^L \theta_{tx} u dx, \end{aligned} \tag{3.2}$$

By utilizing integration by parts along with Young's inequality, we derive

$$\begin{aligned} -\beta \int_0^L \theta_{tx} u dx &= \beta \int_0^L \theta_t u_x dx \\ &\leq \epsilon_1 \int_0^L u_x^2 dx + \frac{\beta^2}{4\epsilon_1} \int_0^L \theta_t^2 dx. \end{aligned} \tag{3.3}$$

From lemma 2.3, we have

$$\begin{cases} \displaystyle\int_0^L m(x) u_t^2 dx = m(\iota_1) \int_0^L u_t^2 dx, \quad \iota_1 \in [0, L], \\ \displaystyle\int_0^L p(x) u_x^2 dx = p(\iota_2) \int_0^L u_x^2 dx, \quad \iota_2 \in [0, L]. \end{cases} \tag{3.4}$$

The substitution of (3.3) and (3.4) into (3.2) gives (3.1). □

**Lemma 3.2** *Let $(u, \theta, \varpi)$ be a solution of the system (2.1)-(2.2), we consider the functional*

$$\mathcal{I}_2(t) := \rho \int_0^L \theta_t \theta dx + \frac{k}{2} \int_0^L \theta_x^2 dx + \gamma \int_0^L u_x \theta dx.$$

*For $\epsilon_2, \epsilon_3 > 0$, we have the following estimate*

$$\mathcal{I}_2'(t) \leq -\left(\sigma - \epsilon_2\right) \int_0^L \theta_x^2 dx + \frac{k}{4\epsilon_2} \int_0^L \int_{\tau_1}^{\tau_2} |g(s)| \varpi^2(x, 1, s, t) ds dx + \left(\rho + \frac{\gamma^2}{4\epsilon_3}\right) \int_0^L \theta_t^2 dx + \epsilon_3 \int_0^L u_x^2 dx. \tag{3.5}$$

**Proof:** Differentiating $\mathcal{I}_2(t)$ with respect to $t$, using equation $(2.1)_2$ and integrating by parts, we obtain

$$\begin{aligned}
\mathcal{I}_2'(t) &= \int_0^L \left(\sigma\theta_{xx} + \int_{\tau_1}^{\tau_2} g(s)\varpi_x(x,1,s,t)ds\right)\theta dx + \rho\int_0^L \theta_t^2 dx + \gamma\int_0^L u_x\theta_t dx \\
&= -\sigma\int_0^L \theta_x^2 dx - \int_0^L \theta_x \int_{\tau_1}^{\tau_2} g(s)\varpi(x,1,s,t)dsdx + \rho\int_0^L \theta_t^2 dx + \gamma\int_0^L u_x\theta_t dx.
\end{aligned} \tag{3.6}$$

By using Young's inequality and (2.3), we get for $\epsilon_2, \epsilon_3 > 0$,

$$\begin{aligned}
-\int_0^L \theta_x \int_{\tau_1}^{\tau_2} g(s)\varpi(x,1,s,t)dsdx &\leq \epsilon_2\int_0^L \theta_x^2 dx + \frac{1}{4\epsilon_2}\int_0^L \left(\int_{\tau_1}^{\tau_2} g(s)\varpi(x,1,s,t)ds\right)^2 dx \\
&\leq \epsilon_2\int_0^L \theta_x^2 dx + \frac{k}{4\epsilon_2}\int_0^L\int_{\tau_1}^{\tau_2} |g(s)|\varpi^2(x,1,s,t)dsdx,
\end{aligned} \tag{3.7}$$

$$\gamma\int_0^L u_x\theta_t dx \leq \epsilon_3\int_0^L u_x^2 dx + \frac{\gamma^2}{4\epsilon_3}\int_0^L \theta_t^2 dx. \tag{3.8}$$

Estimating (3.5) follows by substituting (3.7) and (3.8) into (3.6). □

**Lemma 3.3** *Let $(u,\theta,\varpi)$ be a solution of the system (2.1)-(2.2), we consider the functional*

$$\mathcal{I}_3(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx.$$

*For $n_1 > 0$, we have the following estimate*

$$\mathcal{I}_3'(t) \leq -n_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx + k\int_0^L \theta_{tx}^2 dx - n_1\int_0^L\int_{\tau_1}^{\tau_2} |g(s)|\varpi^2(x,1,s,t)dsdx. \tag{3.9}$$

