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Nonsplit Pendant Domination In Graphs

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ABSTRACT: For a graph G, a dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The least cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. A pendant dominating set S of a graph G is a nonsplit pendant dominating set if the induced graph $< V - S >$ is connected. The nonsplit pendant domination number $\gamma_{nsp}(G)$ is the minimum cardinality of a nonsplit pendant dominating set. In this paper many bounds on $\gamma_{nsp}(G)$ are obtained and exact values for some standard graphs are found. Also, its relationship with other parameters has been investigated.

Key Words: Domination, nonsplit domination, pendant domination number, nonsplit pendant domination.

Contents

1. Introduction

Let $G = (V, E)$ be any graph with $|V(G)| = n$ and $|E(G)| = m$ edges. Then n, m are respectively called the order and the size of the graph G. For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N[v]$ containing v and all the vertices u adjacent to v. Let S be any subset of V, then the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \bigcup S$.

The minimum and maximum of the degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph G is said to be regular if $\delta(G) = \Delta(G)$. A vertex v of a graph G is called a *cut vertex* if its removal increases the number of components. A *bridge* or *cut edge* of a graph is an edge whose removal increases the number of components. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. The corona of two disjoint graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $\mid V(G_1) \mid$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

The graph containing no cycle is called a tree. A unicyclic graph is a connected graph that contains exactly one cycle. A spanning subgraph of a graph G is a subgraph that includes all the vertices of G, but may not include all the edges. A complete bi-partite graph $K_{1,3}$ is a tree called as *claw*. Any graph containing no subgraph isomorphic to $K_{1,3}$ is called a claw-free graph. The n-Pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge. A Barbell graph $B(p, s)$ is the graph obtained by connecting n-copies of a complete graph K_p by a bridge. The Helm graph H_n is the graph with $2n + 1$ vertices obtained from an n-wheel graph by adjoining a pendant edge at each node of the cycle. The basic terminology of graph theory refer [1], [2] [3].

A subset S of $V(G)$ is a dominating set of G if each vertex $u \in V - S$ is adjacent to a vertex in S. The least cardinality of a dominating set in G is called the domination number of G and is usually

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denoted by $\gamma(G)$ [4]. A dominating set D is said to be connected dominating set, if the induced sub graph $< D >$ is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set.
A dominating set S of a graph $G = (V, E)$ is a nonsplit dominating set if the induced graph $< V - S >$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ is the minimum cardinality of a nonsplit domination set. For more details about nonsplit domination refer [6].
A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The least cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. The more details about the pendant domination parameter refer [8], [9], [10].

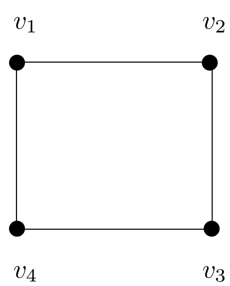

Figure 1. A Cycle Graph C_4

2. Nonsplit Pendant Domination

Definition 2.1 *Let $G = (V, E)$ be a graph. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The least cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. A pendant dominating set S of a graph G is a nonsplit pendant dominating set if the induced graph $< V - S >$ is connected. The nonsplit pendant domination number $\gamma_{nsp}(G)$ is the minimum cardinality of a nonsplit pendant dominating set.*

Remark 2.1 The parameter γ_{nsp} is not defined for a totally disconnected graph. Therefore, throughout this paper, by a graph we assume that G has at least two edges.

Example 2.1 The nonsplit pendant dominating set of cycle graph as shown in the Figure 1 are $\{v_1, v_2\}$, therefore $\gamma_{nsp}(G) = 2$

Remark 2.2

1. For any graph G, $\gamma_{nsp}(G) \leq\mid V(G) \mid$.
2. we observe that $2 \leq \gamma_{nsp}(G) \leq n - 1$.

Example 2.2

1. $\gamma_{nsp}(K_n) = 2$.
2. $\gamma_{nsp}(K_{m,n}) = 2$ where $2 \leq m \leq n$.
3. $\gamma_{pe}(K_n) = \gamma_{nsp}(K_n) = 2$.
4. $\gamma(K_{m,n}) = \gamma_{pe}(K_{m,n}) = \gamma_{nsp}(K_{m,n}) = 2$, $\forall m, n \geq 2$.
5. $\gamma_{nsp}(K_{1,n}) = n$.
6. $\gamma_{nsp}(B_{m,n}) = m + 1$.

