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Generalized Common Proximal Points in Intuitionistic Fuzzy Metric Spaces

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ABSTRACT: This manuscript studies specific criteria for real-valued functions $J, S : (0, 1] \to \mathbb{R}$ for the existence of the best proximal point of generalized $IF_{(S,J)}$-iterative mappings within the framework of intuitionistic fuzzy metric space. After, employ intuitionistic fuzzy versions of (S, J) K-type proximal contraction and fuzzy (S, J)- Hardy Rogers (HR)-type to investigate common best proximal (Cbp) points in intuitionistic fuzzy metric spaces with non-trivial example.

Keywords: Proximal point, fixed point, intuitionistic fuzzy metric spaces.

Contents

1. Introduction

The field of fixed point (FP) theory emerges as a captivating domain of inquiry, particularly focusing on methods for solving non-linear equations. Discussions within FP theory delve into diverse strategies for determining solutions. Despite thorough exploration, instances arise where a unique solution is elusive. Addressing such challenges often entails utilizing best proximity point theorems, which have been subject to various generalizations by numerous researchers, proving pivotal in deriving approximate optimal solutions.

A significant milestone occurred in 1968 when Kannan [17] introduced a novel type of contraction for discontinuous mappings, leading to several breakthroughs in FP results. This innovation provided researchers with an alternative approach to tackle FP problems. Subsequent advancements, such as the iterative Kannan–Mier-type contractions introduced by Karapinar [18], further solidified the foundation. Rus–Reich–Ciric-type contractions were established using simulation functions, along with Hardy–Rogers-type(HR-type) contractions, as referenced [18]- [21].

Altun et al. [2,3] contributed significant insights into best proximity point consequences for proximal contractions, extending these results to interpolative proximal contractions. Shazad et al. [28] presented results regarding Cbp points, while Deep and Betra [5] introduced additional findings in this area. The investigation expanded into proximal F-contraction, where Mondal and Dey [22] established results on Cbp points in complete MS. Shayanpour and Nematizadeh [26] made notable contributions within the realm of complete FMS, subsequently refined by Zhou et al. [33].

The theory of intuitionistic fuzzy set was initiated by Atanassov [4] and after that Alaca et al. [1] characterized the idea of intuitionistic FM-space. Saleem et al. [31] provided integral equation solutions utilizing IFb metric spaces. Nazam et al. [23] investigated generalized interpolative contractions, whereas Hussain et al. [16] delved into FPs in FMS. The analytical exploration of fractional delay differential equations was the primary focus of Naseem et al. [24]. For more important related ideas and detailed concepts, we refer to [6]- [15].

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2. Preliminaries

Definition 2.1 *[26] Let $(Z,\mathbb{A},*)$ be an FMS and $\mathring{C},\mathring{D}\subset Z$. Consider*

$$\mathbb{A}(\mathring{C},\mathring{D},\bar{\xi})=\sup_{\gamma_1\in\mathring{C},\gamma_2\in\mathring{D}}\mathbb{A}(\gamma_1,\gamma_2,\bar{\xi}),\quad \bar{\xi}>0.$$

Then the distance between $\mathring{C}$ and $\mathring{D}$ is called fuzzy distance.

Definition 2.2 *[26] Let $(Z,\mathbb{A},*)$ be an FMS, $\mathring{C},\mathring{D}\subseteq Z$, and $U,G:\mathring{C}\to\mathring{D}$ be two mappings. A point $\gamma\in\mathring{C}$ is called a Cbp point of the mappings U and G, if*

$$\mathbb{A}(\gamma,U\gamma,\bar{\xi})=\mathbb{A}(\mathring{C},\mathring{D},\bar{\xi})=\mathbb{A}(\gamma,G\gamma,\bar{\xi}).$$

Definition 2.3 *[26] Let $(Z,\mathbb{A},*)$ be an FMS, $\mathring{C},\mathring{D}\subseteq Z$, and $U,G:\mathring{C}\to\mathring{D}$ be two mappings are said to be commute proximally if*

$$\mathbb{A}(\gamma_1,U\gamma,\bar{\xi})=\mathbb{A}(\mathring{C},\mathring{D},\bar{\xi})=\mathbb{A}(\gamma_2,G\gamma,\bar{\xi}),\quad\forall\bar{\xi}>0,$$

then $U\gamma_2=G\gamma_1$, where $\gamma,\gamma_1,\gamma_2\in\mathring{C}$.

Definition 2.4 *[26] Let $(Z,\mathbb{A},*)$ be an FMS, $\mathring{C},\mathring{D}\subseteq Z$, and $U,G:\mathring{C}\to\mathring{D}$ be the mappings then U dominate G proximally if*

$$\mathbb{A}(\gamma_1,Uh_1,\bar{\xi})=\mathbb{A}(\mathring{C},\mathring{D},\bar{\xi})=\mathbb{A}(b_1,Gh_2,\bar{\xi}),$$
$$\mathbb{A}(\gamma_2,Uh_1,\bar{\xi})=\mathbb{A}(\mathring{C},\mathring{D},\bar{\xi})=\mathbb{A}(b_2,Gh_2,\bar{\xi}),$$

$\forall\ \bar{\xi}>0$. If, $\exists\ \alpha'\in(0,1)$ such that,

$$\mathbb{A}(\gamma_1,\gamma_2,\alpha'\bar{\xi})\geq\mathbb{A}(b_1,b_2,\bar{\xi}),$$

where $\gamma_1,\gamma_2,b_1,b_2$ and $h_1,h_2\in\mathring{C}$.

Definition 2.5 *[33] Let $\mathcal{L}$ represent the collection of pairs (J,S), where J and S are functions defined on $(0,1]\to\mathbb{R}$ and satisfy the specified properties outlined below:*

(1) $S(\gamma)>J(\gamma)$ for any $\gamma\in(0,1)$;

(2) J is non-decreasing;

(3) $\lim_{\gamma\to T^-}\inf S(\gamma)>\lim_{s\to T^-}\inf J(\gamma)$ for any $0<T^-<1$;

(4) if $\gamma\in(0,1)$ is s.t $S(\gamma)\geq J(1)$ then $\gamma=1$.

Lemma 2.1 *[25] Suppose $J:(0,1]\to\mathbb{R}$. Then the below conditions are mutually equivalent:*

(I) For any $\epsilon\in(0,1)$, $\lim_{\bar{\xi}\to\epsilon^-}\inf J(\bar{\xi})>-\infty$,

(II) $\inf_{\bar{\xi}>\epsilon}J(\bar{\xi})>-\infty\ \forall\ \epsilon$, where $0<\epsilon<1$,

(III) $\lim_{n\to\infty}J(\bar{\xi}_n)=-\infty$ implies that $\lim_{n\to\infty}\bar{\xi}_n=1$.

3. Main Results

Definition 3.1 **([13])** *Let $(Z,\mathbb{A},\mathbb{V},*,\diamond)$ be an IFMS and $\mathring{C},\mathring{D}\subset Z$. Consider*

$$\mathbb{A}(\mathring{C},\mathring{D},\bar{\xi})=\sup_{\gamma_1\in\mathring{C},\gamma_2\in\mathring{D}}\mathbb{A}(\gamma_1,\gamma_2,\bar{\xi}),$$

$$\mathbb{V}(\mathring{C},\mathring{D},\bar{\xi})=1-\sup_{\gamma_1\in\mathring{C},\gamma_2\in\mathring{D}}\mathbb{V}(\gamma_1,\gamma_2,\bar{\xi}),\quad\bar{\xi}>0.$$

Then the distance between $\mathring{C}$ and $\mathring{D}$ is called intuitionistic fuzzy distance.

