Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.77840

On the Existence and Uniqueness of Solutions for a ψ-Caputo Fractional Boundary Value Problem with Fractional Boundary Conditions

Hamza Medjekal* and Besma Berrehail

ABSTRACT: The main purpose of the present work is the investigation of the existence and uniqueness of solutions to an implicit nonlinear boundary value problem of fractional differential equations involving $\psi-$Caputo fractional derivative and subject to fractional boundary conditions. The results of existence and uniqueness are proved thanks to Banach and Scheafer's fixed point theorems. Finally, we present appurtenant examples to illustrate the obtained results.

Keywords: $\psi-$Caputo fractional derivative, $\psi-$Riemann-Liouville fractional integral, boundary value problems, fixed point theorems.

Contents

1. Introduction

Fractional calculus [12,15] is an extension of classical calculus that allows integration and differentiation to arbitrary, non-integer orders. The origins of fractional calculus trace back to a letter from Leibniz in 1695, in which he posed the idea of a half-order derivative. Although initially considered a mathematical curiosity, the theory has steadily matured, especially during the 19^{th} and 20^{th} centuries through the contributions of Liouville, Riemann, Caputo, and others, and found extensive applications across various scientific disciplines including Physics, Mechanics, Bioengineering, and related sciences [8,13]. In recent years, the development of generalized fractional operators has greatly enriched the field by introducing more flexible and adaptive modeling tools [5,9,10,11,17].

The $\psi-$Caputo fractional derivative was first introduced by R. Almeida [2], it extends the classical Caputo derivative by inserting a suitable function that modulates the differential operator. This generalization not only retains the advantages of the classical Caputo derivative such as the use of integer-order initial conditions, but also introduces additional complaisance in modeling real world problems where the underlying time scale or medium is non-uniform. More details on fractional differential equations with $\psi-$Caputo fractional derivatives can be found in [4,3,7,14] and the references therein.

In [1], Mohamed *et al.* investigated existence and uniqueness results of the following boundary value problem

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}y(t) = f(t,y(t)), \quad t\in[a,b],\\ y^{[k]}_{\psi}(a) = y^k_a, \quad k=0,1,\ldots,\ n-2,\ y^{[n-1]}_{\psi}(b) = y_b, \end{cases}$$

where: ${}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}$ is the ψ-Caputo fractional derivative of order $n-1<\alpha<n,\ n\in\mathbb{N},\ n\geq 2,\ \ y^k_a,\ y_b\in\mathbb{R}$, $f\in\mathcal{C}([a,b]\times\mathbb{R},\mathbb{R})$ is a continuous function and $y\in\mathcal{C}^{n-1}([a,b])$.

* Corresponding author.
2020 *Mathematics Subject Classification:* 26A33, 34A08, 34B15, 47H10.
Submitted July 13, 2025. Published March 19, 2026

In [16], the authors carried out the following boundary value problem of fractional differential equation:

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\alpha,\psi}u(t) = f(t,u(t)), \quad t \in J = [0,1] \\ u(0) = u'(0) = 0, \ {}^C\mathcal{D}_{0^+}^{\alpha-1,\psi}u(1) = 0, \ {}^C\mathcal{D}_{0^+}^{\alpha-2,\psi}u(1) = 0, \end{cases}$$

where ${}^C\mathcal{D}_{a+}^{\alpha}$ is the ψ-Caputo fractional derivative of order $3 < \alpha \leq 4$ and $f : J \times [0,\infty) \to \mathbb{R}$ is a given continuous function. They used fixed point theorems to investigate the existence and uniqueness of the solution to the above problem.

Inspired by the above works, we intend to extend the results established in [1,16]. More accurately, we are concerned with the existence and uniqueness of solutions of the following nonlinear fractional differential equation with ψ-Caputo fractional derivative and subject to a fractional boundary condition at the endpoint b:

$$\begin{cases} {}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t) = F(t,\upsilon(t), {}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)), \quad t \in \mathcal{J}=[a,b], \\ \upsilon_{\psi}^{[k]}(a) = \upsilon_k, \ k = 0,1,\ldots,n-2, \ {}^C\mathcal{D}_{a^+}^{\beta,\psi}\upsilon(b) = \upsilon_b, \end{cases} \tag{1.1}$$

where: ${}^C\mathcal{D}_{a^+}^{\alpha,\psi}$ represents the ψ-Caputo fractional derivative of order $\alpha \in (n-1,n)$, $n \in \mathbb{N}$, $n \geq 2$, ${}^C\mathcal{D}_{a^+}^{\beta,\psi}$ is the ψ-Caputo fractional derivative of order $0 < \beta < \alpha - (n-1)$, $\upsilon_k, \upsilon_b \in X$, $F \in \mathcal{C}(\mathcal{J} \times X^2, X)$ is a continuous function and $(X, \|.\|)$ is a Banach space. $\upsilon \in \mathcal{C}^{n-1}(\mathcal{J}, X)$ such that ${}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon$ exists and is continuous.

Our formulation permits boundary conditions that depend on the fractional derivative ${}^C\mathcal{D}_{a^+}^{\beta,\psi}$, resulting in non-trivial analytical complications. To the best of our knowledge, no prior work has addressed the existence of solutions of the fractional boundary problem (1.1) dealing with such a fractional condition.

The rest of the paper is structured as follows: Section 2 is devoted to the necessary definitions and auxiliary lemmas. In section 3, we derive the exact solution formula for the proposed problem and establish existence and uniqueness results via Banach and Scheafer's fixed point theorems. Finally, in Section 4 we provide concrete examples demonstrating the applicability of our results.

2. Preliminaries

Let $\mathcal{J} = [a,b]$, $\psi \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ a strictly increasing function with $\psi'(t) \neq 0$ and $(X, \|.\|)$ be a given Banach space. In what follows, we state key definitions, notations and fundamental results regarding $\psi-$fractional integrals and derivatives that will be utilized all over this article.

