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Fixed point theorems for general contractive mappings in cone Banach algebras with applications

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ABSTRACT: This paper establishes some fixed point theorems for contractive mappings in complete cone metric spaces over Banach algebras. These results extend and unify existing literature, offering good approaches to a series of novel fixed point problems. We illustrate our findings with examples and demonstrate their applicability effectively by proving the unique solution of Urysohn integral and Caputo fractional differential equations using the practical power of Banach contraction principle in these specialized contexts.

Key Words: Fixed point technique, Banach algebra, Urysohn integral equation, fractional derivative.

Contents

1. Scientific introduction

Fixed point theory has a well-furnished and fundamental role in topology and analysis. As applicable, it has not only many applications in mathematics, but also applications in many sciences such as mathematical modeling, economics, chemistry, and biology. The fixed point technique can solve many problems of ordinary differential equations, partial differential equations, integral equations, and modern optimization using Banach's contraction principle [1]. This principle is more important and realistic when the operator is in a continuous state than in a non-continuous state. Continuous operators have valuable characteristics that make it possible for them to maintain important structural components of the underlying space. In contrast, it might be difficult to produce fixed point outcomes when using non-continuous operators since they can result in behavior that is more complicated and erratic.

The concepts of metric space and cone metric space were introduced by Zhang and Huang [2], where the set of real numbers was replaced by real Banach space. Cone metric spaces were established as an extension of metric spaces and have generated a lot of interest because of their applications in fixed point theory, topological vector spaces, and a variety of mathematics and engineering. Recently, many papers have presented the results on cone metric space using identical method to results on ordinary metric space [3,4,5,6,7,9,8,10].

Further, the concept of cone metric space over Banach algebra was introduced by Xu and Liu [11], where the Banach contraction principle was discussed in the context of cone metric space over Banach algebra. Many authors proved Banach's contraction principle in the setting of cone metric space over Banach algebras. The authors proved many fixed point theorems of generalized Lipschitz mappings in the new context without the assumption of normality, which are not equivalent to metric space in terms

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of the existence of the fixed point of the mappings. There are many articles about fixed point in cone b-metric-like spaces over Banach algebra and other spaces (see [12,13,14,15,16,17,18,19,20]). Chiefly, cone metric space over Banach algebras provide a powerful foundation for unifying algebraic structures with metric spaces, understanding and facilitating the study of nonlinear analysis problems and providing thorough knowledge into the behavior of contraction mappings in a lot of mathematical and applied contexts.

Integral equations are mathematical equations that contain integral expressions with unknown functions. Due to their capacity to model complicated processes, these equations have been thoroughly studied and utilized in numerous scientific and technical areas. integral equations provide elegant solutions to many problems that would be difficult to resolve with conventional differential equations. There are two basic categories of integral equations: Fredholm and Volterra equations. In Fredholm integral equations, the unknown function only occurs within the integral, while in Volterra integral equations, the unknown function occurs both within and beyond the integral. Numerous disciplines, such as physics, engineering, biology, economics, and others, have found an extensive and significant utilization for these equations. For example, they are employed in the study of many directions, such as fluid dynamics, signal processing, heat conduction, population dynamics, and electrical circuits, among others. The connection between fixed point theorems and integral equations are quite interesting, as fixed point theorems represent a powerful mathematical technique for showing and illustrating the existence and uniqueness of solutions to specific types of integral equations in order to gain more knowledge (see [21,22,23,24,25]).

In 1695, Leibniz proposed the concept of fractional calculus [26] as one of the developments of ordinary calculus. Lately, according to [27,28,29,30], the fractional calculus theory has become pertinent to fluid mechanics, entropy, engineering, and physics. It is feasible to interpret some physical models and engineering systems more precisely and practically by employing fractional calculus. For instance, fractional calculus-based entropies could be more widely applied than Shannon's entropy [31]. Due to its widespread usage, fractional entropy has been the focal point of significant studies [32]. Furthermore, fractional differential equations are incredibly helpful for describing and modeling a wide range of phenomena [33]. This is due to the fact that a system's eventual state depends on all of its past circumstances, not simply its present form. These equations may be more accurate representations of physical reality than integer-order differential equations. It's essential that we emphasize how widely the theory and applications of fractional calculus have been discussed in the scientific literature [35,36,37,34,38]. According to [39], fractional differential equations have generated a lot of attention in recent years due to their precise explanation of complicated events in polymers, signal processing, system identification, non-Brownian motion, viscoelastic materials, and control issues. According to [40,41,42,43,44,45], the focus of recent studies has been on fractional functional analysis, and various applications to fractional ordinary differential systems, fractional partial differential equations, and fractional ordinary differential equations have been investigated.

The primary objective of this article is to discuss fixed point results within the context of cone metric space over Banach algebras. This framework presents significant advantages over classic metric spaces, as the underlying Banach algebra introduces a richer algebraic structure that allows for more flexible and generalized analysis. Our research aims to advance and provide knowledge of fixed point theory in this powerful, influential, and general context. By investigating fixed point results in the context of cone metric space and Banach algebras, we can establish a comprehensive theoretical basis that can be applicable to a wide range of mathematical and practical problems. To support and illustrate our theoretical results, we provide a wide variety of examples and applications that display the effectiveness and applicability of our approaches.

This paper is structured as follows: Section 1 provides a scientific introduction to cone metric spaces over a Banach algebra and discusses some of their applications. In Section 2, we present the key concepts, definitions, and lemmas necessary for our main results. Section 3 introduces fixed point theorems within this framework, supported by examples and corollaries. The final two sections, Sections 4 and 5, explore applications to integral and fractional differential equations, demonstrating the existence and uniqueness of solutions. We conclude with a summary of our findings.

2. Preliminaries work

This part is concerned with outlining fundamental ideas that assist us in achieving our goal.

Definition 2.1 [46] A Banach algebra $\mathcal{B}$ is a Banach space with an operation of multiplication constructed on it, yielding the following properties:
For every $f, g, h \in \mathcal{B}$, and $\gamma \in \mathcal{B}$, we have

$(\mathcal{BA}_1)$ $f(gh) = (fg)h$;
$(\mathcal{BA}_2)$ $f(g+h) = fg + fh$ and $(f+g)h = fh + gh$;
$(\mathcal{BA}_3)$ $\gamma(fg) = (\gamma f)g = f(\gamma g)$;
$(\mathcal{BA}_4)$ $\|fg\| \leq \|f\|\,\|g\|$.

Proposition 2.1 [46] Let $\mathcal{B}$ be a Banach algebra with a unit $I_{\mathcal{B}}$, and $j \in \mathcal{B}$. If the spectral radius $\rho(j)$ of j is less than 1, i.e.,

$$\rho(j) \;=\; \lim_{n\to\infty} \|j^n\|^{\frac{1}{n}} \;=\; \inf_{n\geq 1} \|j^n\|^{\frac{1}{n}} \;<\; 1,$$

then, $I_{\mathcal{B}} - j$ is invertible. In fact,

$$(I_{\mathcal{B}} - j)^{-1} = \sum_{m=0}^{\infty} j^m, \quad \text{(Neumann series)},$$

is exist and convergence.

Definition 2.2 [46] A subset $\mathcal{P}$ of $\mathcal{B}$ is called a cone if:

1. $\mathcal{P}$ is nonempty, closed and $\{\theta, I_{\mathcal{B}}\} \in \mathcal{P}$;
2. $a\alpha + b\beta \subset \mathcal{P}$ for all non-negative real numbers α, β and $a, b \geq 0$;
3. $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;
4. $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$;

where θ denotes the null of the Banach algebra $\mathcal{B}$.

For a given cone $\mathcal{P} \subset \mathcal{B}$, we can define a partial ordering $\preceq$ with respect to $\mathcal{P}$ by $a \preceq b$ if and only if $b - a \in \mathcal{P}$. The relation $a \prec b$ will stand for $a \preceq b$ and $a \neq b$. While $a \ll b$ will stand for $b - a \in \operatorname{int}\mathcal{P}$ where $\operatorname{int}\mathcal{P}$ denotes the interior of $\mathcal{P}$. If $\operatorname{int}\mathcal{P} \neq \varnothing$, then, $\mathcal{P}$ is a solid cone.

The cone $\mathcal{P}$ is called normal if there is a number $W > 0$ such that for all $a, b \in \mathcal{B}$,

$$\theta \preceq a \preceq b \implies \|a\| \leq W\,\|b\|\,.$$

The least positive number satisfying the above is called the normal constant of $\mathcal{P}$.