Definition 3.2 ([13]) *Let $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be an IFMS and $\mathring{C}, \mathring{D} \subset Z$. The following sets defined by us:*

$$\mathring{C}_0 = \{\gamma_1 \in \mathring{C} : \exists\ \gamma_2 \in \mathring{D}\ s.t.\ \forall\ \bar{\xi} > 0, \mathbb{A}(\gamma_1, \gamma_2, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi})\},$$

$$\mathring{D}_0 = \{\gamma_2 \in \mathring{D} : \exists\ \gamma_1 \in \mathring{C}\ s.t.\ \forall\ \bar{\xi} > 0, \mathbb{A}(\gamma_1, \gamma_2, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi})\},$$
$$\mathring{C}_1 = \{\gamma_1 \in \mathring{C} : \exists\ \gamma_2 \in \mathring{D}\ s.t.\ \forall\ \bar{\xi} > 0, \mathbb{V}(\gamma_1, \gamma_2, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi})\},$$
$$\mathring{D}_1 = \{\gamma_2 \in \mathring{D} : \exists\ \gamma_1 \in \mathring{C}\ s.t.\ \forall\ \bar{\xi} > 0, \mathbb{V}(\gamma_1, \gamma_2, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi})\}.$$

Definition 3.3 ([13]) *Let $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be an IFMS, $\mathring{C}, \mathring{D} \subseteq Z$, and $U, G : \mathring{C} \to \mathring{D}$ be two mappings. A point $\gamma \in \mathring{C}$ is called a Cbp point of the mappings U and G, if*

$$\mathbb{A}(\gamma, U\gamma, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma, G\gamma, \bar{\xi}),$$

$$\mathbb{V}(\gamma, U\gamma, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma, G\gamma, \bar{\xi}).$$

Definition 3.4 ([13]) *Let $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be an IFMS, $\mathring{C}, \mathring{D} \subseteq Z$, and $U, G : \mathring{C} \to \mathring{D}$ be two mappings are said to be commute proximally if*

$$\mathbb{A}(\gamma_1, U\gamma, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_2, G\gamma, \bar{\xi}),$$

$$\mathbb{V}(\gamma_1, U\gamma, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_2, G\gamma, \bar{\xi}), \quad \forall\ \bar{\xi} > 0,$$

then $U\gamma_2 = G\gamma_1$, where $\gamma, \gamma_1, \gamma_2 \in \mathring{C}$.

Definition 3.5 ([13]) *Let $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be an IFMS, $\mathring{C}, \mathring{D} \subseteq Z$, and $U, G : \mathring{C} \to \mathring{D}$ be two mappings then U is to dominate G proximally if*

$$\mathbb{A}(\gamma_1, Uh_1, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(b_1, Gh_2, \bar{\xi}),$$
$$\mathbb{A}(\gamma_2, Uh_1, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(b_2, Gh_2, \bar{\xi})$$

and

$$\mathbb{V}(\gamma_1, Uh_1, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(b_1, Gh_2, \bar{\xi}),$$
$$\mathbb{V}(\gamma_2, Uh_1, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(b_2, Gh_2, \bar{\xi}),$$

$\forall\ \bar{\xi} > 0$, then $\exists\ \alpha' \in (0, 1)$ such that,

$$\mathbb{A}(\gamma_1, \gamma_2, \alpha'\bar{\xi}) \geq \mathbb{A}(b_1, b_2, \bar{\xi}),\ \text{and}\ \mathbb{V}(\gamma_1, \gamma_2, \alpha'\bar{\xi}) \leq \mathbb{V}(b_1, b_2, \bar{\xi}),$$

where $\gamma_1, \gamma_2, b_1, b_2$ and $h_1, h_2 \in \mathring{C}$.

Definition 3.6 *Let $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be a IFMS. The mappings $U, G : \mathring{C} \to \mathring{D}$ are called fuzzy (S, J)-K-type proximal contractions if*

$$\begin{aligned} \mathbb{A}(\gamma_1, Gu_1, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) &= \mathbb{A}(\check{e}_1, Uu_1, \bar{\xi}) \\ \mathbb{A}(\gamma_2, Gu_2, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) &= \mathbb{A}(\check{e}_2, Uu_2, \bar{\xi}) \\ J(\mathbb{A}(\gamma_1, \gamma_2, \bar{\xi})) \geq S\Big((\mathbb{A}(\check{e}_1, \gamma_1, \bar{\xi}))^{\alpha'} &(\mathbb{A}(\check{e}_2, \gamma_2, \bar{\xi}))^{1-\alpha'}\Big) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \mathbb{V}(\gamma_1, Gu_1, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) &= \mathbb{V}(\check{e}_1, Uu_1, \bar{\xi}) \\ \mathbb{V}(\gamma_2, Gu_2, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) &= \mathbb{V}(\check{e}_2, Uu_2, \bar{\xi}) \\ J\left(\mathbb{V}(\gamma_1, \gamma_2, \bar{\xi})\right) \leq S\Big((\mathbb{V}(\check{e}_1, \gamma_1, \bar{\xi}))^{\alpha'} &(\mathbb{V}(\check{e}_2, \gamma_2, \bar{\xi}))^{1-\alpha'}\Big), \end{aligned} \tag{3.2}$$

$\forall\ \gamma_1, \gamma_2, \check{e}_1, \check{e}_2, u_1, u_2 \in \mathring{C}$.

Example 3.1 Let $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be a complete IFMS with $\mathbb{A}(u, \check{n}, \bar{\xi}) = e^{\frac{-|u-\check{n}|}{\xi}}$ and $\mathbb{V}(u, \check{n}, \bar{\xi}) = 1 - e^{\frac{-|u-\check{n}|}{\xi}}$. Let $\mathring{D} = \{1, 3, 5, 7, 9, 11\}$ and $\mathring{C} = \{0, 2, 4, 6, 8, 10\}$. Define mappings $G, U : \mathring{C} \to \mathring{D}$ as

$$U(0) = 3, U(2) = 5, U(4) = 7, U(6) = 3, U(8) = 9, U(10) = 11,$$
$$G(0) = 3, G(2) = 1, G(4) = 9, G(6) = 7, G(8) = 5, G(10) = 11.$$

Also, $\mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = e^{-\frac{1}{\xi}}$ and $\mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = 1 - e^{-\frac{1}{\xi}}$, $\mathring{C}_0 = \mathring{C} = \mathring{C}_1$, $\mathring{D}_0 = \mathring{D} = \mathring{D}_1$. Then clearly $G(\mathring{C}_0) \subseteq \mathring{D}_0$, $U(\mathring{C}_0) \subseteq \mathring{D}_0$, $G(\mathring{C}_1) \subseteq \mathring{D}_1$, $U(\mathring{C}_1) \subseteq \mathring{D}_1$.
Define the functions $S, J : (0, 1] \to (-\infty, \infty)$ by

$$S(\bar{\xi}) = \begin{cases} \frac{1}{\ln \bar{\xi}^2} & \text{if } 0 < \bar{\xi} < 1 \\ 2 & \text{if } \bar{\xi} = 1 \end{cases},$$
$$J(\bar{\xi}) = \begin{cases} \frac{1}{\ln \bar{\xi}} & \text{if } 0 < \bar{\xi} < 1 \\ 1 & \text{if } \bar{\xi} = 1 \end{cases}.\mathbb{A}(0, G2, 1) \quad = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(4, U2, 1),$$