Definition 2.1 *[4] The left-sided $\psi-$Riemann-Liouville fractional integral of order $\alpha \in (n-1,n)$, $n \in \mathbb{N}^*$ of an integrable function $u : \mathcal{J} \to X$ is given by*

$$\mathcal{I}_{a^+}^{\alpha,\psi}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}u(s)\,ds, \ t \in \mathcal{J},$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2 *[4]The left-sided $\psi-$Riemann-Liouville fractional derivative of order $\alpha \in (n-1,n)$, $n \in \mathbb{N}^*$ of an integrable function $u : \mathcal{J} \to X$ is given by*

$$\mathcal{D}_{a^+}^{\alpha,\psi}u(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{n-\alpha,\psi}u(t)\,, \ t \in \mathcal{J}.$$

Definition 2.3 *[4] Let $u \in \mathcal{C}^n(\mathcal{J},X)$. The left-sided $\psi-$Caputo fractional derivative of order α of the function u is given by*

$${}^C\mathcal{D}_{a^+}^{\alpha,\psi}u(t) = D_{a^+}^{\alpha,\psi}\left[u(t) - \sum_{k=0}^{n-1}\frac{u_{\psi}^{[k]}(a)}{k!}(\psi(t) - \psi(a))^k\right],$$

where $u_{\psi}^{[k]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k u(t)$.

If $u \in \mathcal{C}^n(\mathcal{J})$ and $\alpha \notin \mathbb{N}$, then

$$
\begin{aligned}
{}^C\mathcal{D}_{a^+}^{\alpha,\psi} u(t) &= \mathcal{I}_{a^+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} u_\psi^{[k]}(s)\, ds.
\end{aligned}
$$

Remark 2.1 *If $\alpha = n$, then*

$$
{}^C\mathcal{D}_{a^+}^{\alpha,\psi} u(t) = u_\psi^{[k]}(t).
$$

We recall the following useful lemmas:

Lemma 2.1 *[4] Let $\alpha > 0$ and $u : \mathcal{J} \to X$. The following holds:*

1. *If $u \in \mathcal{C}(\mathcal{J}, X)$, then ${}^C\mathcal{D}_{a^+}^{\alpha,\psi} \mathcal{I}_{a^+}^{\alpha,\psi} u(t) = u(t)$, $t \in \mathcal{J}$,*
2. *If $u \in \mathcal{C}^n(\mathcal{J}, X)$, then*

$$
\mathcal{I}_{a^+}^{\alpha,\psi}\ {}^C\mathcal{D}_{a^+}^{\alpha,\psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(a)}{k!} (\psi(t)-\psi(a))^k, \ n = [\alpha]+1,
$$

3. *$(\mathcal{I}_{a^+}^{\alpha,\psi} u)_\psi^{[k]}(t) = (\mathcal{I}_{a^+}^{\alpha-k,\psi} u)(t)$, $t \in \mathcal{J}$, $k = 0, 1, ...n-1$.*

Lemma 2.2 *[2] Let α, $\beta > 0$ and $u : \mathcal{J} \to X$. Then, for every $t \in \mathcal{J}$, we have*

1. *$\mathcal{I}_{a^+}^{\alpha,\psi} (\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\psi(t)-\psi(a))^{\alpha+\beta-1}$,*
2. *${}^C\mathcal{D}_{a^+}^{\alpha,\psi} (\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t)-\psi(a))^{\beta-\alpha-1}$,*
3. *${}^C\mathcal{D}_{a^+}^{\alpha,\psi} (\psi(t)-\psi(a))^k = 0$, $\forall k = 0, 1, \ldots, n-1$, $n = [\alpha]+1$,*
4. *$\mathcal{I}_{a^+}^{\alpha,\psi} \mathcal{I}_{a^+}^{\beta,\psi} u(t) = \mathcal{I}_{a^+}^{\alpha+\beta,\psi} u(t)$.*

3. Main results

In the present section, we investigate the existence and uniqueness of solutions for the fractional boundary value problem (1.1) using appropriate fixed point theorems. In the following auxiliary lemma, we establish the equivalence between the basic version of problem (1.1) and a certain fractional integral equation.

Lemma 3.1 *Let $\alpha \in (n-1, n)$, $n \geq 2$, $\sigma \in \mathcal{C}(\mathcal{J}, X)$ and $\psi \in \mathcal{C}^n(\mathcal{J}, \mathbb{R})$ a strictly increasing function with $\psi'(t) \neq 0$. A function $\upsilon \in \mathcal{C}^{n-1}(\mathcal{J}, X)$ is a solution to the boundary value problem*

$$
\begin{cases}
{}^C\mathcal{D}_{a^+}^{\alpha,\psi} \upsilon(t) = \sigma(t), \quad t \in \mathcal{J}, \\
\upsilon_\psi^{[k]}(a) = \upsilon_k, \quad k = 0, 1, \ldots, n-2, \\
({}^C D_{a^+}^{\beta,\psi} \upsilon)(b) = \upsilon_b, \quad 0 < \beta < \alpha-(n-1),
\end{cases}
\tag{3.1}
$$

if and only if υ possesses the following representation:

$$
\begin{aligned}
\upsilon(t) = \sum_{k=0}^{n-2} \frac{\upsilon_k}{k!} (\psi(t)-\psi(a))^k + \mathcal{I}_{a^+}^{\alpha,\psi} \sigma(t) + \frac{\Gamma(n-\beta)}{(n-1)!} \frac{(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}} \\
\times \left(\upsilon_b - \sum_{k=1}^{n-2} \upsilon_k \frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)} - \mathcal{I}_{a^+}^{\alpha-\beta,\psi} \sigma(b) \right)
\end{aligned}
\tag{3.2}
$$

Proof: Performing the $\psi-$Riemann-Liouville operator $\mathcal{I}_{a^+}^{\alpha,\psi}$ to both sides of the fractional differential equation and using Lemma 2.1, we obtain

$$\begin{aligned} \upsilon(t) &= \sum_{k=0}^{n-1} \frac{c_k}{k!}(\psi(t)-\psi(a))^k + \mathcal{I}_{a^+}^{\alpha,\psi}\sigma(t), \qquad (3.3) \\ &= c_0 + c_1(\psi(t)-\psi(a)) + \frac{c_2}{2!}(\psi(t)-\psi(a))^2 + \ldots + \frac{c_{n-1}}{(n-1)!}(\psi(t)-\psi(a))^{n-1} \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\sigma(s)\,ds \end{aligned}$$