Proposition 2.2 [47] A complex or real Banach algebra $\mathcal{B}$ with a unit $I_{\mathcal{B}}$ is said to be an ordered Banach algebra if $\mathcal{B}$ is ordered by a relation $\preceq$ such that for all $a, b, c \in \mathcal{B}$ and $\lambda \in \mathbb{C}$, we have

1. $a, b \succeq \theta$ implies $a + b \succeq \theta$,
2. $a \succeq \theta$, $\lambda \in \mathbb{C}$ implies $\lambda a \succeq \theta$,
3. $a, b \succeq \theta$ implies $ab \succeq \theta$,
4. $I_{\mathcal{B}} \succeq \theta$.

Thus, if $\mathcal{B}$ is ordered by an algebra cone $\mathcal{P}$, then the pair $(\mathcal{B}, \mathcal{P})$ is an ordered Banach algebra.

Definition 2.3 [2] Let $\varDelta$ be a non-empty set. For all $j, k, w \in \varDelta$, suppose that the mapping $d : \varDelta \times \varDelta \longrightarrow \mathcal{P}$ fulfills

(i) $\theta \prec d(j,k)$ with $j \neq k$ and $d(j,k) = \theta$ iff $j = k$;
(ii) $d(j,k) = d(k,j)$ for all $j, k \in \varDelta$;
(iii) $d(j,k) \preceq d(j,w) + d(w,k)$ for all $j, k, w \in \varDelta$.

Then, d is called a cone metric on $\varDelta$, and $(\varDelta, d)$ is called a cone metric space over Banach algebra.

Example 2.1 [48] Suppose that $\mathcal{B} = \mathbb{R}^2$, $\mathcal{P} = \{(j,k) \in \mathcal{B} : j,k \geq 0\} \subset \mathbb{R}^2$, $\Delta = \mathbb{R}$ and $d : \Delta \times \Delta \longrightarrow \mathcal{P}$ such that $d(j,k) = \big(|j-k|,\ \alpha\,|j-k|\big)$, where $\alpha \geq 0$ is a constant. Then, $(\Delta,\ d)$ is a cone metric space over Banach algebra.

Example 2.2 Assume that $\mathcal{B} = \mathbb{R}^n$, $\mathcal{P} = \{(j_1, j_2, ..., j_n) :\ j_q \in \mathbb{R},\ 0 \leq j_q,\ q = 1,2,...,n\} \subset \mathbb{R}^n$, $\Delta = \mathbb{R}^n$ and $d : \Delta \times \Delta \longrightarrow \mathcal{P}$ such that $d(j,k) = \big(|j_1 - k_1|,\ \alpha_1\,|j_2 - k_2|,\ \alpha_2\,|j_3 - k_3|,\ ...,\ \alpha_{n-1}\,|j_n - k_n|\big)$, where $j = (j_1, j_2, ..., j_n)$, $k = (k_1, k_2, ..., k_n)$ and $\alpha_i \geq 0$ $(i = 1, 2, ..., n-1)$ is a constant. Then, $(\Delta,\ d)$ is a cone metric space over Banach algebra.

Example 2.3 Let $\mathcal{B} = \mathbb{R}^n$, $\mathcal{P} = \{(j_1, j_2, ..., j_n) :\ j_q \in \mathbb{R},\ 0 \leq j_q,\ q = 1,2,...,n\} \subset \mathbb{R}^n$, $\Delta = \mathbb{R}^n$ and $d : \Delta \times \Delta \longrightarrow \mathcal{P}$ such that

$$d(j,k) = \Big(\sum_{q=1}^{n} |j_q - k_q|,\ \alpha_1 \sum_{q=1}^{n} |j_q - k_q|,\ \alpha_2 \sum_{q=1}^{n} |j_q - k_q|,\ ...,\ \alpha_{n-1} \sum_{q=1}^{n} |j_q - k_q|\Big),$$

where $j = (j_1, j_2, ..., j_n)$, $k = (k_1, k_2, ..., k_n)$ and $\alpha_i \geq 0$ $(i = 0, 1, 2, ..., n-1)$, is a constant. Then, (Δ, d) is a cone metric space over Banach algebra.

Definition 2.4 [2] Let $(\Delta, \mathcal{B}, d)$ be a cone metric space over a Banach algebra $\mathcal{B}$, $j_n \in \Delta$ and let $\{j_n\}$ be a sequence in Δ. Then

1. $\{j_n\}$ converges to j if for every $c \in \mathcal{B}$ with $\theta \ll c$ there is a natural number N such that $d(j_n, j) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n \longrightarrow \infty} j_n = j$ or $j_n \longrightarrow j$.
2. $\{j_n\}$ is a Cauchy sequence if for every $c \in \mathcal{B}$ with $\theta \ll c$ there is a natural number N such that $d(j_m, j_n) \ll c$ for all $m, n \geq N$.
3. $(\Delta,\ d)$ is a complete cone metric space if every Cauchy sequence is convergent.

In the follow-up, we will pay our attention to the following lemmas:

Lemma 2.1 [2] Let $(\Delta,\ d)$ be a cone metric space, $\mathcal{P}$ be a normal cone with a normal constant W. Let $\{j_n\}$ be a sequence in Δ. Then $\{j_n\}$ converges to j if and only if $d(j_n, j) \to 0$, as $(n \to \infty)$.

Lemma 2.2 [2] Let $(\Delta,\ d)$ be a cone metric space, $\mathcal{P}$ be a normal cone with a normal constant W. Let $\{j_n\}$ be a sequence in Δ. Then $\{j_n\}$ is a Cauchy sequence if and only if $d(j_n, j_m) \to 0$, as $(n, m \to \infty)$.

Lemma 2.3 Let $(\Delta,\ d)$ be a cone metric space, $\mathcal{P}$ be a normal cone with a normal constant W. Let $\{j_n\}$ and $\{j_m\}$ be two sequences in Δ. Then, $d(j_n, j_m) \longrightarrow \theta$ if and only if $\|d(j_n, j_m)\| \longrightarrow 0$.

Proof: Let the sequences j_n, j_m converges to j.

$(\Longrightarrow)$ Assume that $d(j_n, j_m) \longrightarrow \theta$. Since $\|.\|$ is continuous and positive definite, then

$$\|d(j_n, j_m)\| \longrightarrow \|d(j,j)\| \longrightarrow \|\theta\| = 0.$$

$(\Longleftarrow)$ Conversely, Assume that $\|d(j_n, j_m)\| \longrightarrow 0$. Since $\|\theta\| = 0$ in cone Banach algebra, then

$$\|d(j_n, j_m)\| \longrightarrow \|\theta\|.$$

Hence, $d(j_n, j_m) \longrightarrow \theta$. This completes the proof. □

Lemma 2.4 [49] Let j, k be vectors in $\mathcal{B}$. If j and k commute, then the spectral radius ρ verifies the following assertions:
(i) $\rho(jk) \leq \rho(j)\ \rho(k)$;
(ii) $\rho(j + k) \leq \rho(j) + \rho(k)$;
(iii) $|\rho(j) - \rho(k)| \leq \rho(j - k)$.

Lemma 2.5 [49] Let $k \in \mathcal{B}$. If $0 \leq \rho(k) < 1$, then $I_{\mathcal{B}} - k$ is invertible and $\rho\big((I_{\mathcal{B}} - k)^{-1}\big) \leq \big(1 - \rho(k)\big)^{-1}$.

Remark 2.1 [46] The spectral radius $\rho(\mu)$ of μ verifies $\rho(\mu) \leq \|\mu\|$ for all $\mu \in \mathcal{B}$, where $\mathcal{B}$ is a Banach algebra with a unit $I_{\mathcal{B}}$.

Remark 2.2 [49] If $\rho(\mu) < 1$ then $\|\mu^n\| \to 0$ at $n \to \infty$.
Let us display some major definitions, constructions and basic concepts of fractional calculus (see [50,51]). If $h : [0, \infty) \longrightarrow \mathcal{B}$ is a continuous function, then the fractional derivative of Caputo of order α is defined by

$$^{C}D^{\alpha}(j(t)) = \int_0^t \frac{j^{(n)}(s)}{\Gamma(n-\alpha)\,(t-s)^{-n+\alpha+1}}\, ds,$$

for all $\big(n-1 < \alpha < n,\ n = [\alpha]+1\big)$, where Γ is the Euler gamma function and $[\alpha]$ stands for the integer part of the positive real number α. Also, $^{R-L}J^{\alpha}$ represents the Riemann-Liouville fractional integral of order α, and defined as follows:

$$^{R-L}J^{\alpha}\big(j(t)\big) = \int_0^t \frac{j(s)}{\Gamma(\alpha)\,(t-s)^{-\alpha+1}}\, ds.$$

Theorem 2.1 [49] Let (Δ, d) be a complete cone metric space over Banach algebra $\mathcal{B}$ and let $\mathcal{P}$ be the underlying solid cone with $\mu \in \mathcal{P}$ where $\rho(\mu) < 1$. Suppose that the mapping $\mathcal{T} : \Delta \longrightarrow \Delta$ satisfies the following condition:

$$d(\mathcal{T}j, \mathcal{T}k) \preceq \mu\, d(j,k) \qquad \text{for all} \quad \mathrm{j,k} \in \Delta.$$

Then, $\mathcal{T}$ has a unique fixed point in Δ. Moreover, for any $j_0 \in \Delta$, the sequence $\{\mathcal{T}^n j_0\}$ converges to the fixed point.