3. Main Results

Theorem 3.1 *Let C_n be a cycle graph with n vertices. Then $\gamma_{nsp}(C_n) = n - 2$, for all $n \geq 3$.*

Proof: Let C_n be a cycle graph with n vertices and $V(C_n) = \{v_1, v_2, ..., v_n\}$ such that $deg(v_i) = 2$ for all $i = 1, 2, 3, .., n$. Let the set $D = \{v_1, v_2, ..., v_{n-2}\}$ is the minimum pendant dominating set and $< V - D >= P_2$. Clearly induced graph of $< V - D >$ is connected. This implies that the set D is a nonsplit pendant dominating set. Hence $\gamma_{nsp}(C_n) \leq n - 2$. Suppose $\gamma_{nsp}(C_n) < n - 2$. Let D be a $\gamma_{nsp}(C_n)-$ set of C_n. Then $V - D$ contains 3 vertices. Let $X = \{r, s, t\}$ such that $< V - D >= X$. Clearly $< X >= P_3$, then there exist a vertex v in $V - D$ then v can be dominated by no vertex of D. Hence $\gamma_{nsp}(C_n) = n - 2$

Proposition 3.1 *Let $G \cong C_n$ $(n \geq 4)$ and let H be a spanning subgraph of G such that $< H >$ contains a path of length at least 3. Then $\gamma_{nsp}(G) = \gamma_{nsp}(H)$.*

Proof: We have $\gamma_{nsp}(G) = n - 2$ and since $< H >$ contains a path of length 3 say $(v_k, v_l, v_m, ..., v_n)$, $V(G) - \{v_l, v_m\}$ is a minimum nonsplit pendant dominating set of H and also $\gamma_{nsp}(H) = n - 2$.

Corollary 3.1 *For any cycle C_n and any $v \in V(C_n)$, then*

$$\gamma_{nsp}(C_n - v) = \begin{cases} 2, & if\ n = 4, \\ 3, & if\ n = 5, \\ n - 3, & if\ n \geq 6. \end{cases}$$

Proof: Follows from Proposition 3.1

Theorem 3.2 *Let P_n be a path graph with n vertices. Then $\gamma_{nsp}(P_n) = n - 2$, for all $n \geq 4$.*

Theorem 3.3 *Let G be a Barbell graph. Then $\gamma_{nsp}(G) = n + 1$.*

Theorem 3.4 *Let G be a Pan graph. Then $\gamma_{nsp}(G) = n - 1$.*

Theorem 3.5 *Let W_n be a wheel graph with n vertices. Then $\gamma_{nsp}(W_n) = 2$, for all $n \geq 4$.*

Proof: Consider any wheel graph W_n with n vertices formed by sum of complete graph with one vertex v_1 and cycle graph with $n - 1$ vertices are $v_2, v_3, , , v_{n-1}, v_n$ that is the wheel W_n can be defined as the graph $K_1 + C_{n-1}$. Here the vertex v_1 be the apex vertex and degree is $n - 1$ so it is internal vertex to all other vertices and $d_G(v_2) = d_G(v_3) = ... = d_G(v_n) = 3$. The set $S = \{v_1, v_n\}$ will be the minimum nonsplit pendant dominating set and $< V - S >$ is connected. This implies that the set S is a minimum nonsplit pendant dominating set. Hence $\gamma_{nsp}(W_n) = 2$. Suppose $\gamma_{nsp}(G) \geq 3$ the set $S = \{v_1, v_n, v_i\}$ where v_i is the vertex neither adjacent to v_n nor adjacent to v_2. Then clearly $< V - S >$ is disconnected. Therefore $\gamma_{nsp}(W_n) = 2$.

Theorem 3.6 *For any helm graph H_n, $n \geq 3$, then $\gamma_{nsp}(H_n) = n + 1$.*

Proof: The helm graph H_n is obtained from wheel W_n by attaching a pendant edge of vertex to each of its $n - 1$ rim vertex. So it contains wheel W_n and $n - 1$ pendant vertices, it has $2n + 1$ vertices. The set $S = \{v_i, u\}$ where v_i is the pendant vertices of helm graph and the vertex u is adjacent to any one of the vertex v_i. Therefore, the set S is minimum nonsplit pendant dominating set. Hence $\gamma_{nsp}(G) =\mid S \mid= n + 1$.

Theorem 3.7 *Let G be a Soifer graph, then $\gamma_{nsp}(G) = n - 6$ if $n = 9$.*

Proof: Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ be a vertex set of Soifer graph G with 9 vertices and 20 edges. Let a non empty sunset $D = \{v_1, v_2, v_8\}$ is a nonsplit pendant dominating set of G and induced subgraph of D contains a pendant vertex and $< V - D >$ is connected. If the set $\mid D \mid\geq 4$ then the set D is not a minimal nonsplit pendant dominating set. Therefore, $\gamma_{nsp} =\mid D \mid= n - 6$.