From the initial conditions $\upsilon_\psi^{[k]}(a) = \upsilon_k, \quad k = 0, 1, \ldots, n-2$, we obtain at once

$$c_k = \upsilon_k, \quad k = 0, 1, \ldots, n-2.$$

The remaining constant c_{n-1} is determined by the boundary condition

$$^C D_{a^+}^{\beta,\psi}\upsilon(b) = \upsilon_b. \qquad (3.4)$$

Applying $^C D_{a^+}^{\beta,\psi}$ to both sides of (3.3), we obtain

$$^C D_{a^+}^{\beta,\psi}\upsilon(t) = \sum_{k=0}^{n-1} \upsilon_k \cdot {}^C D_{a^+}^{\beta,\psi}\left(\frac{(\psi(t)-\psi(a))^k}{k!}\right) + {}^C D_{a^+}^{\beta,\psi}\mathcal{I}_{a^+}^{\alpha,\psi}\sigma(t).$$

It follows from Lemma 2.2 that

$$^C D_{a^+}^{\beta,\psi}\left(\frac{(\psi(t)-\psi(a))^k}{k!}\right) = \begin{cases} 0, & k = 0, \\ \dfrac{(\psi(t)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)}, & k \geq 1, \end{cases}$$

so at $t = b$, we get

$$^C D_{a^+}^{\beta,\psi}\left(\frac{(\psi(t)-\psi(a))^k}{k!}\right)\Bigg|_{t=b} = \frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)}, \quad k \geq 1.$$

In regard to the integral term, we use the semigroup property:

$$^C D_{a^+}^{\beta,\psi}\mathcal{I}_{a^+}^{\alpha,\psi}\sigma(t) = \mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(t) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1}\sigma(s)\,ds.$$

Hence, at $t = b$, we have

$$^C D_{a^+}^{\beta,\psi}\mathcal{I}_{a^+}^{\alpha,\psi}\sigma(b) = \mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(b) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1}\sigma(s)\,ds.$$

Substituting into the boundary condition (3.4), we find

$$\sum_{k=1}^{n-2} \upsilon_k \frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)} + c_{n-1}\frac{(\psi(b)-\psi(a))^{n-1-\beta}}{\Gamma(n-\beta)} + \mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(b) = \upsilon_b,$$

and accordingly

$$c_{n-1} = \left(\upsilon_b - \sum_{k=1}^{n-2}\upsilon_k \frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)} - \mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(b)\right)\frac{\Gamma(n-\beta)}{(\psi(b)-\psi(a))^{n-1-\beta}}.$$

Finally, substituting the value of c_{n-1} in equation (3.3) we obtain (3.2).

Conversely, if the function υ satisfies equation (3.2), then

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\left(\frac{(\psi(t)-\psi(a))^k}{k!}\right)=0,\quad k=0,1,\ldots,n-1. $$

Thanks to Lemma 2.2, we get

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathcal{I}_{a^+}^{\alpha,\psi}\sigma(t)=\sigma(t), $$

so

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)=\sigma(t). $$

We need ultimately to check the initial and boundary conditions of (3.1). It's easy to see that

$$ \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k\left(\frac{(\psi(t)-\psi(a))^m}{m!}\right)\Big|_{t=a}=\begin{cases}1, & \text{if } k=m,\\ 0, & \text{if } k<m.\end{cases} $$

So, we obtain at once

$$ \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k\left(\upsilon_k\frac{(\psi(t)-\psi(a))^k}{k!}\right)\Big|_{t=a}=\upsilon_k,\quad k=0,1,\ldots,n-2. $$

For $k=n-1$, the term involving c_{n-1} does not affect the initial conditions, so

$$ \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k\left(\frac{(\psi(t)-\psi(a))^{n-1}}{(n-1)!}\right)\Big|_{t=a}=0. $$

For the integral term, thanks to Lemma 2.1, we have

$$ \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k(\mathcal{I}_{a^+}^{\alpha,\psi}\sigma)(t)\big|_{t=a}=\mathcal{I}_{a^+}^{\alpha-k,\psi}\sigma(t)\big|_{t=a}=0. $$

Next, let us prove that ${}^{C}D_{a^+}^{\beta,\psi}\upsilon(b)=\upsilon_b$. By Lemma 2.2, we have

$$ {}^{C}D_{a^+}^{\beta,\psi}\left(\frac{(\psi(t)-\psi(a))^k}{k!}\right)\Big|_{t=b}=\frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)},\quad k=0,1,\ldots,n-2. $$

For $k=n-1,$ we have

$$ {}^{C}D_{a^+}^{\beta,\psi}\left(\frac{(\psi(t)-\psi(a))^{n-1}}{(n-1)!}\right)\Big|_{t=b}=\frac{(\psi(b)-\psi(a))^{n-1-\beta}}{\Gamma(n-\beta)}. $$

On the other hand, one has

$$ {}^{C}D_{a^+}^{\beta,\psi}(\mathcal{I}_{a^+}^{\alpha,\psi}\sigma)(t)\big|_{t=b}=\mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(b). $$

Substituting into the boundary condition (3.4), with the constant c_{n-1}, we find

$$ \begin{aligned} {}^{C}D_{a^+}^{\beta,\psi}\upsilon(b)&=\sum_{k=1}^{n-2}\upsilon_k\frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)}\\ &\quad+\left(\upsilon_b-\sum_{k=1}^{n-2}\upsilon_k\frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)}-\mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(b)\right)\\ &\quad\times\frac{\Gamma(n-\beta)}{\Gamma(n-\beta)}\frac{(\psi(b)-\psi(a))^{n-1-\beta}}{(\psi(b)-\psi(a))^{n-1-\beta}}+\mathcal{I}_{a^+}^{\alpha-\beta,\psi}\sigma(b)\\ &=\upsilon_b. \end{aligned} $$

Therefore, υ satisfies (3.1). □

Next, we establish the existence and uniqueness of solution to problem (1.1) by applying Banach fixed point theorem. We have

Theorem 3.1 *Assume $F:\mathcal{J}\times X\times X\to X$ is continuous and there exist $\ell_1,\ell_2>0$ such that*

$$\|F(t,x_1,x_2)-F(t,\upsilon_1,\upsilon_2)\|\le \ell_1\|x_1-\upsilon_1\|+\ell_2\|x_2-\upsilon_2\|, \quad x_i,\upsilon_i\in X,\ i=1,2. \tag{3.5}$$

If

$$\ell_1+\ell_2<\frac{1}{2}, \tag{3.6}$$

and

$$\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha<1, \tag{3.7}$$

where

$$\mathbf{A}_\alpha=\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(n-\beta)}{(n-1)!\Gamma(\alpha-\beta+1)},$$

then problem (1.1) has a unique solution on $\mathcal{J}$.