Theorem 2.2 [49] Let (Δ, d) be a complete cone metric space over a Banach algebra $\mathcal{B}$ and let $\mathcal{P}$ be the underlying solid cone with $\mu \in \mathcal{P}$ where $\rho(\mu) < \frac{1}{2}$. Assume the mapping $\mathcal{T} : \Delta \longrightarrow \Delta$ verifies the generalized Lipschitz condition:

$$d(\mathcal{T}j, \mathcal{T}k) \preceq \mu \Big[d(\mathcal{T}j, j) + d(\mathcal{T}k, k)\Big] \qquad \text{for all} \quad \mathrm{j,k} \in \Delta.$$

Then, $\mathcal{T}$ has a unique fixed point in Δ. Moreover, for any $j_0 \in \Delta$, the iterative sequence $\{\mathcal{T}^n j_0\}$ converges to the fixed point.

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Then, $\mathcal{T}$ has a unique fixed point in Δ. Moreover, for any $j_0 \in \Delta$, the iterative sequence $\{\mathcal{T}^n j_0\}$ converges to the fixed point.

3. Main results

In this section, we associate the results of Banach and Chatterjea to get the following (Banach – Chatterjea) theorem in complete cone metric space with Banach algebra:

Theorem 3.1 Suppose that $(\Delta, \mathcal{B}, d)$ is a complete cone metric space with Banach algebra and $\mathcal{P}$ is a cone. Let $\mathcal{T} : \Delta \longrightarrow \Delta$ be a mapping such that:

$$d(\mathcal{T}j, \mathcal{T}k) \preceq \mu \Big[\, d(j,k) + d(j, \mathcal{T}k) + d(k, \mathcal{T}j) \,\Big] \qquad \text{for all} \ \ j, k \in \Delta,$$

where $\mu \in \mathcal{P}$ with $\rho\,(\mu) < \frac{1}{3}$. Then, $\mathcal{T}$ has a unique fixed point in Δ.

Proof: Let j_0 be an arbitrary element in $\mathit{\Delta}$. Put $\mathcal{T}^n j_0 = \mathcal{T} j_{n-1} = j_n,\ \ n \geq 1$. Using the contractive mapping condition, we obtain

$$\begin{aligned} d(j_n, j_{n+1}) &= d(\mathcal{T} j_{n-1}, \mathcal{T} j_n) \\ &\preceq \mu \Big[d(j_{n-1}, j_n) + d(j_{n-1}, \mathcal{T} j_n) + d(j_n, \mathcal{T} j_{n-1}) \Big] \\ &= \mu \Big[d(j_{n-1}, j_n) + d(j_{n-1}, j_{n+1}) + d(j_n, j_n) \Big] \\ &\preceq \mu\, d(j_{n-1}, j_n) \ +\ \mu \Big[d(j_{n-1}, j_n) + d(j_n, j_{n+1}) \Big] \\ &= 2\, \mu\, d(j_{n-1}, j_n) \ +\ \mu\, d(j_n, j_{n+1}). \end{aligned}$$

This tends to,

$$(I_{\mathcal{B}} - \mu)\, d(j_n, j_{n+1}) \ \preceq\ 2\, \mu\, d(j_{n-1}, j_n),$$

that is,

$$\begin{aligned} d(j_n, j_{n+1}) &\preceq \quad (I_{\mathcal{B}} - \mu)^{-1}\, (2\mu)\ \ d(j_{n-1}, j_n) \\ &\preceq \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big)^2\ \ d(j_{n-2}, j_{n-1}) \\ &\ \ \vdots \\ &\preceq \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big)^n\ \ d(j_0, j_1). \end{aligned}$$

Then

$$\|d(j_n, j_{n+1})\| \ \leq \left\| \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big)^n \right\| \ \ \|d(j_0, j_1)\|\,.$$

Now, we deduce that

$$\rho \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big) < 1.$$

Therefore, we pay attention to the fact that both of $(I_{\mathcal{B}} - \mu)^{-1}$ and (2μ) commute. Since $\rho\,(\mu) < \frac{1}{3}$, we obtain

$$\begin{aligned} \rho \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big) &\leq \rho\,\big((I_{\mathcal{B}} - \mu)^{-1} \big)\ \ \rho\,(2\mu) \\ &\leq \rho\big(\sum_{i=0}^{\infty} \mu^i \big)\ \ \rho(2\mu) \\ &= \sum_{i=0}^{\infty} \rho(\mu^i)\ \ \rho(2\mu) \\ &\leq (1 - \rho\,(\mu))^{-1}\ \ \rho\,(2\mu) \\ &\leq \frac{2\,\rho\,(\mu)}{1 - \rho\,(\mu)}\ \ <\ \ 1. \end{aligned}$$

This lead us to, $\left\| \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big)^n \right\| \longrightarrow 0$, whenever $\rho \Big((I_{\mathcal{B}} - \mu)^{-1}\, (2\mu) \Big) < 1$ (by Remark 2.2). Hence, $\|d(j_n, j_{n+1})\| \ \leq\ 0$ which is contradiction. Consequently,

$$\|d(j_n, j_{n+1})\| \ \longrightarrow\ 0 \ \ \Longrightarrow\ \ d(j_n, j_{n+1}) \ \longrightarrow\ \theta.$$

Accordingly, $\{j_n\}$ is a Cauchy sequence. Since $\varDelta$ is complete, then there is $j^* \in \varDelta$ such that $j_n \longrightarrow j^*$. We now show that $j^* = \mathcal{T}j^*$. Indeed,

$$\begin{aligned} d(j^*, \mathcal{T}j^*) &\preceq d(j^*, \mathcal{T}j_n) + d(\mathcal{T}j_n, \mathcal{T}j^*) \\ &\preceq d(j^*, j_{n+1}) + \mu\Big[d(j_n, j^*) + d(j_n, \mathcal{T}j^*) + d(j^*, \mathcal{T}j_n)\Big] \\ &\preceq d(j^*, j_{n+1}) + \mu\Big[d(j_n, j^*) + d(j_n, \mathcal{T}j^*) + d(j^*, j_{n+1})\Big] \\ &\preceq d(j^*, j_{n+1}) + \mu\Big[d(j_n, j^*) + d(j_n, j^*) + d(j^*, \mathcal{T}j^*) + d(j^*, j_{n+1})\Big] \end{aligned}$$

that is,

$$(I_{\mathcal{B}} - \mu)\, d(j^*, \mathcal{T}j^*) \preceq (I_{\mathcal{B}} + \mu)\, d(j^*, j_{n+1}) + 2\,\mu\, d(j_n, j^*).$$

This leads to

$$d(j^*, \mathcal{T}j^*) \preceq (I_{\mathcal{B}} - \mu)^{-1}\,(I_{\mathcal{B}} + \mu)\, d(j^*, j_{n+1}) + (I_{\mathcal{B}} - \mu)^{-1}\,(2\,\mu)\, d(j^*, j_n).$$

Then, we obtain

$$\begin{aligned} &\|d(j^*, \mathcal{T}j^*)\| \\ \leq\ &\left\|(I_{\mathcal{B}} - \mu)^{-1}\right\| \Big(\|I_{\mathcal{B}} + \mu\|\ \|d(j^*, j_{n+1})\| + \|2\,\mu\|\ \|d(j^*, j_n)\| \Big) \longrightarrow 0, \quad (n \to \infty). \end{aligned}$$

Hence, $d(j^*, \mathcal{T}j^*) \longrightarrow \theta$. Then, j^* is a fixed point of the mapping $\mathcal{T}$.