Since, G and U are intuitionistic fuzzy (S, J)- K-type proximal in IFMS. Let $\gamma_1 = 0$, $\gamma_2 = 8$, $\check{e}_1 = 4$, $\check{e}_2 = 6$, $u_1 = 2$, $u_2 = 4$, and $\bar{\xi} = 1$.

$$\mathbb{A}(8, G4, 1) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(6, U4, 1)$$

and

$$\mathbb{V}(8, G4, 1) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(6, U4, 1),$$
$$\mathbb{V}(0, G2, 1) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(4, U2, 1).$$

Then, there exists a number $\lambda = 0.2$ such that

$$\mathbb{A}(\gamma_1, \gamma_2, \lambda\bar{\xi}) \geq \Big((\mathbb{A}(\check{e}_1, \gamma_1, \bar{\xi}))^{\alpha'} * (\mathbb{A}(\check{e}_2, \gamma_2, \bar{\xi}))^{1-\alpha'} \Big)$$
$$\mathbb{A}(0, 8, (0.2)1) \geq \Big((\mathbb{A}(4, 2, 1))^{\frac{1}{2}} * (\mathbb{A}(6, 8, 1))^{\frac{1}{2}} \Big)$$

and

$$\mathbb{V}(\gamma_1, \gamma_2, \lambda\bar{\xi}) \leq S\Big((\mathbb{V}(\check{e}_1, \gamma_1, \bar{\xi}))^{\alpha'} \diamond \mathbb{V}(\check{e}_2, \gamma_2, 1)^{1-\alpha'} \Big)$$
$$\mathbb{V}(0, 8, (0.2)1) \leq S\Big((\mathbb{V}(4, 2, 1))^{\frac{1}{2}} \diamond \mathbb{V}(6, 8, 1)^{\frac{1}{2}} \Big).$$

This leads to a contradiction. Therefore, G and U do not exhibit the properties of a fuzzy K-type proximal.

Lemma 3.1 ([13]) *The mappings $U, G : \mathring{C} \to \mathring{D}$ satisfying Equations (3.1-3.2). Assume $\{\gamma_n\}$ is a sequence such that*

$$\lim_{n\to\infty} \mathbb{A}(\gamma_n, \gamma_{n+1}, \bar{\xi}) = 1, \ \textit{and} \ \lim_{n\to\infty} \mathbb{V}(\gamma_n, \gamma_{n+1}, \bar{\xi}) = 0 \ \textit{for any} \ \epsilon > 0.$$

If the functions $S, J : (0, 1] \to \mathbb{R}$ with

$$\limsup_{\bar{\xi}\to\epsilon+} S(\bar{\xi}) < J(\epsilon+), \ \textit{and} \ \lim_{\bar{\xi}\to\epsilon+} (1 - \sup S(\bar{\xi})) > J(\epsilon+).$$

Then $\{\gamma_n\}$ is a Cauchy sequence.

Theorem 3.1 *Let $\mathring{C}, \mathring{D} \subseteq (Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ is a complete IFMS such that $\mathring{D}$ is AC w.r.t $\mathring{C}$. Also, assume that $\lim_{\bar{\xi}\to\infty} \mathbb{A}(\gamma_1, \gamma_2, \bar{\xi}) = 1$, $\lim_{\bar{\xi}\to\infty} \mathbb{V}(\gamma_1, \gamma_2, \bar{\xi}) = 0$ and $\mathring{C}_0, \mathring{D}_0, \mathring{C}_1, \mathring{D}_1 \neq \emptyset$. Let $G, U : \mathring{C} \to \mathring{D}$ satisfy the following circumstance:*

(i) U dominates G and is intuitionistic fuzzy (S,J)- K-type proximal;

(ii) G and U are compact proximal and continue;

(iii) J is a non-decreasing and $\lim_{\bar{\xi}\to\epsilon+} S(\bar{\xi}) > J(\epsilon+)$ for any $\epsilon > 0$;

(iv) $G(\mathring{C}_0) \subseteq \mathring{D}_0$, $G(\mathring{C}_0) \subseteq U(\mathring{C}_0)$ and $G(\mathring{C}_1) \subseteq \mathring{D}_1$, $G(\mathring{C}_1) \subseteq U(\mathring{C}_1)$.

Then, U and G have a unique $u \in \mathring{C}$ s.t

$$\mathbb{A}(u, Uu, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \quad and \quad \mathbb{A}(u, Gu, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}),$$

also,

$$\mathbb{V}(u, Uu, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) \quad and \quad \mathbb{V}(u, Gu, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}).$$

Proof: *Let $u_0 \in \mathring{C}_0, \mathring{C}_1$. Since $G(\mathring{C}_0) \subseteq U(\mathring{C}_0)$ and $G(\mathring{C}_1) \subseteq U(\mathring{C}_1)$ then, $\exists$ an element $u_1 \in \mathring{C}_0, \mathring{C}_1$ such that*

$$Gu_0 = Uu_1.$$

Also, $\exists$ an element $u_2 \in \mathring{C}_0, \mathring{C}_1$ such that

$$Gu_1 = Uu_2.$$

This process of iteration we get a sequence $\{u_n\} \in \mathring{C}_0, \mathring{C}_1$ such that

$$Gu_{n-1} = Uu_n, \forall n \in \mathbb{N}.$$

Similarly, $\exists$ an element γ_n in $\mathring{C}_0, \mathring{C}_1$ such that

$$\mathbb{A}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{V}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}).$$

Further, choosing u_n and γ_n s.t

$$\begin{aligned} \mathbb{A}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) &= \mathbb{A}(\gamma_n, U(u_{n+1}), \bar{\xi}), \\ \mathbb{A}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) &= \mathbb{A}(\gamma_{n-1}, U(u_n), \bar{\xi}) \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_n, U(u_{n+1}), \bar{\xi}), \\ \mathbb{V}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_{n-1}, U(u_n), \bar{\xi}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{A}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_{n-1}, U(u_n), \bar{\xi}), \\ \mathbb{V}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_{n-1}, U(u_n), \bar{\xi}). \end{aligned} \tag{3.3}$$

If, $\exists$ some $n \in \mathbb{N}$ s.t $\gamma_n = \gamma_{n-1}$, then from Equation (3.3), γ_n is a Cbp point of G and U. On the other side, if $\gamma_{n-1} \neq \gamma_n$ $\forall$ $n \in \mathbb{N}$ then,

$$\begin{aligned} \mathbb{A}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_n, U(u_n), \bar{\xi}), \\ \mathbb{A}(\gamma_n, G(u_n), \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_{n-1}, U(u_{n-1}), \bar{\xi}) \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_n, U(u_n), \bar{\xi}), \\ \mathbb{V}(\gamma_n, G(u_n), \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_{n-1}, U(u_{n-1}), \bar{\xi}). \end{aligned}$$

Thus, from Equation (3.1-3.2) we have,

$$\begin{aligned} J(\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi})) \geq S\Big((\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi}))^{\alpha'} (\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}\Big) \text{ and} \\ J(\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi})) \leq S\Big((\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi}))^{\alpha'} (\mathbb{V}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}\Big), \end{aligned} \tag{3.4}$$