Proof: According to Lemma 3.1, a function $\upsilon\in\mathcal{C}^{n-1}(\mathcal{J},X)$ is a solution to the problem (1.1) if it satisfies

$$\begin{aligned}\upsilon(t)&=\sum_{k=0}^{n-2}\frac{\upsilon_k}{k!}(\psi(t)-\psi(a))^k\\&+\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}F(s,\upsilon(s),{}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(s))\,ds\\&+\left(\upsilon_b-\sum_{k=1}^{n-2}\upsilon_k\frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)}-\mathcal{I}_{a^+}^{\alpha-\beta,\psi}F(b,\upsilon(b),{}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(b))\right)\\&\times\frac{\Gamma(n-\beta)(\psi(t)-\psi(a))^{n-1}}{(n-1)!(\psi(b)-\psi(a))^{n-1-\beta}}.\end{aligned}$$

Let E be the space of all functions $\upsilon\in\mathcal{C}^{n-1}(\mathcal{J},X)$ such that ${}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon\ \in\mathcal{C}(\mathcal{J},X)$, endowed with the norm:

$$\|\upsilon\|_E=\sup_{t\in\mathcal{J}}\|\upsilon(t)\|+\sup_{t\in\mathcal{J}}\left\|{}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)\right\|.$$

It is not hard to conclude that $(E,\ \|.\|_E)$ is a Banach space.

Set

$$\begin{aligned}\mathcal{F}_\upsilon(t)&=F(t,\upsilon(t),{}^C\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)),\\\mathcal{Y}_{b,\psi}&=\upsilon_b-\sum_{k=1}^{n-2}\upsilon_k\frac{(\psi(b)-\psi(a))^{k-\beta}}{\Gamma(k+1-\beta)},\\\mathbf{C}_\beta&=\frac{\Gamma(n-\beta)}{(n-1)!},\end{aligned}$$

and define the operator $\mathfrak{T}:E\to E$ by

$$\begin{aligned}\mathfrak{T}\upsilon(t)&=\sum_{k=0}^{n-2}\frac{\upsilon_k}{k!}(\psi(t)-\psi(a))^k+\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathcal{F}_\upsilon(s)\,ds\\&-\frac{\mathbf{C}_\beta}{\Gamma(\alpha-\beta)}\frac{(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}}\int_a^b\psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1}\mathcal{F}_\upsilon(s)\\&+\mathcal{Y}_{b,\psi}\mathbf{C}_\beta\frac{(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}}.\end{aligned} \tag{3.8}$$

Let us show that $\mathfrak{T}$ maps E into itself. Indeed, consider first $\upsilon \in \mathcal{C}^{n-1}(\mathcal{J}, X)$. It is not hard to check that $\mathfrak{T}\upsilon \in \mathcal{C}^{n-1}(\mathcal{J}, X)$. On the other hand, thanks to Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
{}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t) &= \sum_{k=0}^{n-2} \upsilon_k \; {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\left(\frac{(\psi(t)-\psi(a))^k}{k!}\right) + {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathcal{I}_{a^+}^{\alpha,\psi}\mathcal{F}_\upsilon(t) \\
&\quad - \mathbf{C}_\beta \frac{{}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}((\psi(t)-\psi(a))^{n-1})}{(\psi(b)-\psi(a))^{n-1-\beta}} \mathcal{I}_{a^+}^{\alpha-\beta,\psi}\mathcal{F}_\upsilon(b) \\
&\quad + \frac{\mathcal{Y}_{b,\psi}\mathbf{C}_\beta}{(\psi(b)-\psi(a))^{n-1-\beta}} \; {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}((\psi(t)-\psi(a))^{n-1}) \\
&= \mathcal{F}_\upsilon(t).
\end{aligned} \tag{3.9}$$

Since F is continuous on $\mathcal{J}$, then ${}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon \in \mathcal{C}(\mathcal{J}, X)$.

Next, we prove that $\mathfrak{T}$ is a contraction on E. Indeed, let $\upsilon_1, \upsilon_2 \in E$, then

$$\begin{aligned}
\|\mathfrak{T}\upsilon_1(t) - \mathfrak{T}\upsilon_2(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|\mathcal{F}_{\upsilon_1}(s) - \mathcal{F}_{\upsilon_2}(s)\| \, ds \\
&\quad + \frac{\mathbf{C}_\beta}{\Gamma(\alpha-\beta)} \frac{(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}} \int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1} \\
&\quad \times \|\mathcal{F}_{\upsilon_1}(s) - \mathcal{F}_{\upsilon_2}(s)\| \, ds.
\end{aligned}$$

Thanks to assumption (3.5), we obtain

$$\begin{aligned}
\|\mathfrak{T}\upsilon_1(t) - \mathfrak{T}\upsilon_2(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \Big(\ell_1 \|\upsilon_1(s) - \upsilon_2(s)\| \\
&\qquad + \ell_2 \left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_1(s) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_2(s) \right\| \Big) \, ds \\
&\quad + \frac{\mathbf{C}_\beta}{\Gamma(\alpha-\beta)} \frac{(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}} \int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1} \\
&\quad \times \left(\ell_1 \|\upsilon_1(s) - \upsilon_2(s)\| + \ell_2 \left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_1(s) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_2(s) \right\| \right) ds \\
&\le \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \Big(\ell_1 \sup_{t\in\mathcal{J}} \|\upsilon_1(t) - \upsilon_2(t)\| \\
&\qquad + \ell_2 \sup_{t\in\mathcal{J}} \left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_1(t) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_2(t) \right\| \Big) \\
&\quad + \frac{\mathbf{C}_\beta(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha-\beta+1)} \Big(\ell_1 \sup_{t\in\mathcal{J}} \|\upsilon_1(t) - \upsilon_2(t)\| \\
&\qquad + \ell_2 \sup_{t\in\mathcal{J}} \left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_1(t) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_2(t) \right\| \Big)
\end{aligned}$$

$$\begin{aligned}
&\le \mathbf{A}_\alpha(\ell_1+\ell_2)(\psi(b)-\psi(a))^\alpha \\
&\times \left(\sup_{t\in\mathcal{J}} \|\upsilon_1(t) - \upsilon_2(t)\| + \sup_{t\in\mathcal{J}} \left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_1(t) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon_2(t) \right\| \right)
\end{aligned}$$