Now, if $k^* \neq j^*$ is another fixed point of the mapping $\mathcal{T}$, then

$$\begin{aligned} d(j^*, k^*) &= d(\mathcal{T}j^*, \mathcal{T}k^*) \\ &\preceq \mu\Big[d(j^*, k^*) + d(j^*, \mathcal{T}k^*) + d(k^*, \mathcal{T}j^*)\Big] \\ &= 3\,\mu\, d(j^*, k^*) \\ &\ \ \vdots \\ &\preceq (3\,\mu)^n\, d(j^*, k^*), \qquad \text{for all } n \geq 1. \end{aligned}$$

Again, we get

$$\|d(j^*, k^*)\| \leq \|(3\,\mu)^n\|\ \|d(j^*, k^*)\| \longrightarrow 0, \qquad \text{as } (n \longrightarrow \infty),$$

where $\lim_{n \longleftrightarrow \infty} \|(3\,\mu)^n\| = 0$. Thus,

$$\|d(j^*, k^*)\| \leq 0.$$

which is a contradiction. Then, $d(j^*, k^*) = \theta \implies j^* = k^*$. Hence, the fixed point is unique. □

In the following, we deal with the results of Banach and Kannan to obtain the following (Banach – Kannan) theorem in complete cone metric space over Banach algebra:

Theorem 3.2 Suppose that $(\varDelta, \mathcal{B}, d)$ be a complete cone metric space with Banach algebra and $\mathcal{P}$ be a cone. Let $\mathcal{T} : \varDelta \longrightarrow \varDelta$ be a mapping such that:

$$d(\mathcal{T}j, \mathcal{T}k) \preceq \mu\Big[d(j, k) + d(j, \mathcal{T}j) + d(k, \mathcal{T}k)\Big] \qquad \text{for all } j, k \in \varDelta,$$

where $\mu \in \mathcal{P}$ with $\rho\,(\mu) < \frac{1}{3}$. Then, $\mathcal{T}$ has a unique fixed point in $\varDelta$.

Proof: Let j_0 be arbitrary element in $\mathit{\Delta}$. Put $\mathcal{T}^n j_0 = \mathcal{T} j_{n-1} = j_n, \ \ n \geq 1$. Using the contractive mapping condition, we obtain

$$\begin{aligned} d(j_n, j_{n+1}) &= d(\mathcal{T} j_{n-1}, \mathcal{T} j_n) \\ &\preceq \mu \Big[d(j_{n-1}, j_n) + d(j_{n-1}, \mathcal{T} j_{n-1}) + d(j_n, \mathcal{T} j_n) \Big] \\ &= \mu \Big[d(j_{n-1}, j_n) + d(j_{n-1}, j_n) + d(j_n, j_{n+1}) \Big] \\ &\preceq 2\,\mu\, d(j_{n-1}, j_n) \ + \ \mu\, d(j_n, j_{n+1}). \end{aligned}$$

This implies that,

$$(I_{\mathcal{B}} - \mu)\, d(j_n, j_{n+1}) \ \preceq \ 2\,\mu\, d(j_{n-1}, j_n),$$

that is,

$$\begin{aligned} d(j_n, j_{n+1}) &\preceq (I_{\mathcal{B}} - \mu)^{-1}\,(2\mu)\ \ d(j_{n-1}, j_n) \\ &\preceq \Big((I_{\mathcal{B}} - \mu)^{-1}\,(2\mu) \Big)^2\ \ d(j_{n-2}, j_{n-1}) \\ &\ \vdots \\ &\preceq \Big((I_{\mathcal{B}} - \mu)^{-1}\,(2\mu) \Big)^n\ \ d(j_0, j_1). \end{aligned}$$

Implying thereby

$$\|d(j_n, j_{n+1})\| \ \leq \ \left\| \Big((I_{\mathcal{B}} - \mu)^{-1}\,(2\mu) \Big)^n \right\| \ \ \|d(j_0, j_1)\|\,,$$

where

$$\rho \Big((I_{\mathcal{B}} - \mu)^{-1}\,(2\mu) \Big) < 1.$$

This lead us to, $\left\| \Big((I_{\mathcal{B}} - \mu)^{-1}\,(2\mu) \Big)^n \right\| \longrightarrow 0$, whenever $\rho \Big((I_{\mathcal{B}} - \mu)^{-1}\,(2\mu) \Big) < 1$ (by Remark 2.2). Hence, $\|d(j_n, j_{n+1})\| \ \leq \ 0$ which is contradiction. Consequently,

$$\|d(j_n, j_{n+1})\| \ \longrightarrow \ 0 \quad \Longrightarrow \quad d(j_n, j_{n+1}) \ \longrightarrow \ \theta.$$

Accordingly, $\{j_n\}$ is a Cauchy sequence . Since $\mathit{\Delta}$ is complete, then there is $j^* \in \mathit{\Delta}$ such that $j_n \longrightarrow j^*$. We now show $j^* = \mathcal{T} j^*$.

$$\begin{aligned} d(j^*, \mathcal{T} j^*) &\preceq d(j^*, \mathcal{T} j_n) + d(\mathcal{T} j_n, \mathcal{T} j^*) \\ &\preceq d(j^*, j_{n+1}) + \mu \Big[d(j_n, j^*) + d(j_n, \mathcal{T} j^*) + d(j^*, \mathcal{T} j_n) \Big] \\ &\preceq d(j^*, j_{n+1}) + \mu \Big[d(j_n, j^*) + d(j_n, j^*) + d(j^*, \mathcal{T} j^*) + d(j^*, j_{n+1}) \Big], \end{aligned}$$

that is,

$$(I_{\mathcal{B}} - \mu)\, d(j^*, \mathcal{T} j^*) \ \ \preceq \ \ (I_{\mathcal{B}} + \mu)\, d(j^*, j_{n+1}) + 2\,\mu\, d(j_n, j^*).$$

This leads to

$$d(j^*, \mathcal{T} j^*) \ \preceq \ (I_{\mathcal{B}} - \mu)^{-1}\,(I_{\mathcal{B}} + \mu)\, d(j^*, j_{n+1}) + (I_{\mathcal{B}} - \mu)^{-1}\,(2\,\mu)\, d(j^*, j_n).$$

Thus, we obtain

$$\begin{aligned} &\|d(j^*, \mathcal{T} j^*)\| \\ \leq\ \ &\left\| (I_{\mathcal{B}} - \mu)^{-1} \right\| \Bigg(\|I_{\mathcal{B}} + \mu\| \ \|d(j^*, j_{n+1})\| + \|2\,\mu\| \ \|d(j^*, j_n)\| \Bigg) \longrightarrow 0, \quad (n \to \infty). \end{aligned}$$

Hence, $d(j^*, \mathcal{T}j^*) \longrightarrow \theta$. Then, j^* is a fixed point of the mapping $\mathcal{T}$.

Now, if $k^* \neq j^*$ is another fixed point of the mapping $\mathcal{T}$, then

$$\begin{aligned} d(j^*, k^*) &= d(\mathcal{T}j^*, \mathcal{T}k^*) \\ &\preceq \mu \Big[d(j^*, k^*) + d(j^*, \mathcal{T}k^*) + d(k^*, \mathcal{T}j^*) \Big] \\ &= 3\,\mu\, d(j^*, k^*) \\ &\vdots \\ &\preceq (3\,\mu)^n\, d(j^*, k^*), \qquad \text{for all } n \geq 1. \end{aligned}$$

Again, we get

$$\|d(j^*, k^*)\| \;\leq\; \|(3\,\mu)^n\|\; \|d(j^*, k^*)\| \longrightarrow 0, \qquad \text{as } (n \longrightarrow \infty);$$

where $\lim\limits_{n \longleftrightarrow \infty} \|(3\,\mu)^n\| = 0$. Thus,

$$\|d(j^*, k^*)\| \;\leq\; 0.$$

which is a contradiction. Then, $d(j^*, k^*) = \theta \implies j^* = k^*$. Therefore, the fixed point is unique. □

In the next, we integrate the results of Banach, Kannan and Chatterjea to get the next theorem in complete cone metric space with Banach algebra:

Theorem 3.3 Suppose that $(\Delta, \mathcal{B}, d)$ be a complete cone metric space with Banach algebra and $\mathcal{P}$ be a cone. Let $\mathcal{T} : \Delta \longrightarrow \Delta$ be a mapping such that:

$$d(\mathcal{T}j, \mathcal{T}k) \;\preceq\; \mu \Big[d(j, k) + d(j, \mathcal{T}k) + d(k, \mathcal{T}j) + d(j, \mathcal{T}j) + d(k, \mathcal{T}k) \Big] \qquad \text{for all } j, k \in \Delta,$$

where $\mu \in \mathcal{P}$ with $\rho\,(\mu) < \frac{1}{5}$. Then, $\mathcal{T}$ has a unique fixed point in Δ.