$\forall\ \gamma_{n-1}, u_n, \gamma_n, \gamma_{n+1}, u_{n+1} \in \mathring{C}$.
Since, $S(\bar{\xi}) > J(\bar{\xi})$, by using Equation (3.4) we get,

$$J(\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi})) > J\Big((\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi}))^{\alpha'}(\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}\Big)$$

and

$$J(\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi})) < J\Big((\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi}))^{\alpha'}(\mathbb{V}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}\Big).$$

Now, using the property of J we have,

$$\mathbb{A}(\gamma_{n+1}, \gamma_n, \lambda\bar{\xi}) > (\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi}))^{\alpha'}(\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}$$

and

$$\mathbb{V}(\gamma_{n+1}, \gamma_n, \lambda\bar{\xi}) < (\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi}))^{\alpha'}(\mathbb{V}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}.$$

This implies that

$$(\mathbb{A}(\gamma_{n+1}, \gamma_n, \lambda\bar{\xi}))^{1-\alpha'} > (\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}$$

and

$$(\mathbb{V}(\gamma_{n+1}, \gamma_n, \lambda\bar{\xi}))^{1-\alpha'} < (\mathbb{V}(\gamma_n, \gamma_{n-1}, \bar{\xi}))^{1-\alpha'}.$$

Let $\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi}) = \Pi_n$, $\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi}) = \Pi'_n$ we have,

$$J(\Pi_n) \geq S\left((\Pi_n)^{\alpha'}(A_{n-1})^{1-\alpha'}\right) > J\left(((\Pi_n)^{\alpha'}(A_{n-1})^{1-\alpha'})\right)$$

and

$$J(\Pi'_n) \leq S\left((\Pi'_n)^{\alpha'}(\Pi'_{n-1})^{1-\alpha'}\right) < J\left(((\Pi'_n)^{\alpha'}(\Pi'_{n-1})^{1-\alpha'})\right).$$

Therefore $\{\Pi_n\}$ is non-decreasing and converges to $\Pi \geq 1$. For this using Equation (3.4) we have,

$$J(\epsilon+) = \lim_{n\to\infty} J(\Pi_n) \geq \lim_{n\to\infty} S\left((\Pi_n)^{\alpha'}(\Pi_{n-1})^{1-\alpha'}\right) \geq \lim_{t\to\Pi^+} \inf S(t).$$

Similarly, $\Pi'_n < \Pi'_{n-1}$. Therefore $\{\Pi'_n\}$ is non increasing and it converges to $\Pi' \leq 1$. We show that $\Pi' = 0$ we have,

$$J(\epsilon+) = \lim_{n\to\infty} J(\Pi'_n) \leq \lim_{n\to\infty} S\left((\Pi'_n)^{\alpha'}(\Pi'_{n-1})^{1-\alpha'}\right) \leq \lim_{t\to\Pi^+} \inf S(t).$$

which comes contradicts to condition (iii). Hence, $\Pi = 1$, and $\Pi' = 0$ and

$$\lim_{n\to\infty} \mathbb{A}(\gamma_n, \gamma_{n+1}, \bar{\xi}) = 1 \text{ and } \lim_{n\to\infty} \mathbb{V}(\gamma_n, \gamma_{n+1}, \bar{\xi}) = 0.$$

From Lemma (3.1) and by condition (iii), $\{\gamma_n\}$ is a Cauchy sequence.
Also, $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ is a complete IFMS and $\mathring{C} \subseteq Z$. Since, $G(\mathring{C}_0) \subseteq \mathring{D}_0, G(\mathring{C}_1) \subseteq \mathring{D}_1$ $\exists$ an element γ^* in $\mathring{C}$ such that

$$\lim_{n\to\infty} \mathbb{A}(\gamma_n, \gamma^*) = 0 \text{ and } \lim_{n\to\infty} \mathbb{V}(\gamma_n, \gamma^*) = 1.$$

Moreover,

$$\mathbb{A}(\gamma^*, G(u_n), \bar{\xi}) \geq \mathbb{A}(\gamma^*, \gamma_n, \bar{\xi}) * \mathbb{A}(\gamma_n, G(u_n), \bar{\xi}),$$
$$\mathbb{A}(\gamma^*, U(u_n), \bar{\xi}) \geq \mathbb{A}(\gamma^*, \gamma_n, \bar{\xi}) * \mathbb{A}(\gamma_n, U(u_n), \bar{\xi})$$

and

$$\mathbb{V}(\gamma^*, G(u_n), \bar{\xi}) \leq \mathbb{V}(\gamma^*, \gamma_n, \bar{\xi}) \diamond \mathbb{V}(\gamma_n, G(u_n), \bar{\xi}),$$
$$\mathbb{V}(\gamma^*, U(u_n), \bar{\xi}) \leq \mathbb{V}(\gamma^*, \gamma_n, \bar{\xi}) \diamond \mathbb{V}(\gamma_n, U(u_n), \bar{\xi}).$$

Therefore,

$$\mathbb{A}(\gamma^*, U(u_n), \bar{\xi}) \to \mathbb{A}(\gamma^*, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{A}(\gamma^*, G(u_n), \bar{\xi}) \to \mathbb{A}(\gamma^*, \mathring{D}, \bar{\xi}) \text{ and}$$
$$\mathbb{V}(\gamma^*, U(u_n), \bar{\xi}) \to \mathbb{V}(\gamma^*, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{V}(\gamma^*, G(u_n), \bar{\xi}) \to \mathbb{V}(\gamma^*, \mathring{D}, \bar{\xi}) \text{ as } n \to \infty.$$

As G and U are compact proximal, $U\gamma^*$ and $G\gamma^*$ are same. Further $\mathring{D}$ is AC w.r.t $\mathring{C}$, $\exists$ a sub-sequence $\{U(u_{n_{\bar{\xi}}})\}$ of $\{U(u_n)\}$ and $\{G(u_{n_{\bar{\xi}}})\}$ of $\{G(u_n)\}$ such that

$$U(u_{n_{\bar{\xi}}}) \to \hat{e}^* \in \mathring{D} \quad \text{and} \quad G(u_{n_{\bar{\xi}}}) \to \hat{e}^* \in \mathring{D} \quad \text{as} \quad \bar{\xi} \to \infty.$$

Now,

$$\mathbb{A}(\hat{e}^*, G(u_{n_{\bar{\xi}}}), \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{A}\left(\hat{e}^*, Uu_{n_{\bar{\xi}}}, \bar{\xi}\right) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}), \tag{3.5}$$

$$\mathbb{V}(\hat{e}^*, G(u_{n_{\bar{\xi}}}), \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{V}\left(\hat{e}^*, Uu_{n_{\bar{\xi}}}, \bar{\xi}\right) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}). \tag{3.6}$$

We get,

$$\mathbb{A}(\hat{e}^*, \gamma^*, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{V}(\hat{e}^*, \gamma^*, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}).$$