Hence,

$$\|\mathfrak{T}\upsilon_1(t) - \mathfrak{T}\upsilon_2(t)\| \le \mathbf{A}_\alpha(\ell_1+\ell_2)(\psi(b)-\psi(a))^\alpha \|\upsilon_1 - \upsilon_2\|_E .$$

It follows from assumptions (3.6) and (3.7) that

$$\sup_{t\in\mathcal{J}} \|\mathfrak{T}\upsilon_1 - \mathfrak{T}\upsilon_2\| < \frac{1}{2} \|\mathfrak{T}\upsilon_1 - \mathfrak{T}\upsilon_2\|_E . \tag{3.10}$$

Furthermore, from assumption (3.5) and equality (3.9), we have

$$\begin{aligned}
&\left\| {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\mathfrak{T}\upsilon_1(t) - {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\mathfrak{T}\upsilon_2(t)\right\| \\
&\leq \ell_1 \left\|\upsilon_1(t)-\upsilon_2(t)\right\| + \ell_2 \left\| {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\upsilon_1(t) - {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\upsilon_2(t)\right\| \\
&\leq (\ell_1+\ell_2)\left(\sup_{t\in\mathcal{J}}\left\|\upsilon_1(t)-\upsilon_2(t)\right\| + \sup_{t\in\mathcal{J}}\left\| {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\upsilon_1(t) - {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\upsilon_2(t)\right\|\right).
\end{aligned}$$

By assumption (3.6), we obtain

$$\sup_{t\in\mathcal{J}}\left\| {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\mathfrak{T}\upsilon_1(t) - {}^C\mathcal{D}^{\alpha,\psi}_{a^+}\mathfrak{T}\upsilon_2(t)\right\| < \frac{1}{2}\left\|\upsilon_1-\upsilon_2\right\|_E. \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$\left\|\mathfrak{T}\upsilon_1-\mathfrak{T}\upsilon_2\right\|_E < \left\|\upsilon_1-\upsilon_2\right\|_E,$$

which shows that $\mathfrak{T}$ is a contraction mapping. We conclude by Banach fixed point theorem, that $\mathfrak{T}$ has a unique fixed point $\upsilon\in E$ which is the unique solution to problem (1.1). □

The next result is based on Scheafer's fixed point theorem. We have

Theorem 3.2 *Assume $F:\mathcal{J}\times X\times X\to X$ is continuous and there exist three positive constants η, γ and δ such that*

$$\left\|F(t,x_1,x_2)\right\| \leq \eta+\gamma\left\|x_1\right\|+\delta\left\|x_2\right\|, \quad t\in\mathcal{J},\ x_1,x_2\in X, \tag{3.12}$$

and

$$\boldsymbol{M}(1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha)<1 \tag{3.13}$$

where $\boldsymbol{M}=\max(\eta,\gamma+\delta)$, $\mathbf{A}_\alpha$ and $\mathbf{C}_\beta$ are defined above.

Then problem (1.1) has at least one solution in the space E.

Proof: Let $\mathfrak{T}:E\to E$ be the operator defined by (3.8). We proceed in several steps.

Step 1: Let us show that $\mathfrak{T}$ is continuous. Let $(\upsilon_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\upsilon_n\to\upsilon$ in E, as $n\to\infty$. Then, for every $n=0,1,...,$ and $t\in\mathcal{J}$, we have

$$\begin{aligned}
\left\|\mathfrak{T}\upsilon_n(t)-\mathfrak{T}\upsilon(t)\right\| &\leq \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left\|\mathcal{F}_{\upsilon_n}(s)-\mathcal{F}_\upsilon(s)\right\| ds \\
&+\frac{\mathbf{C}_\beta}{\Gamma(\alpha-\beta)}\frac{(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1} \\
&\times\left\|\mathcal{F}_{\upsilon_n}(s)-\mathcal{F}_\upsilon(s)\right\| ds \\
&\leq \left\|\mathcal{F}_{\upsilon_n}(.)-\mathcal{F}_\upsilon(.)\right\|\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} ds \\
&+\left\|\mathcal{F}_{\upsilon_n}(.)-\mathcal{F}_\upsilon(.)\right\|\frac{\mathbf{C}_\beta(\psi(t)-\psi(a))^{n-1}}{(\psi(b)-\psi(a))^{n-1-\beta}}. \\
&\times\frac{\mathbf{1}}{\Gamma(\alpha-\beta)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1} ds
\end{aligned}$$

Thus

$$\sup_{t\in\mathcal{J}}\left\|\mathfrak{T}\upsilon_n-\mathfrak{T}\upsilon\right\| \leq \mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha\left\|\mathcal{F}_{\upsilon_n}(.)-\mathcal{F}_\upsilon(.)\right\|,$$

In addition, using (3.9), we get

$$\sup_{t\in\mathcal{J}}\left\|{}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon_n(t) - {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t)\right\| \leq \left\|\mathcal{F}_{\upsilon_n}(.) - \mathcal{F}_{\upsilon}(.)\right\|.$$

Since F is continuous, then the right-hand sides of the above inequalities tend to zero, as $n \to \infty$, and so,

$$\|\mathfrak{T}\upsilon_n - \mathfrak{T}\upsilon\|_E \to 0, \text{ as } n \to \infty,$$

which means that $\mathfrak{T}$ is continuous.