**Proof:** Differentiating $\mathcal{I}_3$, and using the equation $(2.1)_3$, we obtain

$$\begin{aligned}
\mathcal{I}_3'(t) &= 2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi_t(x,\xi,s,t)\varpi(x,\xi,s,t)dsd\xi dx \\
&= -\int_0^L\int_{\tau_1}^{\tau_2}\int_0^1 |g(s)|\frac{\partial}{\partial\xi}\left(e^{-s\xi}\varpi^2(x,\xi,s,t)\right)d\xi dsdx - \int_0^L\int_{\tau_1}^{\tau_2}\int_0^1 se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)d\xi dsdx \\
&= -\int_0^L\int_{\tau_1}^{\tau_2} |g(s)|\left(e^{-s}\varpi^2(x,1,s,t) - \varpi^2(x,0,s,t)\right)dsdx - \int_0^L\int_{\tau_1}^{\tau_2}\int_0^1 se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)d\xi dsdx.
\end{aligned}$$

We have $\varpi(x,0,s,t) = \theta_{tx}(x,t)$ and for all $\xi$ in $[0,1]$ we also have $e^{-s} \leq e^{-s\xi} \leq 1$, we obtain

$$\mathcal{I}_3'(t) \leq -n_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx + \int_{\tau_1}^{\tau_2}|g(s)|ds\int_0^L \theta_{tx}^2 dx - n_1\int_0^L\int_{\tau_1}^{\tau_2} |g(s)|\varpi^2(x,1,s,t)dsdx,$$

where $n_1 = -e^{-\tau_2}$ ($-e^{-s}$ is an increasing function on $[\tau_1,\tau_2]$). We obtain (3.9) by virtue of (2.3). □

**Lemma 3.4** *For $\mathcal{N}$ sufficiently large, the functional defined by*

$$\mathcal{L}(t) := \mathcal{N}\mathcal{E}(t) + \mathcal{N}_1\mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{N}_3\mathcal{I}_3(t), \tag{3.10}$$

*where $\mathcal{N}$, $\mathcal{N}_1$ and $\mathcal{N}_3$ are positive real numbers to be chosen appropriately later, satisfies*

$$C_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq C_2\mathcal{E}(t), \tag{3.11}$$

*for all $t > 0$, where $C_1$ and $C_2$ are positive constants.*

**Proof:** We have

$$\begin{aligned}|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| = \quad & \Bigg|\mathcal{N}_1\left(\int_0^L m(x)u_t u dx + \int_0^L \delta(x)u_x^2 dx\right) + (\rho\int_0^L \theta_t\theta dx + \frac{k}{2}\int_0^L \theta_x^2 dx + \gamma\int_0^L u_x\theta dx\Big) \\ & + \mathcal{N}_3\left(\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx\right)\Bigg|,\end{aligned}$$

then

$$\begin{aligned}|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| \leq \quad & \mathcal{N}_1\left|\int_0^L m(x)u_t u dx\right| + \mathcal{N}_1\int_0^L \delta(x)u_x^2 dx + \rho\left|\int_0^L \theta_t\theta dx\right| + \frac{k}{2}\int_0^L \theta_x^2 dx \\ & + \gamma\left|\int_0^L u_x\theta dx\right| + \mathcal{N}_3\left|\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx\right|,\end{aligned}$$

through a straightforward calculation, we obtain

$$\begin{aligned}|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| \leq \quad & \frac{\mathcal{N}_1\|m\|_\infty}{2}\int_0^L u_t^2 dx + \frac{\mathcal{N}_1\|m\|_\infty}{2}\int_0^L u^2 dx + \frac{\mathcal{N}_1\|\delta\|_\infty}{2}\int_0^L u_x^2 dx \\ & + \frac{\rho}{2}\int_0^L \theta_t^2 dx + \frac{\rho}{2}\int_0^L \theta^2 dx + \frac{k}{2}\int_0^L \theta_x^2 dx + \frac{\gamma}{2}\int_0^L u_x^2 dx + \frac{\gamma}{2}\int_0^L \theta^2 dx \\ & + \mathcal{N}_3\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx,\end{aligned}$$