Since, $\gamma^* \in \mathring{C}_0, \mathring{C}_1$, so $G(\gamma^*) \in G(\mathring{C}_0) \subseteq \mathring{D}_0$, $G(\gamma^*) \in G(\mathring{C}_1) \subseteq \mathring{D}_1$, $U(\gamma^*) \in U(\mathring{C}_0) \subseteq \mathring{D}_0$, $U(\gamma^*) \in U(\mathring{C}_1) \subseteq \mathring{D}_1$ then, $\exists\ x \in \mathring{C}_0$ and $x' \in \mathring{C}_1$ such that

$$\begin{aligned} J(\gamma^*, G(\gamma^*), \bar{\xi}) &= J(x, G(\gamma^*), \bar{\xi}) = J(\mathring{C}, \mathring{D}, \bar{\xi}), \\ J(\gamma^*, U(\gamma^*), \bar{\xi}) &= J(x, U(\gamma^*), \bar{\xi}) = J(\mathring{C}, \mathring{D}, \bar{\xi}) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} J(\gamma^*, G(\gamma^*), \bar{\xi}) &= J(x', G(\gamma^*), \bar{\xi}) = J(\mathring{C}, \mathring{D}, \bar{\xi}), \\ J(\gamma^*, U(\gamma^*), \bar{\xi}) &= J(x', U(\gamma^*), \bar{\xi}) = J(\mathring{C}, \mathring{D}, \bar{\xi}). \end{aligned} \tag{3.8}$$

Now, by considering Equations (3.7-3.8) and (3.1-3.2) we have,

$$\begin{aligned} J\left(\mathbb{A}(\gamma^*, x, \bar{\xi})\right) &\geq S\Big(\mathbb{A}(\gamma^*, x, \bar{\xi})^{\alpha'} \mathbb{A}(\gamma^*, x, \bar{\xi})^{1-\alpha'}\Big), \\ &\geq S(\mathbb{A}(\gamma^*, x, \bar{\xi})) > \mathbb{A}(\gamma^*, x, \bar{\xi}) \end{aligned}$$

and

$$\begin{aligned} J\left(\mathbb{V}(\gamma^*, x', \bar{\xi})\right) &\leq S\Big(\mathbb{V}(\gamma^*, x', \bar{\xi})^{\alpha'} \mathbb{V}(\gamma^*, x', \bar{\xi})^{1-\alpha'}\Big), \\ &\leq S(\mathbb{V}(\gamma^*, x', \bar{\xi})) < \mathbb{V}(\gamma^*, x', \bar{\xi}). \end{aligned}$$

We know that J is a non-decreasing function, we get

$$\mathbb{A}(\gamma^*, x, \alpha'\bar{\xi}) \geq \mathbb{A}(\gamma^*, x, \bar{\xi}) > \mathbb{A}(\gamma^*, x, \bar{\xi}),$$

$$\mathbb{V}(\gamma^*, x', \alpha'\bar{\xi}) \leq \mathbb{V}(\gamma^*, x', \bar{\xi}) > \mathbb{V}(\gamma^*, x', \bar{\xi}).$$

This implies γ^* and x, x' are same. Therefore, from Equation (3.3) we have

$$\mathbb{A}(\gamma^*, U(\gamma^*), \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma^*, G(\gamma^*), \bar{\xi})$$

and

$$\mathbb{V}(\gamma^*, U(\gamma^*), \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma^*, G(\gamma^*), \bar{\xi}).$$

This shows that the γ^* is a Cbp point of U and G.

□

Corollary 3.1 *Let $\mathring{C}, \mathring{D} \subseteq (Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ be a complete IFMS such that $\mathring{D}$ is AC w.r.t $\mathring{C}$. Suppose that $\lim_{\bar{\xi}\to\infty} \mathbb{A}(\gamma_1, \gamma_2, \bar{\xi}) = 1$, $\lim_{\bar{\xi}\to\infty} \mathbb{V}(\gamma_1, \gamma_2, \bar{\xi}) = 0$ and $\mathring{C}_0, \mathring{D}_0, \mathring{C}_1, \mathring{D}_1 \neq \emptyset$. Let $G, U : \mathring{C} \to \mathring{D}$ be mappings satisfying the following circumstance:*

(i) U *dominates* G *and* $IF_{(S,J)}$*- K-type proximal;*

(ii) G *and* U *are compact proximal and continues;*

(iii) J *is non-decreasing and* $\{J(\bar{\xi}_n)\}$ *and* $\{S(\bar{\xi}_n)\}$ *are convergent sequences s.t* $\lim_{n\to\infty} J(\bar{\xi}_n) = \lim_{n\to\infty} S(\bar{\xi}_n)$*, then* $\lim_{n\to\infty} \bar{\xi}_n = 1$*;*

(iv) $G(\mathring{C}_0) \subseteq \mathring{D}_0$*,* $G(\mathring{C}_0) \subseteq U(\mathring{C}_0)$ *and* $G(\mathring{C}_1) \subseteq \mathring{D}_1$*,* $G(\mathring{C}_1) \subseteq U(C_1)$*.*

Then, U *and* G *have a unique* $u \in \mathring{C}$ *s.t*

$$\mathbb{A}(u, Uu, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}), \quad \mathbb{A}(u, Gu, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \text{ and}$$
$$\mathbb{V}(u, Uu, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}), \quad \mathbb{V}(u, Gu, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}).$$

Definition 3.7 *Let* $\mathring{C}, \mathring{D} \subseteq (Z, \mathbb{A}, \mathbb{Q}, *, \diamond)$*. The mappings* $U : \mathring{C} \to \mathring{D}$ *and* $G : \mathring{C} \to \mathring{D}$ *are called* $IF_(S, J)$*-HR's type proximal contraction if*

$$\begin{aligned}
&\mathbb{A}(\gamma_1, Gu_1, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\check{e}_1, Uu_1, \bar{\xi}) \\
&\mathbb{A}(\gamma_2, Gu_2, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\check{e}_2, Uu_2, \bar{\xi}) \\
&J\big(\mathbb{A}(\gamma_1, \gamma_2, \bar{\xi})\big) \leq S((\mathbb{A}(\check{e}_1, \check{e}_2, \bar{\xi}))^{\alpha'} (\mathbb{A}(\check{e}_1, \gamma_1, \bar{\xi}))^{\beta'} (\mathbb{A}(\check{e}_2, \gamma_2, \bar{\xi}))^{\gamma'} \\
&\big(\mathbb{A}(\check{e}_1, \gamma_2, \bar{\xi})^{\delta'} \mathbb{A}(\check{e}_2, \gamma_1, \bar{\xi})^{1-\alpha'-\beta'-\gamma'}\big))
\end{aligned} \tag{3.9}$$

$\forall\ \gamma_1, \gamma_2, \check{e}_1, \check{e}_2, u_1, u_2 \in \mathring{C}$.

Example 3.2 Let $(Z, \mathbb{A}, \mathbb{Q}, *, \diamond)$ be a complete IFMS. In addition to Example (3.1), the mappings G and U are $IF_(S, J)$-HR-type proximal in IFMS. If we take, $\lambda = 0.2$, $\alpha' = 0.01, \beta' = 0.02, \gamma' = 0.03, \delta' = 0.04$, then G and U are not intuitionistic fuzzy HR-type proximal.