Proof: Assume j_0 be an arbitrary element in Δ. Put $\mathcal{T}^n j_0 = \mathcal{T} j_{n-1} = j_n,\ n \geq 1$. Using the contractive mapping condition, we obtain

$$\begin{aligned} d(j_n, j_{n+1}) &= d(\mathcal{T}j_{n-1}, \mathcal{T}j_n) \\ &\preceq \mu \Big[d(j_{n-1}, j_n) + d(j_{n-1}, \mathcal{T}j_n) \\ &\qquad\qquad + d(j_n, \mathcal{T}j_{n-1}) \;+\; d(j_{n-1}, \mathcal{T}j_{n-1}) + d(j_n, \mathcal{T}j_n) \Big] \\ &= \mu \Big[d(j_{n-1}, j_n) + d(j_{n-1}, j_{n+1}) + d(j_n, j_n) + d(j_{n-1}, j_n) + d(j_n, j_{n+1}) \Big] \\ &\preceq 2\,\mu\, d(j_{n-1}, j_n) \;+\; \mu\, d(j_n, j_{n+1}) \;+\; \mu \Big[d(j_{n-1}, j_n) + d(j_n, j_{n+1}) \Big] \\ &= 3\,\mu\, d(j_{n-1}, j_n) \;+\; 2\,\mu\, d(j_n, j_{n+1}). \end{aligned}$$

This tends to,

$$(I_{\mathcal{B}} - 2\mu)\, d(j_n, j_{n+1}) \;\preceq\; 3\,\mu\, d(j_{n-1}, j_n),$$

that is,

$$\begin{aligned} d(j_n, j_{n+1}) &\preceq (I_{\mathcal{B}} - 2\mu)^{-1}\, (3\mu)\; d(j_{n-1}, j_n) \\ &\preceq \Big((I_{\mathcal{B}} - 2\mu)^{-1}\, (3\mu) \Big)^2\; d(j_{n-2}, j_{n-1}) \\ &\vdots \\ &\preceq \Big((I_{\mathcal{B}} - 2\mu)^{-1}\, (3\mu) \Big)^n\; d(j_0, j_1). \end{aligned}$$

This tends to

$$\|d(j_n, j_{n+1})\| \leq \left\| \Big((I_{\mathcal{B}} - 2\mu)^{-1} (3\mu) \Big)^n \right\| \|d(j_0, j_1)\| .$$

Now, we show that

$$\rho\left((I_{\mathcal{B}} - 2\mu)^{-1} (3\mu) \right) < 1.$$

Therefore, we pay attention to the fact that both of $(I_{\mathcal{B}} - 2\mu)^{-1}$ and (3μ) commute. Since $\rho(\mu) < \frac{1}{5}$, we obtain

$$\begin{aligned} \rho\left((I_{\mathcal{B}} - 2\mu)^{-1} (3\mu) \right) &\leq \rho\left((I_{\mathcal{B}} - 2\mu)^{-1} \right) \rho(3\mu) \\ &\leq (1 - \rho(2\mu))^{-1} \rho(3\mu) \\ &\leq \frac{3\rho(\mu)}{1 - 2\rho(\mu)} < 1. \end{aligned}$$

This lead us to, $\left\| \Big((I_{\mathcal{B}} - 2\mu)^{-1} (3\mu) \Big)^n \right\| \longrightarrow 0$, whenever $\rho\left((I_{\mathcal{B}} - 2\mu)^{-1} (3\mu) \right) < 1$ (by Remark 2.2). Hence, $\|d(j_n, j_{n+1})\| \leq 0$ which is contradiction. Consequently,

$$\|d(j_n, j_{n+1})\| \longrightarrow 0 \implies d(j_n, j_{n+1}) \longrightarrow \theta.$$

Accordingly, $\{j_n\}$ is a Cauchy sequence . Since Δ is complete, then there is $j^* \in \Delta$ such that $j_n \longrightarrow j^*$. Now, we prove that $j^* = \mathcal{T} j^*$.

$$\begin{aligned} d(j^*, \mathcal{T} j^*) &\preceq d(j^*, \mathcal{T} j_n) + d(\mathcal{T} j_n, \mathcal{T} j^*) \\ &\preceq d(j^*, j_{n+1}) + \mu \Big[d(j_n, j^*) + d(j_n, \mathcal{T} j^*) \\ &\qquad + d(j^*, \mathcal{T} j_n) + d(j_n, \mathcal{T} j_n) + d(j^*, \mathcal{T} j^*) \Big] \\ &\preceq d(j^*, j_{n+1}) \\ &\quad + \mu \Big[d(j_n, j^*) + d(j_n, j^*) + d(j^*, \mathcal{T} j^*) + d(j^*, j_{n+1}) + d(j_n, j_{n+1}) + d(j^*, \mathcal{T} j^*) \Big], \end{aligned}$$

that is,

$$(I_{\mathcal{B}} - 2\mu)\, d(j^*, \mathcal{T} j^*) \preceq (I_{\mathcal{B}} + \mu)\, d(j^*, j_{n+1}) + 2\mu\, d(j_n, j^*) + \mu\, d(j_n, j_{n+1}).$$

This leads to

$$d(j^*, \mathcal{T} j^*) \preceq (I_{\mathcal{B}} - 2\mu)^{-1} \Big((I_{\mathcal{B}} + \mu)\, d(j^*, j_{n+1}) + (2\mu)\, d(j^*, j_n) + \mu\, d(j_n, j_{n+1}) \Big).$$

Then, we obtain

$$\begin{aligned} &\|d(j^*, \mathcal{T} j^*)\| \\ &\leq \left\|(I_{\mathcal{B}} - 2\mu)^{-1}\right\| \Big(\|I_{\mathcal{B}} + \mu\| \, \|d(j^*, j_{n+1})\| + \|2\mu\| \, \|d(j^*, j_n)\| + \|\mu\| \, \|d(j_n, j_{n+1})\| \Big) \longrightarrow 0, \quad (n \to \infty). \end{aligned}$$

Hence, $d(j^*, \mathcal{T} j^*) \longrightarrow \theta$. Then, j^* is a fixed point of the mapping $\mathcal{T}$.

Now, if $k^* \neq j^*$ is another fixed point of the mapping $\mathcal{T}$, then

$$\begin{aligned} d(j^*, k^*) &= d(\mathcal{T} j^*, \mathcal{T} k^*) \\ &\preceq \mu \Big[d(j^*, k^*) + d(j^*, \mathcal{T} k^*) + d(k^*, \mathcal{T} j^*) + d(j^*, \mathcal{T} j^*) + d(k^*, \mathcal{T} k^*) \Big] \\ &= 3\mu\, d(j^*, k^*) \\ &\ \ \vdots \\ &\preceq (3\mu)^n\, d(j^*, k^*), \qquad \text{for all} \quad n \geq 1. \end{aligned}$$

Again, using the normality of $\mathcal{P}$, we get

$$\|d(j^*,k^*)\| \ \leq \ \|(3\,\mu)^n\| \ \|d(j^*,k^*)\| \longrightarrow 0, \qquad \text{as } (n \longrightarrow \infty);$$

where $\lim\limits_{n \longleftrightarrow \infty} \|(3\,\mu)^n\| = 0$. Thus,

$$\|d(j^*,k^*)\| \ \leq \ 0.$$

which is a contradiction. Then, $d(j^*,k^*) = \theta \implies j^* = k^*$. Therefore, the fixed point is unique. □

Example 3.1 Let $\mathcal{B} = \mathbb{R}^3$. For all $(j_1, j_2, j_3) \in \mathcal{B}$, $\|(j_1, j_2, j_3)\| = \sum_{i=1}^{3} |j_i| = |j_1| + |j_2| + |j_3|$. The operation of multiplication is defined by

$$jk = (j_1, k_1, w_1).(j_2, k_2, w_2) = (j_1 j_2 \ , \ j_1 k_2 + j_2 k_1 \ , \ j_1 w_2 + j_2 w_1 + w_1 w_2).$$

Then $\mathcal{B}$ is a Banach algebra with unit $I_{\mathcal{B}} = (1, 0, 0)$.

Let $\mathcal{P} = \{ \ (j_1, j_2, j_3) \in \mathcal{B}^3 \ : \ j_1, j_2, j_3 \geq 0\}$.

Let $\Delta = \mathcal{B}$ and the metric be defined by

$$d(j,k) = \big(\, |j-k| \, , 2\,|j-k| \, , 3\,|j-k| \, \big) \in \mathcal{P}.$$

Then, $(\Delta, \mathcal{B}, d)$ is a complete cone metric space over Banach algebra.

Now, we define the mapping $\mathcal{T} : \Delta \longrightarrow \Delta$ by

$$\mathcal{T}j \ = \ \frac{1}{2}\, j.$$

Then, we get

$$\begin{aligned} d(\mathcal{T}j, \mathcal{T}k) \ &= \ \Big(\, |\mathcal{T}j - \mathcal{T}k| \, , |\mathcal{T}j - \mathcal{T}k| \, , |\mathcal{T}j - \mathcal{T}k| \Big) \\ &= \ \big(\frac{1}{2}\,|j-k| \, , |j-k| \, , \frac{3}{2}\,|j-k| \, \big) \\ &= \ \frac{1}{2}\Big(\, |j-k| \, , 2\,|j-k| \, , 3\,|j-k| \Big) \\ &\preceq \ \mu \ d(j,k), \qquad \mu = \frac{1}{2} \in (0,1). \end{aligned}$$

Therefore all hypotheses of Theorem 3.1 are satisfied and 0 is a unique **FP** of $\mathcal{T}$.