by employing lemma 2.2, we get

$$\begin{aligned}|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| \leq \quad & \frac{\mathcal{N}_1\|m\|_\infty}{2}\int_0^L u_t^2 dx + \frac{\mathcal{N}_1\|m\|_\infty}{2}\frac{L^2}{\pi^2}\int_0^L u_x^2 dx + \frac{\mathcal{N}_1\|\delta\|_\infty}{2}\int_0^L u_x^2 dx \\ & + \frac{\rho}{2}\int_0^L \theta_t^2 dx + \frac{\rho L^2}{2\pi^2}\int_0^L \theta_x^2 dx + \frac{k}{2}\int_0^L \theta_x^2 dx + \frac{\gamma}{2}\int_0^L u_x^2 dx \\ & + \frac{\gamma L^2}{2\pi^2}\int_0^L \theta_x^2 dx + \mathcal{N}_3\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx,\end{aligned}$$

after all the above, we find the required result

$$\begin{aligned}|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| \leq \quad & \frac{\mathcal{N}_1\|m\|_\infty}{2}\int_0^L u_t^2 dx + \left(\frac{\mathcal{N}_1\|m\|_\infty}{2}\frac{L^2}{\pi^2} + \frac{\mathcal{N}_1\|\delta\|_\infty}{2} + \frac{\gamma}{2}\right)\int_0^L u_x^2 dx + \frac{\rho}{2}\int_0^L \theta_t^2 dx \\ & + \left(\frac{\rho L^2}{2\pi^2} + \frac{k}{2} + \frac{\gamma L^2}{2\pi^2}\right)\int_0^L \theta_x^2 dx + \mathcal{N}_3\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\xi}|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx,\end{aligned}$$

consequently,

$$\begin{aligned}|\mathcal{L}(t)-\mathcal{N}\mathcal{E}(t)| \leq\ & \left(\frac{\mathcal{N}_1\|m\|_\infty}{2}+\frac{\mathcal{N}_1\|m\|_\infty}{2}\frac{L^2}{\pi^2}+\frac{\mathcal{N}_1\|\delta\|_\infty}{2}+\frac{\gamma}{2}+\frac{\rho}{2}+\frac{\rho L^2}{2\pi^2}+\frac{k}{2}+\frac{\gamma L^2}{2\pi^2}+\mathcal{N}_3\right)\mathcal{E}(t)\\ &= S_1\mathcal{E}(t),\end{aligned}$$

which yields

$$C_1\mathcal{E}(t)=(\mathcal{N}-S_1)\mathcal{E}(t)\leq\mathcal{L}(t)\leq(\mathcal{N}+S_1)\mathcal{E}(t)=C_2\mathcal{E}(t).$$

By choosing $\mathcal{N}$ large enough, the equation (3.11) follows. □

The following theorem is the main result of this section:

**Theorem 3.1** *Let $(u,\theta,\varpi)$ be the solution of system (2.1)-(2.2), then the energy satisfies:*

$$\mathcal{E}(t)\leq Q_1e^{-Q_2t},\ \forall t\geq 0, \tag{3.12}$$

*where $Q_1$ and $Q_2$ are positive constants.*

**Proof:** By differentiating the Lyapounov function with respect to $t$, and using lemma.2.1, lemma.3.1, lemma.3.2 and lemma.3.3, we have

$$\begin{aligned}\mathcal{L}'(t)\ \leq\ & -2\gamma\mathcal{N}\int_0^L\delta(x)u_{tx}^2dx-\beta\mathcal{N}\left(k-\int_{\tau_1}^{\tau_2}|g(s)|ds\right)\int_0^L\theta_{tx}^2dx\\ & -(p(\iota_2)-\epsilon_1)\mathcal{N}_1\int_0^L u_x^2dx+m(\iota_1)\mathcal{N}_1\int_0^L u_t^2dx+\frac{\beta^2}{4\epsilon_1}\mathcal{N}_1\int_0^L\theta_t^2dx\\ & -(\sigma-\epsilon_2)\int_0^L\theta_x^2dx+\frac{k}{4\epsilon_2}\int_0^L\int_{\tau_1}^{\tau_2}|g(s)|\varpi^2(x,1,s,t)dsdx+\left(\rho+\frac{\gamma^2}{4\epsilon_3}\right)\int_0^L\theta_t^2dx\\ & +\epsilon_3\int_0^L u_x^2dx-n_1\mathcal{N}_3\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx+k\mathcal{N}_3\int_0^L\theta_{tx}^2dx\\ & -n_1\mathcal{N}_3\int_0^L\int_{\tau_1}^{\tau_2}|g(s)|\varpi^2(x,1,s,t)dsdx.\end{aligned}$$