Step 2: Let us show that $\mathfrak{T}$ maps bounded subsets of E into uniformly bounded subsets of E. Indeed, Let $\rho > 0$, and set

$$\mathcal{B}_\rho = \{\upsilon \in E,\ \ \|\upsilon\|_E \leq \rho\}.$$

We need to prove that, for all $\upsilon \in \mathcal{B}_\rho$, there exists $\rho' > 0$, such that $\|\mathfrak{T}\upsilon\|_E \leq \rho'$. Indeed, for every $t \in \mathcal{J}$ and $\upsilon \in \mathcal{B}_\rho$, we have

$$\begin{aligned}\|\mathfrak{T}\upsilon(t)\| \leq{}& \sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(t)-\psi(a))^k + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,\|\mathcal{F}_\upsilon(s)\|\,ds\\ &+\frac{\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta}{\Gamma(\alpha-\beta)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1}\,\|\mathcal{F}_\upsilon(s)\|\,ds\\ &+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta.\end{aligned}$$

Thanks to assumption (3.12), we get

$$\begin{aligned}\|\mathfrak{T}\upsilon(t)\| \leq{}& \sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k\\ &+\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\left(\eta+\gamma\,\|\upsilon(t)\|+\delta\left\|{}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)\right\|\right)\\ &+\frac{\mathbf{C}_\beta(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha-\beta+1)}\left(\eta+\gamma\,\|\upsilon(t)\|+\delta\left\|{}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)\right\|\right)\\ &+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta\\ \leq{}& \sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\left(\eta+(\gamma+\delta)\,\|\upsilon\|_E\right)\\ &+\frac{\mathbf{C}_\beta(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha-\beta+1)}\left(\eta+(\gamma+\delta)\,\|\upsilon\|_E\right)+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta.\\ \leq{}& \sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\boldsymbol{M}(\|\upsilon\|_E+1)\\ &+\frac{\mathbf{C}_\beta(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha-\beta+1)}\boldsymbol{M}(\|\upsilon\|_E+1)+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{t\in\mathcal{J}}\|\mathfrak{T}\upsilon(t)\| \leq{}& \sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k + \|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta\\ &+\boldsymbol{M}(\rho+1)\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha.\end{aligned} \tag{3.15}$$

Moreover, from (3.9) and assumption (3.12), we have

$$\begin{aligned}\left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t)\right\| &= \left\| F(t,\upsilon(t),{}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t))\right\| \\ &\leq \eta+\gamma\left\|\upsilon(t)\right\|+\delta\left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)\right\| \\ &\leq \eta+(\gamma+\delta)\left(\sup_{t\in\mathcal{J}}\left\|\upsilon(t)\right\|+\sup_{t\in\mathcal{J}}\left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\upsilon(t)\right\|\right) \\ &\leq \boldsymbol{M}(\left\|\upsilon\right\|_E+1).\end{aligned}$$

Hence

$$\sup_{t\in\mathcal{J}}\left\| {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t)\right\| \leq \boldsymbol{M}(\rho+1). \tag{3.16}$$

From (3.15) and (3.16), we have

$$\begin{aligned}\left\|\mathfrak{T}\upsilon\right\|_E \leq{}& \sum_{k=0}^{n-2}\frac{\left\|\upsilon_k\right\|}{k!}(\psi(b)-\psi(a))^k+\left\|\mathcal{Y}_{b,\psi}\right\|\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta \\ &+\boldsymbol{M}(\rho+1)\left[1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha\right].\end{aligned}$$

The right-hand side of this inequality is independent of t and υ. Hence, there exists

$$\begin{aligned}\rho' ={}& \sum_{k=0}^{n-2}\frac{\left\|\upsilon_k\right\|}{k!}(\psi(b)-\psi(a))^k+\left\|\mathcal{Y}_{b,\psi}\right\|\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta \\ &+\boldsymbol{M}(\rho+1)\left[1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha\right],\end{aligned}$$

satisfying

$$\left\|\mathfrak{T}\upsilon\right\| \leq \rho',$$

which means that $\mathfrak{T}(\mathcal{B}_\rho)$ is uniformly bounded.

Step 3: Let us show that $\mathfrak{T}$ maps bounded subsets of E into equicontinuous subsets in E. Define the function $\varpi:(0,\infty)\backslash\{1\}\to\mathbb{R}$ by

$$\varpi(\alpha)=\begin{cases}1, & \text{if } \alpha>1,\\ -1, & \text{if } 0<\alpha<1.\end{cases}$$

For every $t_1,\ t_2\in\mathcal{J},\ t_1<t_2$ and $\upsilon\in\mathcal{B}_\rho$, we have

$$\begin{aligned}&\left\|\mathfrak{T}\upsilon(t_2)-\mathfrak{T}\upsilon(t_1)\right\| \leq \sum_{k=0}^{n-2}\frac{\left\|\upsilon_k\right\|}{k!}(\psi(t_2)-\psi(a))^k-(\psi(t_1)-\psi(a))^k \\ &+\frac{1}{\Gamma(\alpha)}\left\|\int_a^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathcal{F}_\upsilon(s)ds\right. \\ &\left.-\int_a^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathcal{F}_\upsilon(s)ds\right\| \\ &+\frac{\left\|\mathcal{Y}_{b,\psi}\right\|\mathbf{C}_\beta}{(\psi(b)-\psi(a))^{n-1-\beta}}\left[\psi(t_2)-\psi(a))^{n-1}-(\psi(t_1)-\psi(a))^{n-1}\right] \\ &+\frac{\mathbf{C}_\beta}{(\psi(b)-\psi(a))^{n-1-\beta}}\left[(\psi(t_2)-\psi(a))^{n-1}-(\psi(t_1)-\psi(a))^{n-1}\right] \\ &\times\frac{1}{\Gamma(\alpha-\beta)}\int_a^b\psi'(s)(\psi(b)-\psi(s))^{\alpha-\beta-1}\left\|\mathcal{F}_\upsilon(s)\right\|ds.\end{aligned}$$