Theorem 3.2 *Let* $\mathring{C}, \mathring{D} \subseteq (Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ *is a complete IFMS such that* $\mathring{D}$ *is AC w.r.t* $\mathring{C}$*. Also, assume that* $\lim_{\bar{\xi}\to\infty} \mathbb{A}(\gamma_1, \gamma_2, \bar{\xi}) = 1$*, and* $\lim_{\bar{\xi}\to\infty} \mathbb{V}(\gamma_1, \gamma_2, \bar{\xi}) = 0$*. Let* $G : \mathring{C} \to \mathring{D}$ *and* $U : \mathring{C} \to \mathring{D}$*, satisfied the following conditions:*

(i) U *dominates* G *and* $IF_{(J,S)}$ *- HR-type proximal;*

(ii) G *and* U *are compact proximal and continuous;*

(iii) J *is non-decreasing function and* $\limsup_{\bar{\xi}\to\varepsilon+} S(\bar{\xi}) < J(\varepsilon)$ *for any* $\varepsilon > 0$*;*

(iv) $G(\mathring{C}_0) \subset \mathring{D}_0$ *and* $G(\mathring{C}_0) \subset U(\mathring{C}_0)$ *also* $G(\mathring{C}_1) \subset \mathring{D}_1$ *and* $G(\mathring{C}_1) \subset U(\mathring{C}_1)$ *where* $\mathring{C}_0, \mathring{D}_0, \mathring{C}_1, \mathring{D}_1 \neq \emptyset$*.*

Then, U *and* G *have a unique element* $u \in \mathring{C}$ *such that*

$$\mathbb{A}(u, Uu) = \mathbb{A}(\mathring{C}, \mathring{D}) \quad \text{and} \quad \mathbb{A}(u, Gu) = \mathbb{A}(\mathring{C}, \mathring{D}),$$

$$\mathbb{V}(u, Uu) = \mathbb{V}(\mathring{C}, \mathring{D}) \quad \text{and} \quad \mathbb{V}(u, Gu) = \mathbb{V}(\mathring{C}, \mathring{D}).$$

Proof: *Let* $u_0 \in \mathring{C}_0, \mathring{C}_1$*. Since* $G(\mathring{C}_0) \subset U(\mathring{C}_0)$ *and* $G(\mathring{C}_1) \subset U(\mathring{C}_1)$ *then* $\exists$*, an element* $u_1 \in \mathring{C}_0, \mathring{C}_1$ *such that*

$$Gu_0 = Uu_1.$$

Also, we have $G(\mathring{C}_0) \subset U(\mathring{C}_0)$ *and* $G(\mathring{C}_1) \subset U(\mathring{C}_1)$*,* $\exists$ *an element* $u_2 \in \mathring{C}_0$ *such that*

$$Gu_1 = Uu_2.$$

This process of iteration we get a sequence $\{u_n\} \subset \mathring{C}_0$ *such that*

$$Gu_{n-1} = Uu_n, \forall n \in \mathbb{N}$$

Since $G(\mathring{C}_0) \subset \mathring{D}_0$ *and* $G(\mathring{C}_1) \subset \mathring{D}_1$, $\exists$ *an element* $\gamma_n \in \mathring{C}_0, \mathring{C}_1$ *such that*

$$\mathbb{A}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \quad and \quad \mathbb{V}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}).$$

Further, choice of u_n *and* γ_n *we have,*

$$\begin{aligned}\mathbb{A}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_n, Gu_n, \bar{\xi}),\\ \mathbb{A}(\gamma_n, U(u_{n+1}), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_{n-1}, U(u_n), \bar{\xi})\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_n, Gu_n, \bar{\xi}),\\ \mathbb{V}(\gamma_n, U(u_{n+1}), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_{n-1}, U(u_n), \bar{\xi}).\end{aligned}$$

Also,

$$\begin{aligned}\mathbb{A}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}) = \mathbb{A}(\gamma_{n-1}, U(u_n), \bar{\xi}),\ and\\ \mathbb{V}(\gamma_n, Gu_n, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}) = \mathbb{V}(\gamma_{n-1}, U(u_n), \bar{\xi}).\end{aligned} \tag{3.10}$$

If, $\exists$ *some* $n \in \mathbb{N}$ *such that* $\gamma_n = \gamma_{n-1}$, *then by Equation (3.10), we get* γ_n *is a Cbp point of* G *and* U. *Conversely, if* $\gamma_{n-1} \neq \gamma_n$, $\forall\ n \in \mathbb{N}$, *then from Equation (3.10) we have,*

$$\begin{aligned}\mathbb{A}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_n, U(u_n), \bar{\xi}),\\ \mathbb{A}(\gamma_n, G(u_n), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma_{n-1}, U(u_{n-1}), \bar{\xi})\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}(\gamma_{n+1}, G(u_{n+1}), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_n, U(u_n), \bar{\xi}),\\ \mathbb{V}(\gamma_n, G(u_n), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma_{n-1}, U(u_{n-1}), \bar{\xi}).\end{aligned}$$

Thus, from Equation (3.10), we have

$$\begin{aligned}J\left(\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi})\right) &\geq S\Big(\left(\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi})\right)^{\alpha'} \left(\mathbb{A}(\gamma_n, \gamma_{n+1}, \bar{\xi})\right)^{\beta'} \left(\mathbb{A}(\gamma_{n-1}, \gamma_n, \bar{\xi})\right)^{\gamma'}\\ &\qquad \left(\mathbb{A}(\gamma_n, \gamma_n, \bar{\xi})\right)^{\delta'} \left(\mathbb{A}(\gamma_{n-1}, \gamma_{n+1}, \bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'}\Big)\\ &\geq S\Big(\left(\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi})\right)^{\alpha'} \left(\mathbb{A}(\gamma_n, \gamma_{n+1}, \bar{\xi})\right)^{\beta'} \left(\mathbb{A}(\gamma_{n-1}, \gamma_n, \bar{\xi})\right)^{\gamma'}\\ &\qquad \left(\mathbb{A}(\gamma_{n-1}, \gamma_{n+1}, \bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'}\Big),\end{aligned} \tag{3.11}$$

$$\begin{aligned}J\left(\mathbb{V}(\gamma_{n+1}, \gamma_n, \bar{\xi})\right) &\leq S\Big(\left(\mathbb{V}(\gamma_n, \gamma_{n-1}, \bar{\xi})\right)^{\alpha'} \left(\mathbb{V}(\gamma_n, \gamma_{n+1}, \bar{\xi})\right)^{\beta'} \left(\mathbb{V}(\gamma_{n-1}, \gamma_n, \bar{\xi})\right)^{\gamma'}\\ &\qquad \left(\mathbb{V}(\gamma_n, \gamma_n, \bar{\xi})\right)^{\delta'} \left(\mathbb{V}(\gamma_{n-1}, \gamma_{n+1}, \bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'}\Big)\\ &\leq S\Big(\left(\mathbb{V}(\gamma_n, \gamma_{n-1}, \bar{\xi})\right)^{\alpha'} \left(\mathbb{V}(\gamma_n, \gamma_{n+1}, \bar{\xi})\right)^{\beta'} \left(\mathbb{V}(\gamma_{n-1}, \gamma_n, \bar{\xi})\right)^{\gamma'}\\ &\qquad \left(\mathbb{V}(\gamma_{n-1}, \gamma_{n+1}, \bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'}\Big)\end{aligned} \tag{3.12}$$