Using lemma 2.2 and lemma 2.3 gives

$$\begin{aligned}\mathcal{L}'(t)\ \leq\ & -\left[\frac{2\gamma\pi^2\delta(\iota_3)}{L^2}\mathcal{N}-m(\iota_1)\mathcal{N}_1\right]\int_0^L u_t^2dx-\left[(p(\iota_2)-\epsilon_1)\mathcal{N}_1-\epsilon_3\right]\int_0^L u_x^2dx\\ & -\left[\beta\mathcal{N}\left(k-\int_{\tau_1}^{\tau_2}|g(s)|ds\right)-\frac{\beta^2L^2}{4\pi^2\epsilon_1}\mathcal{N}_1-\left(\rho+\frac{\gamma^2}{4\epsilon_3}\right)\frac{L^2}{\pi^2}-k\mathcal{N}_3\right]\int_0^L\theta_{tx}^2dx\\ & -(\sigma-\epsilon_2)\int_0^L\theta_x^2dx-\left[n_1\mathcal{N}_3-\frac{k}{4\epsilon_2}\right]\int_0^L\int_{\tau_1}^{\tau_2}|g(s)|\varpi^2(x,1,s,t)dsdx\\ & -n_1\mathcal{N}_3\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s|g(s)|\varpi^2(x,\xi,s,t)dsd\xi dx.\end{aligned} \tag{3.13}$$

At this point, we first choose $\epsilon_3=1$ and $\epsilon_1$, $\epsilon_2$ small enough such that

$$p(\iota_2)-\epsilon_1>0,\qquad \sigma-\epsilon_2>0.$$

We then choose large constants $\mathcal{N}_1$ and $\mathcal{N}_3$ such that

$$(p(\iota_2)-\epsilon_1)\mathcal{N}_1-\epsilon_3>0,\quad n_1\mathcal{N}_3-\frac{k}{4\epsilon_2}>0.$$

For fixed $\mathcal{N}_1$ and $\mathcal{N}_3$, we then choose $\mathcal{N}$ large enough so that

$$\frac{2\gamma\pi^2\delta(\iota_3)}{L^2}\mathcal{N}-m(\iota_1)\mathcal{N}_1>0,$$

$$\beta\mathcal{N}\left(k-\int_{\tau_1}^{\tau_2}|g(s)|ds\right)-\frac{\beta^2L^2}{4\pi^2\epsilon_1}\mathcal{N}_1-\left(\rho+\frac{\gamma^2}{4\epsilon_3}\right)\frac{L^2}{\pi^2}-k\mathcal{N}_3>0.$$

By (2.4), we deduce that there exists a positive constant $\mu$ such that (3.13) becomes

$$\mathcal{L}'(t)\le-\mu\mathcal{E}(t),\quad\forall t\ge0, \tag{3.14}$$

A combination of (3.11) and (3.14) gives

$$\mathcal{L}'(t)\le-Q_2\mathcal{L}(t),\quad\forall t\ge0, \tag{3.15}$$

where $Q_2=\dfrac{\mu}{C_2}$. A simple integration of (3.14) over $(0,t)$ yields

$$\mathcal{L}(t)\le\mathcal{L}(0)e^{-Q_2t},\quad\forall t\ge0. \tag{3.16}$$

Finally, by combining (3.11) and (3.16) we obtain (3.12) with $Q_1=\dfrac{C_2\mathcal{E}(0)}{C_1}$, which completes the proof. □

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*Younes Bidi,*
*Department of Mathematics, Higher College of Teachers (ENS) of Laghouat,*
*Pure and applied Mathematics Laboratory, University of Laghouat,*
*Algeria.*
*E-mail address:* `y.bidi@ens-lagh.dz`

*and*

*Madani Douib,*
*Department of Mathematics,*
*Higher College of Teachers (ENS) of Laghouat,*
*Algeria.*
*E-mail address:* `madanidouib@gmail.com`