Using assumption (3.12), we obtain

$$
\begin{aligned}
&\|\mathfrak{T}\upsilon(t_2)-\mathfrak{T}\upsilon(t_1)\| \\
&\le \sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}\left[(\psi(t_2)-\psi(a))^k-(\psi(t_1)-\psi(a))^k\right] \\
&+\varpi(\alpha)\frac{\boldsymbol{M}(\rho+1)}{\Gamma(\alpha+1)}\left[(\psi(t_2)-\psi(a))^\alpha-(\psi(t_2)-\psi(t_1))^\alpha-(\psi(t_1)-\psi(a))^\alpha\right] \\
&+\frac{\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta}{(\psi(b)-\psi(a))^{n-1-\beta}}(\psi(t_2)-\psi(a))^{n-1}-(\psi(t_1)-\psi(a))^{n-1} \\
&+\frac{\mathbf{C}_\beta\boldsymbol{M}(\rho+1)(\psi(b)-\psi(a))^{\alpha-n+1}}{\Gamma(\alpha-\beta+1)}\left[(\psi(t_2)-\psi(a))^{n-1}-(\psi(t_1)-\psi(a))^{n-1}\right].
\end{aligned}
$$

Also, for every $t_2\in\mathcal{J},\ t_1<t_2$ we have

$$
\left\|{}^C\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t_1)-{}^C\,\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t_2)\right\|=\|\mathcal{F}_\upsilon(t_2)-\mathcal{F}_\upsilon(t_1)\|.
$$

In consequence, we obtain

$$
\|\mathfrak{T}\upsilon(t_2)-\mathfrak{T}\upsilon(t_1)\|+\left\|{}^C\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t_1)-{}^C\,\mathcal{D}_{a^+}^{\alpha,\psi}\mathfrak{T}\upsilon(t_2)\right\|\to 0,
$$

as $t_1\to t_2$ independent of υ. Therefore, $\mathfrak{T}(\mathcal{B}_\rho)$ is equicontinuous in E. As a consequence of steps 1 to 3 with Arzelà-Ascoli theorem, we conclude that $\mathfrak{T}$ is a completely continuous operator.

Step 4: Let us show that the set

$$
\Theta=\{\upsilon\in E:\upsilon=\mu\mathfrak{T}\upsilon,\quad \mu\in(0,1)\}.
$$

is bounded. Indeed, let $\upsilon\in\Theta$ and $\mu\in(0,1)$, be such that $\upsilon=\mu T\upsilon$. From step 2, for all $t\in\mathcal{J}$ we have

$$
\begin{aligned}
\|\mathfrak{T}\upsilon\|_E\le{}&\sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta \\
&+\boldsymbol{M}(\|\boldsymbol{\upsilon}\|_{\boldsymbol{E}}+\mathbf{1})(1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha).
\end{aligned}
$$

Since $\mu\in(0,1)$ then $\upsilon<\mathfrak{T}\upsilon$. Hence,

$$
\begin{aligned}
\|\upsilon\|_E&<\|\mathfrak{T}\upsilon\|_E \\
&\le\sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta \\
&+\boldsymbol{M}(\mathbf{1}+\|\boldsymbol{\upsilon}\|_{\boldsymbol{E}})(1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha)
\end{aligned}
$$

from which we get

$$
\|\upsilon\|_E\le\frac{\sum_{k=0}^{n-2}\frac{\|\upsilon_k\|}{k!}(\psi(b)-\psi(a))^k+\|\mathcal{Y}_{b,\psi}\|\,\mathbf{C}_\beta(\psi(b)-\psi(a))^\beta+\boldsymbol{M}(1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha)}{1-\boldsymbol{M}(1+\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha)},
$$

which means that Θ is bounded. By virtue of Scheafer's fixed point theorem, $\mathfrak{T}$ has at least one fixed point $\upsilon\in E$ which is, the solution of the boundary value problem (1.1). □

4. Illustrative examples

In this section, we give two concrete examples to illustrate our results.

Example 4.1 *Consider the following fractional boundary problem*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{5}{2},\ln(t+1)}\upsilon(t) = \dfrac{\frac{1}{10}\sin(\upsilon(t)) + \frac{1}{5}{}^{C}D_{0^+}^{\frac{5}{2},\ln(t+1)}\upsilon(t)}{1+t^2}, \quad t\in[0,2] \\ \upsilon_{\ln(t+1)}^{[0]}(0) = \frac{1}{5},\ \upsilon_{\ln(t+1)}^{[1]}(0) = \frac{1}{10}, \\ {}^{C}D_{0^+}^{\frac{2}{5},\ln(t+1)}\upsilon(2) = \frac{3}{10}, \end{cases} \tag{4.1}$$

In this example, we have

$$2<\alpha=\frac{5}{2}<3\ (n=3),\quad \beta=\frac{2}{5}<\alpha-(n-1)=\frac{1}{2},\ (X,\|.\|)=(\mathbb{R},|.|),\ \psi(t)=\ln(t+1),\ t\in[0,2].$$

The function F is given by

$$F(t,x,\upsilon)=\frac{\frac{1}{10}\sin(x)+\frac{1}{5}\upsilon}{1+t^2},\quad (t,x,\upsilon)\ \in[0,2]\times\mathbb{R}\times\mathbb{R}.$$

It is obvious that F is continuous and, for all $x_1,x_2,\upsilon_1,\upsilon_2\in\mathbb{R}$ and $t\in[0,1]$, we have

$$\begin{aligned} |F(t,x_1,\upsilon_1)-F(t,x_2,\upsilon_2)| &= \left|\frac{\frac{1}{10}\sin(x_1)+\frac{1}{5}\upsilon_1-\frac{1}{10}\sin(x_2)-\frac{1}{5}\upsilon_2}{1+t^2}\right| \\ &\le \frac{1}{10}\left|\sin(x_1)-\sin(x_2)\right|+\frac{1}{5}\left|\upsilon_1-\upsilon_2\right| \\ &\le \frac{1}{10}\left|x_1-x_2\right|+\frac{1}{5}\left|\upsilon_1-\upsilon_2\right|. \end{aligned}$$

Hence, conditions (3.5) and (3.6) are satisfied with $\ell_1=\frac{1}{10}$, $\ell_2=\frac{1}{5}$, and

$$\ell_1+\ \ell_2=\frac{1}{10}+\frac{1}{5}=\frac{3}{10}<\frac{1}{2}.$$

Easy computations show that

$$\mathbf{A}_\alpha(\psi(b)-\psi(a))^\alpha=\left(\frac{1}{\Gamma(2.5+1)}+\frac{\Gamma(2.6)}{2!\Gamma(2.5-0.4+1)}\right)(\ln(3))^{\frac{5}{2}}\approx 0.792\,14<1.$$

So, condition (3.7) is also satisfied. Hence, all of the assumptions of Theorem 3.1 are satisfied, we conclude that the boundary value problem (4.1) has at least one solution on the interval $[0,2]$.