$\forall\ \gamma_{n-1}, u_n, \gamma_n, \gamma_{n+1}, u_{n+1} \in \mathring{C}$. *In Equation (3.11-3.12), using* $S(\bar{\xi}) > J(\bar{\xi})\ \forall\ \bar{\xi} > 0$, *we get,*

$$\begin{aligned}J\left(\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi})\right) > J\Big(&\left(\mathbb{A}(\gamma_n, \gamma_{n-1}, \bar{\xi})\right)^{\alpha'} \left(\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi})\right)^{\beta'}\\ &\left(\mathbb{A}(\gamma_{n+1}, \gamma_n, \bar{\xi})\right)^{\gamma'} \left(\mathbb{A}(\gamma_{n-1}, \gamma_{n+1}, \bar{\xi})^{1-\alpha'-\beta'-\gamma'-\delta'}\right)\Big)\end{aligned}$$

and

$$J\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right) < J\Big(\left(\mathbb{V}(\gamma_n,\gamma_{n-1},\bar{\xi})\right)^{\alpha'}\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\beta'}$$
$$\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\gamma'}\left(\mathbb{V}(\gamma_{n-1},\gamma_{n+1},\bar{\xi})^{1-\alpha'-\beta'-\gamma'-\delta'}\right)\Big).$$

Thus, J is a non-decreasing function and we have

$$\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi}) > \Big(\left(\mathbb{A}(\gamma_n,\gamma_{n-1},\bar{\xi})\right)^{\alpha'}\left(\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\beta'}\left(\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\gamma'}$$
$$(\left(\mathbb{A}(\gamma_{n-1},\gamma_{n+1},\bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'})\Big),$$
$$\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi}) > \Big(\left(\mathbb{A}(\gamma_n,\gamma_{n-1},\bar{\xi})\right)^{\alpha'}\left(\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\beta'}\left(\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\gamma'}$$
$$(\left(\mathbb{A}(\gamma_{n-1},\gamma_n,\bar{\xi})\mathbb{A}(\gamma_n,\gamma_{n+1},\bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'})\Big),$$
$$\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi}) > \left(\mathbb{A}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\beta'-\gamma'-\delta'}\left(\mathbb{A}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\alpha'-\delta'}$$

and

$$\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi}) < \Big(\left(\mathbb{V}(\gamma_n,\gamma_{n-1},\bar{\xi})\right)^{\alpha'}\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\beta'}\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\gamma'}$$
$$(\left(\mathbb{V}(\gamma_{n-1},\gamma_{n+1},\bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'})\Big),$$
$$\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi}) < \Big(\left(\mathbb{V}(\gamma_n,\gamma_{n-1},\bar{\xi})\right)^{\alpha'}\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\beta'}\left(\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})\right)^{\gamma'}$$
$$(\left(\mathbb{V}(\gamma_{n-1},\gamma_n,\bar{\xi})\mathbb{V}(\gamma_n,\gamma_{n+1},\bar{\xi})\right)^{1-\alpha'-\beta'-\gamma'-\delta'})\Big),$$
$$\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi}) < \left(\mathbb{V}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\beta'-\gamma'-\delta'}\left(\mathbb{V}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\alpha'-\delta'}.$$

This implies that

$$\mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi}) > \left(\mathbb{A}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\beta'-\gamma'-\delta'}\left(\mathbb{A}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\alpha'-\delta'}$$

and

$$\mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi}) < \left(\mathbb{V}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\beta'-\gamma'-\delta'}\left(\mathbb{V}(\gamma_{n-1},\gamma_n,\bar{\xi})\right)^{1-\alpha'-\delta'}.$$

Let $\Pi_n = \mathbb{A}(\gamma_{n+1},\gamma_n,\bar{\xi})$ and $\Pi'_n = \mathbb{V}(\gamma_{n+1},\gamma_n,\bar{\xi})$, we have

$$J(\Pi_n) > S\left(\Pi_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi_n\right)^{1-\alpha'-\delta'} > J\left(\left(\Pi_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi_n\right)^{1-\alpha'-\delta'}\right)$$

and

$$J(\Pi'_n) < S\left(\Pi'_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi'_n\right)^{1-\alpha'-\delta'} < J\left(\left(\Pi'_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi'_n\right)^{1-\alpha'-\delta'}\right).$$

Suppose that $\Pi_n > \Pi_{n-1}$ and $\Pi'_n < \Pi'_{n-1}$ for some $n \geq 1$. Since J is non-decreasing, then by (3.13-3.14), we get,

$$(\Pi_n) > \left(\Pi_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi_n\right)^{1-\alpha'-\delta'},$$

and

$$(\Pi'_n) < \left(\Pi'_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi'_n\right)^{1-\alpha'-\delta'}.$$

This is not possible. Hence, we obtain $\Pi_n > \Pi_{n-1}$, $\Pi'_n < \Pi'_{n-1}$ $\forall\ n \geq 1$. Thus, it converges to some element $\Pi \geq 1$ and $\Pi' \leq 0$. We show that $\Pi = 1, \Pi' = 0$. On the contrary, let $\Pi > 1$ and $\Pi' < 0$, from Equation (3.13-3.14) we have,

$$J(\epsilon^+) = \lim_{n\to\infty} J(\Pi_n) \geq \lim_{n\to\infty} S\left(\left(\Pi_{n-1}\right)^{1-\beta'-\gamma'-\delta'}\left(\Pi_n\right)^{1-\alpha'-\delta'}\right) \geq \lim_{t\to\Pi+}\inf S(t)$$

and

$$J(\epsilon^+) = \lim_{n\to\infty} J(\Pi'_n) \leq \lim_{n\to\infty} S\left(\left(\Pi'_{n-1}\right)^{1-\beta'-\gamma'-\delta'} \left(\Pi'_n\right)^{1-\alpha'-\delta'} \right) \leq \lim_{t\to\Pi+} \inf S(t).$$

This comes contraction to condition (iii), hence $\Pi = 1, \Pi' = 0$ *and* $\lim_{n\to\infty} \mathbb{A}(\gamma_n, \gamma_{n+1}, \bar{\xi}) = 1$ *and* $\lim_{n\to\infty} \mathbb{V}(\gamma_n, \gamma_{n+1}, \bar{\xi}) = 0$. *By Lemma (3.1) and condition (iii),* $\{\gamma_n\}$ *is a Cauchy sequence.*
Since $(Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ *is a complete IFMS and* $\mathring{C}, \mathring{D} \subseteq Z$. *Since* $G(\mathring{C}_0) \subseteq \mathring{D}_0$, $G(\mathring{C}_1) \subseteq \mathring{D}_1$, $\exists$ *an element* $\gamma^* \in \mathring{C}, \mathring{D}$ *such that* $\lim_{n\to\infty} \mathbb{A}(\gamma_n, \gamma^*, \bar{\xi}) = 1$, *and* $\lim_{n\to\infty} \mathbb{V}(\gamma_n, \gamma^*, \bar{\xi}) = 0$.
Moreover,

$$\mathbb{A}(\gamma^*, G(u_n), \bar{\xi}) \geq \mathbb{A}(\gamma^*, \gamma_n, \bar{\xi}) \cdot \mathbb{A}(\gamma_n, G(u_n), \bar{\xi}),$$

$$\mathbb{V}(\gamma^*, G(u_n), \bar{\xi}) \leq \mathbb{V}(\gamma^*, \gamma_n, \bar{\xi}) \cdot \mathbb{A}(\gamma_n, G(u_n), \bar{\xi}).$$

Also,

$$\mathbb{A}(\gamma^*, U(u_n), \bar{\xi}) \geq \mathbb{A}(\gamma^*, \gamma_n) \cdot \mathbb{A}(\gamma_n, U(u_n)),$$
$$\mathbb{V}(\gamma^*, U(u_n), \bar{\xi}) \leq \mathbb{V}(\gamma^*, \gamma_n) \cdot \mathbb{V}(\gamma_n, U(u_n)).$$