Example 3.2 Let $\mathcal{B}$, $\mathcal{P}$ and the metric d are defined as Example 3.1. Now, we define the mapping $\mathcal{T} : \Delta \longrightarrow \Delta$ by

$$\mathcal{T}j \ = \ j.$$

Then, we have

$$\begin{aligned} d(j,k) + d(j,\mathcal{T}k) + d(k,\mathcal{T}j) \ &= \ \Big(\ |j-k| \, , 2\,|j-k| \, , 3\,|j-k| \Big) \\ &+ \ \Big(|j - \mathcal{T}k| \, , 2\,|j - \mathcal{T}k| \, , 3\,|j - \mathcal{T}k| \Big) \\ &+ \ \Big(|k - \mathcal{T}j| \, , 2\,|k - \mathcal{T}j| \, , 3\,|k - \mathcal{T}j| \Big) \\ &= \ \Big(\ |\mathcal{T}j - \mathcal{T}k| \, , 2\,|\mathcal{T}j - \mathcal{T}k| \, , 3\,|\mathcal{T}j - \mathcal{T}k| \Big) \\ &+ \ \Big(|\mathcal{T}j - \mathcal{T}k| \, , 2\,|\mathcal{T}j - \mathcal{T}k| \, , 3\,|\mathcal{T}j - \mathcal{T}k| \Big) \\ &+ \ \Big(|\mathcal{T}j - \mathcal{T}k| \, , 2\,|\mathcal{T}j - \mathcal{T}k| \, , 3\,|\mathcal{T}j - \mathcal{T}k| \Big) \\ &\succeq \ 3 \, \Big(|\mathcal{T}j - \mathcal{T}k| \, , 2 \ |\mathcal{T}j - \mathcal{T}k| \, , 3 \ |\mathcal{T}j - \mathcal{T}k| \Big) \\ &= \ 3\, d(\mathcal{T}j, \mathcal{T}k). \end{aligned}$$

Then,

$$d(\mathcal{T}j,\mathcal{T}k) \preceq \frac{1}{3}\Big[d(j,k)+d(j,\mathcal{T}k)+d(k,\mathcal{T}j)\Big],$$

where $\mu=\dfrac{1}{3}$. Therefore, all conditions of Theorem 3.2 are verified and 0 is a unique fixed point of $\mathcal{T}$.

4. Solving Urysohn integral equations

In this section, we prove the applicability of Theorem 3.2.

Let $\mathcal{B}$ and $\mathcal{P}$ be defined as Example 3.1. Let $\Delta = C([0,1],\mathcal{B})$ where $C[0,1]$ denotes the set of all continuous functions, $\mathcal{P}$ be a cone and $d:\varDelta\times\varDelta\longrightarrow\mathcal{B}$ be defined as follows:

$$\begin{aligned} d(j,k) &= \big(\, \|j-k\|_\infty\,,\ \|j-k\|_\infty\,,\ \|j-k\|_\infty\,\big)\\ &= \big(\sup_{t\in[0,1]} |j(t)-k(t)|\,,\ \sup_{t\in[0,1]} |j(t)-k(t)|\,,\ \sup_{t\in[0,1]} |j(t)-k(t)|\,\big)\ .\ I_{\mathcal{B}}\ \in\mathcal{P}, \end{aligned}$$

where $j,k\in\varDelta$. Then, $(\varDelta,\ \mathcal{B},\ d)$ is a complete cone metric space over Banach algebra.

Consider the following Urysohn integral equations:

$$j(t) = \Big(h(t)+\int_0^1 \mathcal{K}(t,s,j(s))\,ds\Big)\ .\ I_{\mathcal{B}}, \tag{4.1}$$

$$k(t) = \Big(h(t)+\int_0^1 \mathcal{K}(t,s,k(s))\,ds\Big)\ .\ I_{\mathcal{B}}, \tag{4.2}$$

where
(i) $j(t)$ and $k(t)$ are unknown variables for every $t\in[0,1]$,
(ii) $h:[0,1]\longrightarrow\mathcal{B}$ is the deterministic free term defined for all $t\in[0,1]$,
(iii) $\mathcal{K}:[0,1]\times[0,1]\times\mathcal{B}\longrightarrow\mathcal{B}$ is measurable kernel defined for all $t,s\in[0,1]$.

Theorem 4.1 Define the map $\mathcal{T}:\Delta\longrightarrow\Delta$ by

$$\begin{aligned} \mathcal{T}\,j(t) &= \Big(h(t)+\int_0^1 \mathcal{K}(t,s,j(s))\,ds\Big)\ .\ I_{\mathcal{B}},\\ \mathcal{T}\,k(t) &= \Big(h(t)+\int_0^1 \mathcal{K}(t,s,k(s))\,ds\Big)\ .\ I_{\mathcal{B}}, \end{aligned}$$

for all $t\in[0,1]$. Assume the following inequality holds:

$$|\mathcal{K}(t,s,j(s))-\mathcal{K}(t,s,k(s))| \le \mu\ |j(s)-k(s)|\,.$$

Then, the integral equations (4.1) and (4.2) have a unique solution.

Proof: Consider

$$\begin{aligned}
& d(\mathcal{T}j, \mathcal{T}k) \\
= & \left(\|\mathcal{T}j \;-\; \mathcal{T}k\|_{\infty}, \; \|\mathcal{T}j \;-\; \mathcal{T}k\|_{\infty}, \; \|\mathcal{T}j \;-\; \mathcal{T}k\|_{\infty} \right) \\
= & \left(\sup_{t\in[0,1]} |\mathcal{T}j(t) - \mathcal{T}k(t)|, \; \sup_{t\in[0,1]} |\mathcal{T}j(t) - \mathcal{T}k(t)|, \; \sup_{t\in[0,1]} |\mathcal{T}j(t) - \mathcal{T}k(t)| \right) . I_{\mathcal{B}} \\
\preceq & \left(\sup_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s,j(s)) \;-\; \mathcal{K}(t,s,k(s))| \, ds, \right. \\
& \qquad \sup_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s,j(s)) - \mathcal{K}(t,s,k(s))| \, ds, \\
& \qquad\qquad \left. \sup_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s,j(s)) - \mathcal{K}(t,s,k(s))| \, ds \right) . I_{\mathcal{B}} \\
\preceq & \; \mu \left(\sup_{t\in[0,1]} \int_0^1 |j(t) - k(t)| \, ds, \; \sup_{t\in[0,1]} \int_0^1 |j(t) - k(t)| \, ds, \; \sup_{t\in[0,1]} \int_0^1 |j(t) - k(t)| \, ds \right) . I_{\mathcal{B}} \\
\preceq & \; \mu \left(\sup_{t\in[0,1]} |j(t) - k(t)| \int_0^1 ds, \; \sup_{t\in[0,1]} |j(t) - k(t)| \int_0^1 ds, \; \sup_{t\in[0,1]} |j(t) - k(t)| \int_0^1 ds \right) . I_{\mathcal{B}} \\
= & \; \mu \left(\|j(t) - k(t)\|_{\infty}, \; \|j(t) - k(t)\|_{\infty}, \; \|j(t) - k(t)\|_{\infty} \right) \\
= & \; \mu \; d(j,k),
\end{aligned}$$

where $\mu \in [0,1)$. Then, all requirements of Theorem 3.1 are satisfied. Then, the integral equations (4.1) and (4.2) have a unique solution. □

The example below support Theorem 4.1.

Example 4.1 Define the metric as follows:

$$\begin{aligned}
d(j,k) &= \left(\|j - k\|_{\infty}, \; \|j - k\|_{\infty}, \; \|j - k\|_{\infty} \right) \\
&= \left(\sup_{t\in[0,1]} |j(t) - k(t)|, \; \sup_{t\in[0,1]} |j(t) - k(t)|, \; \sup_{t\in[0,1]} |j(t) - k(t)| \right) . I_{\mathcal{B}} \; \in \; \mathcal{P}.
\end{aligned}$$

Also, let $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{P}$ are defined as Example 3.1. Consider

$$\mathcal{T}\, j(t) = \left(e^t + \int_0^1 e^{-st} \, \frac{j(s)}{5} \; ds \right) . I_{\mathcal{B}}, \tag{4.3}$$

$$\mathcal{T}\, k(t) = \left(e^t + \int_0^1 e^{-st} \, \frac{k(s)}{5} \; ds \right) . I_{\mathcal{B}}, \tag{4.4}$$

for all $t \in [0,1]$. Clearly,

$$|\mathcal{K}(t,s,j(t)) - \mathcal{K}(t,s,k(t))| \;\leq\; \frac{e^{-st}}{5} \; |j(t) - k(t)| \, .$$