Our second example is the following:

Example 4.2 *Consider the following fractional boundary problem*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{7}{2},t^2+1}\upsilon(t) = \dfrac{\frac{1}{5}+\frac{3}{20}\upsilon(t)+\frac{1}{10}{}^{C}D_{0^+}^{\frac{7}{2},t^2+1}\upsilon(t)}{e^t+\left|{}^{C}D_{0^+}^{\frac{7}{2},t^2+1}\upsilon(t)\right|}, \quad t\in[0,1] \\ \upsilon_{t^2+1}^{[0]}(0)=\frac{1}{2},\ \upsilon_{t^2+1}^{[1]}(0)=\frac{3}{10},\ \upsilon_{t^2+1}^{[2]}(0)=\frac{1}{5} \\ {}^{C}D_{0^+}^{\frac{3}{10},t^2+1}\upsilon(1)=\frac{7}{10}, \end{cases} \tag{4.2}$$

In this example, we have

$$1<\alpha=\frac{7}{2}<2\ (n=4),\quad \beta=\frac{3}{10}<\alpha-(n-1)=\frac{1}{2},(X,\|.\|)=(R,|.|),\ \psi(t)=t^2+1,\ t\in[0,1].$$

The function F is given by

$$F(t,x,\upsilon)=\frac{\frac{1}{5}+\frac{3}{20}x+\frac{1}{10}\upsilon}{e^{t}+|\upsilon|},\quad (t,x,\upsilon)\ \in[0,1]\times\mathbb{R}\times\mathbb{R}.$$

As we see, the function F is continuous and, for all $x,\upsilon\in\mathbb{R}$ and $t\in[0,1]$, we have

$$|F(t,x,\upsilon)|=\left|\frac{\frac{1}{5}+\frac{3}{20}x+\frac{1}{10}\upsilon}{e^{t}+|\upsilon|}\right|\leq\left|\frac{1}{5}+\frac{3}{20}x+\frac{1}{10}\upsilon\right|\leq\frac{1}{5}+\frac{3}{20}|x|+\frac{1}{10}|\upsilon|.$$

Hence, condition (3.12) is satisfied with $\eta=\frac{1}{5}$, $\gamma=\frac{3}{20}$ and $\delta=\frac{1}{10}$. On the other hand, for these values of η,γ and δ, we have

$$\boldsymbol{M}=\max(\frac{1}{5},\frac{3}{20}+\frac{1}{10})=\frac{1}{4}$$

and

$$\boldsymbol{M}(1+\mathbf{A}_{\alpha}(\psi(b)-\psi(a))^{\alpha})=\frac{1}{4}\left(1+\left(\frac{1}{\Gamma(\frac{7}{2}+1)}+\frac{\Gamma(3.7)}{(3)!\Gamma(4.2)}\right)\times(2-1)^{\frac{7}{2}}\right)\approx 0.293\,90<1.$$

Hence, condition (3.13) holds too. Actually, the hypotheses of theorem 3.2 are fulfilled, we therefore conclude that problem (4.2) has at least one solution on the interval $[0,1]$.

References

1. M. S. Abdo, S. K. Panchal and A. M. Saeed, Fractional boundary value problem with $\psi-$Caputo fractional derivative, Proc. Indian Acad. Sci. (Math. Sci.) 129 (2019), 65.

2. R. Almeida, A Caputo fractional derivative of a function with respect to another function. Commun. Nonlinear Sci. Numer. Simul. 44, $460-481$ (2017).

3. R. Almeida, Fractional differential equations with mixed boundary conditions. Bull. Malays. Math. Sci. Soc. 2019, 42, $1687-1697$.

4. R. Almeida, A. B. Malinowska, T. Odzijewicz, Optimal Leader-Follower control for the fractional opinion Formation Model. J. Optim. Theory Appl. 2019, 182, $1171-1185$.

5. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), $73-85$ (2015).

6. k. Deimling, Nonlinear functional analysis, Springer, Berlin 1985.

7. C. Derbazi , Z. Baitiche , M. Benchohra , A. Cabada, Initial value problem for nonlinear fractional differential equations with ψ-Caputo derivative via monotone iterative technique. Axioms. 2020, 9 (2) $1-13$.

8. R. Hilfer, Applications of fractional calculus in Physics. World Scientific Publishing Co, Singapore, 2003.

9. Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives. Adv. Differ. Equ. 2014, 10 (2014).

10. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transforms. Discrete Contin. Dyn. Syst. Ser. S 13 (3), $709-722$ (2020).

11. U. Katugampola, A new approach to a generalized fractional integral. Appl. Math. Comput. 218, $860-865$ (2011).

12. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).

13. R. L. Magin, Fractional calculus in Bioengineering. Begell House Inc. Publisher, 2006.

14. B. Samet, H. Aydi, Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative. J. Inequal. Appl. 2018, 2018, 286.

15. S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, translated from the 1987 Russian original. Gordon and Breach, Yverdon, 1993.

16. S. Solhi, E. Anaama, A. Kajouni, K. Hilal, Existence and uniqueness of solutions for nonlinear fractional boundary value problems with $\psi-$Caputo derivatives, Bol. Soc. Paran. Mat. 2025 (43) $1:1-12$.

17. J. V. C. Sousa, E. C. Oliveira, On the ψ-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. 60, $72-91$ (2018).

Hamza Medjekal,
Department CPST,
National Higher School of Technology and Engineering, Annaba 23005,
Algeria.
E-mail address: h.medjekal@ensti-annaba.dz

and

Besma Berrehail,
Department CPST,
National Higher School of Technology and Engineering, Annaba 23005,
Algeria.
E-mail address: b.berrehail@ensti-annaba.dz