Therefore,

$$\mathbb{A}(\gamma^*, U(u_n), \bar{\xi}) \to \mathbb{A}(\gamma^*, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{A}(\gamma^*, G(u_n), \bar{\xi}) \to \mathbb{A}(\gamma^*, \mathring{D}, \bar{\xi}),$$
$$\mathbb{V}(\gamma^*, U(u_n), \bar{\xi}) \to \mathbb{V}(\gamma^*, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{V}(\gamma^*, G(u_n), \bar{\xi}) \to \mathbb{V}(\gamma^*, \mathring{D}, \bar{\xi}).$$

As G *and* U *are compact proximal,* $U\gamma^*$ *and* $G\gamma^*$ *are same. Since* $\mathring{D}$ *is AC w.r.t* $\mathring{C}$, $\exists$ *a sub-sequence* $U(u_{n_{\bar{\xi}}})$ *of* $U(u_n)$ *and* $G(u_{n_{\bar{\xi}}})$ *of* $G(u_n)$ *such that* $U(u_{n_{\bar{\xi}}}) \to \check{e}^*$ *and* $G(u_{n_{\bar{\xi}}}) \to \check{e}^*$ *as* $\bar{\xi} \to \infty$. *Therefore,*

$$\begin{aligned} \mathbb{A}(\check{e}^*, G(u_{n_{\bar{\xi}}}), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}), \\ \mathbb{A}(\check{e}^*, U(u_{n_{\bar{\xi}}}), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \mathbb{V}(\check{e}^*, G(u_{n_{\bar{\xi}}}), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}), \\ \mathbb{V}(\check{e}^*, U(u_{n_{\bar{\xi}}}), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}), \end{aligned} \tag{3.14}$$

Taking, $\bar{\xi} \to \infty$ *in Equations (3.13-3.14) we have,*

$$\mathbb{A}(\check{e}^*, \gamma^*, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) \text{ and } \mathbb{V}(\check{e}^*, \gamma^*, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}). \tag{3.15}$$

Since, $\gamma^* \in \mathring{C}_0, \mathring{C}_1$, *so* $G(\gamma^*) \in G(\mathring{C}_0) \subseteq \mathring{D}_0$, $G(\gamma^*) \in G(\mathring{C}_1) \subseteq \mathring{D}_1$ *and* $\exists x \in \mathring{C}_0$ *and* $x' \in \mathring{C}_1$. *Similarly,* $U(\gamma^*) \in U(\mathring{C}_0) \subseteq \mathring{D}_0$ *such that*

$$\begin{aligned} \mathbb{A}(\gamma^*, G(\gamma^*), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma^*, U(\gamma^*), \bar{\xi}), \\ \mathbb{A}(x, G(\gamma^*), \bar{\xi}) &= \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(x, U(\gamma^*), \bar{\xi}), \\ \mathbb{V}(\gamma^*, G(\gamma^*), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma^*, U(\gamma^*), \bar{\xi}), \\ \mathbb{V}(x', G(\gamma^*), \bar{\xi}) &= \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(x', U(\gamma^*), \bar{\xi}), \end{aligned} \tag{3.16}$$

Now, bearing in mind Equations (3.15), (3.16), from (3.11), we get,

$$J\left(\mathbb{A}(\gamma^*, x, \bar{\xi})\right) \geq S\left(\ \mathbb{A}(\gamma^*, x, \bar{\xi})^{\alpha'} \mathbb{A}(\gamma^*, x, \bar{\xi})^{\beta'} \ \right) \geq S(\mathbb{A}(\gamma^*, x, \bar{\xi})) > \mathbb{A}(\gamma^*, x, \bar{\xi}),$$

$$J\left(\mathbb{V}(\gamma^*, x', \bar{\xi})\right) \leq S\left(\ \mathbb{V}(\gamma^*, x', \bar{\xi})^{\alpha'} \mathbb{V}(\gamma^*, x', \bar{\xi})^{\beta'} \ \right) \leq S(\mathbb{V}(\gamma^*, x', \bar{\xi})) < \mathbb{V}(\gamma^*, x', \bar{\xi}).$$

Since J *is a non-decreasing function, we have,*

$$\mathbb{A}(\gamma^*, x, \alpha'\bar{\xi}) \geq \mathbb{A}(\gamma^*, x, \bar{\xi}) > \mathbb{A}(\gamma^*, x, \bar{\xi}),$$

$$\mathbb{V}(\gamma^*, x', \alpha'\bar{\xi}) \leq \mathbb{V}(\gamma^*, x', \bar{\xi}) < \mathbb{V}(\gamma^*, x', \bar{\xi}).$$

This implies γ^* and x, x' are same. Finally, from Equation (3.12-3.13) we get,

$$\mathbb{A}(\gamma^*, U(\gamma^*), \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{A}(\gamma^*, G(\gamma^*), \bar{\xi}),$$

$$\mathbb{V}(\gamma^*, U(\gamma^*), \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}) = \mathbb{V}(\gamma^*, G(\gamma^*), \bar{\xi}).$$

This shows that the point γ^* is a Cbp point of the pair of mappings G and U. □

Corollary 3.2 *Let $\mathring{C}, \mathring{D} \subseteq (Z, \mathbb{A}, \mathbb{V}, *, \diamond)$ such that $\mathring{D}$ is AC w.r.t $\mathring{C}$. Also, assume that $\lim_{\bar{\xi}\to\infty} \mathbb{A}(\gamma_1, \gamma_2, \bar{\xi}) = 1$ and $\lim_{\bar{\xi}\to\infty} \mathbb{V}(\gamma_1, \gamma_2, \bar{\xi}) = 0$. Let $G : \mathring{C} \to \mathring{D}$ and $U : \mathring{C} \to \mathring{D}$ satisfy the following circumstances:*

(i) U dominates G and $IF_{(S,J)}$- HR-type proximal;

(ii) G and U are compact and continuous;

(iii) J is non-decreasing and $J(\bar{\xi}_n)$ and $S(\bar{\xi}_n)$ are convergent sequences such that $\lim_{n\to\infty} J(\bar{\xi}_n) = \lim_{n\to\infty} S(\bar{\xi}_n)$, then $\lim_{n\to\infty} \bar{\xi}_n = 1$;

(iv) $G(\mathring{C}_0) \subseteq \mathring{D}_0$ and $G(\mathring{C}_0) \subseteq U(\mathring{C}_0)$ also $G(\mathring{C}_1) \subseteq \mathring{D}_1$ and $G(\mathring{C}_1) \subseteq U(\mathring{C}_1)$ where $\mathring{C}_0, \mathring{D}_0, \mathring{C}_1, \mathring{D}_1 \neq \emptyset$.

Then, U and G have a unique element $u \in \mathring{C}$ such that

$$\mathbb{A}(u, Uu, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}), \quad \mathbb{A}(u, Gu, \bar{\xi}) = \mathbb{A}(\mathring{C}, \mathring{D}, \bar{\xi}),$$

$$\mathbb{V}(u, Uu, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}), \quad \mathbb{V}(u, Gu, \bar{\xi}) = \mathbb{V}(\mathring{C}, \mathring{D}, \bar{\xi}).$$

4. List of Abberivations

Cbp –common best proximity
HR –Hardy Rogers
K-type – Kannan -type
FMS –fuzzy metric space
IFMS –intuitionistic fuzzy metric space
MIFMS – modified intuitionistic fuzzy metric space
$IF_{(J,S)}$ – intuitionistic fuzzy (J, S)

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