Further,

$$
\begin{aligned}
& d(\mathcal{T}j, \mathcal{T}k) \\
= & \left(\|\mathcal{T}j - \mathcal{T}k\|_{\infty}, \|\mathcal{T}j - \mathcal{T}k\|_{\infty}, \|\mathcal{T}j - \mathcal{T}k\|_{\infty} \right) \\
= & \left(\sup_{t\in[0,1]} |\mathcal{T}j(t) - \mathcal{T}k(t)|, \sup_{t\in[0,1]} |\mathcal{T}j(t) - \mathcal{T}k(t)|, \sup_{t\in[0,1]} |\mathcal{T}j(t) - \mathcal{T}k(t)| \right) . I_{\mathcal{B}} \\
\preceq & \left(\sup_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s,j(s)) - \mathcal{K}(t,s,k(s))| \, ds, \right. \\
& \sup_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s,j(s)) - \mathcal{K}(t,s,k(s))| \, ds, \\
& \left. \sup_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s,j(s)) - \mathcal{K}(t,s,k(s))| \, ds \right) . I_{\mathcal{B}} \\
\preceq & \frac{1}{5} \left| e^{-st} \right| \left(\sup_{t\in[0,1]} \int_0^1 |j(t) - k(t)| \, ds, \right. \\
& \left. \sup_{t\in[0,1]} \int_0^1 |j(t) - k(t)| \, ds, \ \sup_{t\in[0,1]} \int_0^1 |j(t) - k(t)| \, ds \right) . I_{\mathcal{B}} \\
\preceq & \frac{1}{5} \left(\sup_{t\in[0,1]} |j(t) - k(t)| \int_0^1 ds, \right. \\
& \left. \sup_{t\in[0,1]} |j(t) - k(t)| \int_0^1 ds, \ \sup_{t\in[0,1]} |j(t) - k(t)| \int_0^1 ds \right) . I_{\mathcal{B}} \\
= & \frac{1}{5} \left(\|j - k\|_{\infty}, \ \|j - k\|_{\infty}, \ \|j - k\|_{\infty} \right) \\
= & \frac{1}{5} \left(\frac{\|j - k\|_{\infty}}{3} + \frac{\|j - k\|_{\infty}}{3} + \frac{\|j - k\|_{\infty}}{3}, \right. \\
& \frac{\|j - k\|_{\infty}}{3} + \frac{\|j - k\|_{\infty}}{3} + \frac{\|j - k\|_{\infty}}{3}, \\
& \left. \frac{\|j - k\|_{\infty}}{3} + \frac{\|j - k\|_{\infty}}{3} + \frac{\|j - k\|_{\infty}}{3} \right) \\
= & \frac{1}{5} \left(\frac{\|j - k\|_{\infty}}{3} + \frac{\|\mathcal{T}j - k\|_{\infty}}{3} + \frac{\|j - \mathcal{T}k\|_{\infty}}{3}, \right. \\
& \frac{\|j - k\|_{\infty}}{3} + \frac{\|\mathcal{T}j - k\|_{\infty}}{3} + \frac{\|j - \mathcal{T}k\|_{\infty}}{3}, \\
& \left. \frac{\|j - k\|_{\infty}}{3} + \frac{\|\mathcal{T}j - k\|_{\infty}}{3} + \frac{\|j - \mathcal{T}k\|_{\infty}}{3} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{15}\Bigg[\Big(\|j-k\|_\infty,\ \|j-k\|_\infty,\ \|j-k\|_\infty\Big) \\
&\qquad + \Big(\|\mathcal{T}j-k\|_\infty, \|\mathcal{T}j-k\|_\infty, \|\mathcal{T}j-k\|_\infty\Big) \\
&\qquad\qquad + \Big(\|j-\mathcal{T}k\|_\infty, \|j-\mathcal{T}k\|_\infty, \|j-\mathcal{T}k\|_\infty\Big)\Bigg] \\
&= \mu\ \Big[d(j,k)\ +\ d(\mathcal{T}j,k)\ +\ d(\mathcal{T}k,j)\Big].
\end{aligned}
$$

Therefore, all conditions of Theorem 4.1 are fulfilled with $\mu = \dfrac{1}{15} < 1$.

5. Solving a Caputo fractional derivative

In this part, according to [50,51], we will recollect some theoretical results to study the existence and uniqueness of the solution to a fractional derivative of Caputo of order α, which takes the following form:

$$
{}^{C}D^{\alpha}(j(t)) \ = \ f(t, j(t)) \qquad (1 \geq t \geq 0, \quad 2 \geq \alpha > 1), \tag{5.1}
$$

via the integral boundary assertions

$$
j(1) \ = \ 0 \ = \ j(0), \qquad \text{with} \quad j \in C([0,1], \mathcal{B}),
$$

where $C([0,1],\mathcal{B})$ is the set of all real continuous functions from $[0,1]$ into $\mathcal{B}$, ${}^{C}D^{\alpha}$ represents the fractional derivative of Caputo of order α and $f : [0,1] \times \mathcal{B} \longrightarrow \mathcal{B}$ is a continuous function (see [52]).

Let $\mathcal{B}$ and $\mathcal{P}$ be defined as Example 3.1. Let $\Delta = C([0,1],\mathcal{B})$ where $C[0,1]$ denotes the set of all continuous functions, $\mathcal{P}$ be a cone and $d : \Delta \times \Delta \longrightarrow \mathcal{B}$ be defined as:

$$
\begin{aligned}
d(j,k) &= \big(\|j-k\|_\infty, \lambda\|j-k\|_\infty, \beta\|j-k\|_\infty\big) \\
&= \big(\max_{t\in[0,1]}|j(t)-k(t)|^2, \lambda\max_{t\in[0,1]}|j(t)-k(t)|^2, \beta\max_{t\in[0,1]}|j(t)-k(t)|^2\big) \ . \ I_{\mathcal{B}} \ \in \ \mathcal{P},
\end{aligned}
$$

where $j,k \in \Delta$ and $\lambda, \beta \geq 0$. Then, $(\Delta, \mathcal{B}, d)$ is a complete cone metric space over Banach algebra.

Now, our main theorem in this part is as follows:

Theorem 5.1 Problem (5.1) has a solution provided that the following assertions are true:

(A) there exists the continuous function $f : [0,1] \times \mathcal{B} \longrightarrow \mathcal{B}$ such that

$$
|f(t,j(t)) - f(t,k(t))|^2 \ \leq \ \frac{\Gamma(\alpha+1)}{5}\ \big(|j(t)-k(t)|^2,\ \lambda|j(t)-k(t)|^2,\ \beta|j(t)-k(t)|^2\big),
$$

for all $j,k \in \Delta$, $\lambda, \beta \geq 0$ and $t \in [0,1]$,

(B) there exists $v > 5$ such that

$$
\max_{t\in[0,1]} \int_0^1 |\mathcal{G}(t,s)|^2 \ ds \ \leq \ \frac{1}{v},
$$

where

$$
\mathcal{G}(t,s) = \begin{cases} \dfrac{\big[t(1-s)\big]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1; \\[2ex] \dfrac{\big[t(1-s)\big]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1; \end{cases}
$$

(C) for all $j, k \in C([0,1], \mathcal{B})$, let the metric $d : C([0,1], \mathcal{B}) \times C([0,1], \mathcal{B}) \to \mathcal{B}^3_+$ is defined as

$$\begin{aligned} d(j,k) &= \Big(\|j-k\|_\infty \,,\, \lambda \|j-k\|_\infty \,,\, \beta \|j-k\|_\infty \Big) \\ &= \Big(\max_{t\in[0,1]} |j(t)-k(t)|^2 \,,\, \lambda \max_{t\in[0,1]} |j(t)-k(t)|^2 \,,\, \beta \max_{t\in[0,1]} |j(t)-k(t)|^2 \Big) \,.\, I_{\mathcal{B}} \in \mathcal{P}, \end{aligned}$$

(D) if $\{j_n\}$ is a sequence in $C([0,1], \mathcal{B})$ and $j \in C([0,1], \mathcal{B})$, then $j_n \longrightarrow j$ in $C([0,1], \mathcal{B})$.

Then, the problem (5.1) has at least one solution.

Proof: Let $\varDelta = C([0,1], \mathcal{B})$ where $\varDelta$ is the Banach space equipped with the norm $\|j\|_\infty = \Big(\max_{t\in[0,1]} |j(t)|^2 \Big) \,.\, I_{\mathcal{B}}$ for all $j \in \varDelta$. Then, $j \in \varDelta$ is a solution of the problem (5.1) if and only if $j \in \varDelta$ is a solution of the following **IE** to:

$$j(t) = \Big(\int_0^1 \mathcal{G}(t,s) \; f(s, j(s)) \; ds \Big) \,.\, I_{\mathcal{B}}, \qquad t \in [0,1].$$

Define the mapping $\mathcal{T}$ by:

$$\mathcal{T} j(t) = \Big(\int_0^1 \mathcal{G}(t,s) \; f(s, j(s)) \; ds \Big) \,.\, I_{\mathcal{B}}.$$

From condition **(A)**, we have

$$\begin{aligned} |\mathcal{T} j(t) - \mathcal{T} k(t)|^2 &= \Big| \int_0^1 \mathcal{G}(t,s) \; f(s,j(s)) \; ds - \int_0^1 \mathcal{G}(t,s) \; f(s,k(s)) \; ds \Big|^2 \\ &= \Big| \int_0^1 \mathcal{G}(t,s) \; \big[f(s,j(s)) - f(s,k(s)) \big] \; ds \Big|^2 \\ &\leq \int_0^1 |\, \mathcal{G}(t,s)|^2 \; ds \,.\, \int_0^1 |f(s,j(s)) - f(s,k(s))|^2 \; ds. \end{aligned}$$

Taking maximum on both sides and multiplying by $I_{\mathcal{B}}$, we obtain

$$\begin{aligned} & \Big[\max_{t\in[0,1]} |\mathcal{T} j(t) - \mathcal{T} k(t)|^2 \Big] \,.\, I_{\mathcal{B}} \\ \preceq\ & \Big[\max_{t\in[0,1]} \Big\{ \int_0^1 |\, \mathcal{G}(t,s)|^2 \; ds \,.\, \int_0^1 |f(s,j(s)) - f(s,k(s))|^2 \; ds \Big\} \Big] \,.\, I_{\mathcal{B}} \\ \preceq\ & \Big[\max_{t\in[0,1]} \Big\{ \int_0^1 |\mathcal{G}(t,s)|^2 \; ds \Big\} \,.\, \max_{t\in[0,1]} \Big\{ \int_0^1 |f(s,j(s)) - f(s,k(s))|^2 \; ds \Big\} \Big] \,.\, I_{\mathcal{B}} \\ =\ & \Big[\max_{t\in[0,1]} \Big\{ \int_0^1 |\, \mathcal{G}(t,s)|^2 \; ds \Big\} \,.\, \max_{t\in[0,1]} |f(s,j(s)) - f(s,k(s))|^2 \Big\{ \int_0^1 ds \Big\} \Big] \,.\, I_{\mathcal{B}} \\ \preceq\ & \frac{1}{v} \,.\, \Big[\max_{t\in[0,1]} \; |f(s,j(s)) - f(s,k(s))|^2 \Big] \,.\, I_{\mathcal{B}} \\ =\ & \frac{\varGamma(\alpha+1)}{5v} \,.\, \Big[\max_{t\in[0,1]} \; \big(|j(t)-k(t)|^2 , \lambda |j(t)-k(t)|^2 , \beta |j(t)-k(t)|^2 \big) \Big] \,.\, I_{\mathcal{B}} \\ =\ & \frac{\varGamma(\alpha+1)}{5v} \\ & \,.\, \big(\max_{t\in[0,1]} |j(t)-k(t)|^2 \,,\, \lambda \max_{t\in[0,1]} |j(t)-k(t)|^2 \,,\, \beta \max_{t\in[0,1]} |j(t)-k(t)|^2 \big) \,.\, I_{\mathcal{B}}, \end{aligned}$$

which tends to

$$\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty \;\leq\; \frac{\Gamma(\alpha+1)}{5v}\,.\,\big(\|j(t)-k(t)\|_\infty,\lambda\,\|j(t)-k(t)\|_\infty,\beta\,\|j(t)-k(t)\|_\infty\big).$$

It follows that

$$\begin{aligned}
d(\mathcal{T}j,\mathcal{T}k) &= \Big(\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty,\ \lambda\,\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty,\ \beta\,\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty\Big)\\
&\preceq \frac{\Gamma(\alpha+1)}{5v}\,.\,(1,\lambda,\beta)\,.\,\big(\|j(t)-k(t)\|_\infty,\lambda\,\|j(t)-k(t)\|_\infty,\beta\,\|j(t)-k(t)\|_\infty\big)\\
&= \mu\, d(j,k).
\end{aligned}$$

where $\mu=\dfrac{\Gamma(\alpha+1)}{5v}\,(1,\lambda,\beta)$. Therefore, all assertions of Theorem 3.1 are satisfied. Hence, the problem (5.1) has a unique solution. □

Remark 5.1 By a similar way of Theorem 5.1, we can show that all conditions of Theorem 3.2 are verified as follows: Since

$$\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty \;\leq\; \frac{\Gamma(\alpha+1)}{5v}\,.\,\big(\|j(t)-k(t)\|_\infty,\lambda\,\|j(t)-k(t)\|_\infty,\beta\,\|j(t)-k(t)\|_\infty\big).$$

Therefore,

$$\begin{aligned}
d(\mathcal{T}j,\mathcal{T}k) &= \Big(\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty,\ \lambda\,\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty,\ \beta\,\|\mathcal{T}j(t)-\mathcal{T}k(t)\|_\infty\Big)\\
&\preceq \frac{\Gamma(\alpha+1)}{5v}\,(1,\lambda,\beta)\,.\,\big(\|j(t)-k(t)\|_\infty,\lambda\,\|j(t)-k(t)\|_\infty,\beta\,\|j(t)-k(t)\|_\infty\big)\\
&\preceq \frac{\Gamma(\alpha+1)}{5v}\,(1,\lambda,\beta)\\
&\quad .\Bigg(\frac{\|j(t)-k(t)\|_\infty}{3}+\frac{\|\mathcal{T}j(t)-k(t)\|_\infty}{3}+\frac{\|j(t)-\mathcal{T}k(t)\|_\infty}{3},\\
&\qquad \lambda\left[\frac{\|j(t)-k(t)\|_\infty}{3}+\frac{\|\mathcal{T}j(t)-k(t)\|_\infty}{3}+\frac{\|j(t)-\mathcal{T}k(t)\|_\infty}{3}\right],\\
&\qquad\quad \beta\left[\frac{\|j(t)-k(t)\|_\infty}{3}+\frac{\|\mathcal{T}j(t)-k(t)\|_\infty}{3}+\frac{\|j(t)-\mathcal{T}k(t)\|_\infty}{3}\right]\Bigg)\\
&\preceq \frac{\Gamma(\alpha+1)}{15v}\,(1,\lambda,\beta)\\
&\quad .\Bigg[\Big(\|j(t)-k(t)\|_\infty,\ \lambda\,\|j(t)-k(t)\|_\infty,\ \beta\,\|j(t)-k(t)\|_\infty\Big)\\
&\qquad +\Big(\|\mathcal{T}j(t)-k(t)\|_\infty,\ \lambda\,\|\mathcal{T}j(t)-k(t)\|_\infty,\ \beta\,\|\mathcal{T}j(t)-k(t)\|_\infty\Big)\\
&\qquad\quad +\Big(\|j(t)-\mathcal{T}k(t)\|_\infty,\ \lambda\,\|j(t)-\mathcal{T}k(t)\|_\infty,\ \beta\,\|j(t)-\mathcal{T}k(t)\|_\infty\Big)\Bigg]\\
&\preceq \mu\,\Big[d(j,k)\ +\ d(\mathcal{T}j,k)\ +\ d(\mathcal{T}k,j)\Big],
\end{aligned}$$

where $\dfrac{\Gamma(\alpha+1)}{15v}\ (1, \lambda, \beta)\ =\ \mu.$

6. Conclusions

In this manuscript, we introduced the results of Banach, Kannan, and Chatterjea in the context of cone metric space over Banach algebra. We presented some new definitions in cone metric space over Banach algebra, such as Banach-Kannan type mapping, Banach-Chatterjea type mapping, and Banach-Kannan-Chatterjea type mapping. These definitions are considered a generalization and extension to the results in the literature. Furthermore, we provided some nontrivial examples and application to the result related to our main theorems. While our results have applicability to integral equations, more studies are required to expand the framework, improve computing features, and investigate additional applications in a variety of areas. Also, our work helps us comprehend cone metric space over Banach algebra and its broader ramifications in a variety of areas. By creating a general framework for the application of the established fixed point results in this article, researchers will be better equipped to explore some results in cone metric spaces over Banach algebra in the future by investigating the characteristics and behavior of mappings on such spaces, forge linkages with other mathematical constructions in this area of study, and develop methods and strategies for solving practical problems in mathematics and other disciplines. As a future work, we can study fractional integrals and derivatives in other space such as C^*-algebra metric space, fuzzy metric space and so on. also, we can introduce some applications of nonlinear fractional differential equations, ordinary or partial differential equations, integral equation in such spaces. We can specifically focus and concentrate on understanding their behavior and developing approximation methods within these novel spaces.

Declarations

Availability of data and material Not applicable.

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