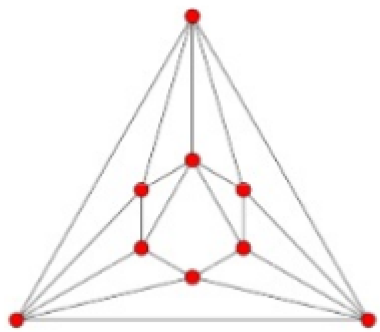

Figure 1: Soifer Graph

Theorem 3.8 *For any graph G of order n, we have $2 \leq \gamma(G) \leq \gamma_{ns}(G) \leq \gamma_{pe}(G) \leq \gamma_{nsp}(G) \leq n$. Equality holds if and only if $G \cong C_4, K_{m,n}$.*

Proposition 3.2 *For any connected graph G, $\gamma_{nsp}(G) \leq n-1$. Further equality holds if and only if G is a star.*

Proof: Every set $S \subseteq V(G)$ with $\mid S \mid = n-1$ is a nonsplit pendant dominating set of G and so $\gamma_{nsp}(G) \leq n-1$.
If G is a star, clearly $\gamma_{nsp}(G) = n-1$. Suppose $\gamma_{nsp}(G) = n-1$. If G is not a star, the G has an edge say $e = uv$ such that both u and v are non pendant vertices. Now $V(G) - \{u, v\}$ is a nonsplit pendant dominating set of G and so $\gamma_{nsp}(G) \leq n-2$ which is a contradiction. Hence G is a star.

Corollary 3.2 *For any graph G, $\gamma_{nsp}(G) = n-1$ if and only if G is a star.*

Theorem 3.9 *Let G be a unicyclic graph with cycle C_n and $\delta(G) = 1$. Then $\gamma_{nsp}(\overline{G}) = \chi(G) = 2$ if and only if n is even.*

Proof: If $\chi(G) = 2$ then n is even. Conversely, suppose that n is even. If G has two pendant vertices u, v with two supports u_1, v_1 and $u_1 \neq v_1$, then for any other vertex in C_n, $\{u, x\}$ is a minimum nonsplit pendant dominating of $\overline{G}$. If $u_1 = v_1$, then $\{u, x\}$ is a minimum nonsplit pendant dominating set of $\overline{G}$.

Theorem 3.10 *Let G be any connected graph and $\chi(G)$ be the chromatic number of G. Then $\gamma_{nsp}(G) + \chi(G) \leq n + \Delta(G)$ and equality holds if $G \cong K_3$*

Proof: Since $\gamma_{nsp}(G) \leq n-1$ and $\chi(G) \leq 1 + \Delta(G)$ we have $\gamma_{nsp}(G) + \chi(G) \leq n + \Delta(G)$.

Theorem 3.11 *For a non trivial tree T, $\gamma_{nsp}(T) \geq \Delta(T)$ and $\gamma_{nsp}(T) = \Delta(T)$ if and only if $T \cong$ star.*

Proof: Since T is a tree, T has at least $\Delta(T)$ pendant vertices. If $T \cong$ star then $\gamma_{nsp}(T) = \Delta(T)$. If T is not a star the every nonsplit pendant dominating set must contains all the pendent vertices and so $\gamma_{nsp}(T) \geq \Delta(T)$.

Theorem 3.12 *For any tree T not isomorphic to P_2, $\gamma_{nsp}(\overline{T}) = 2 \, or \, 3$.*

Proof: If $diam(T) = 2$, then $T = K_{1,n-1}$. If u is the support vertex and $v_1, v_2, ..., v_{n-1}$ are the pendant vertices then $\{u, v_1, v_2\}$ are nonsplit pendant dominating sets in $\overline{T}$.
If $diam(T) = 3$ and if u, v are the supports and v_1 is the pendant vertex and adjacent to any one of the support vertex then $\{u, v, v_1\}$ is a nonsplit pendant dominating set in $\overline{T}$. If the graph $\overline{T}$ doesnot contain a pendant vertex then $\{v_1, v_2\}$ where the vertices v_1 are v_2 are the adjacent to support vertices in T, then $\{v_1, v_2\}$ is a nonsplit pendant dominating set in $\overline{T}$. If $diam(T) = 4$, let $P = \{u_1, u_2, u_3, u_4, u_5\}$ be the diametrical path in T. Then $\{u_2, u_5\}$ is a nonsplit pendant dominating set in $\overline{T}$. If $diam(T) \geq 5$, let $P = \{u_1, u_2, ..., u_n\}$ $(n \geq 6)$ be the diametrical path in T. Then $\{v_1, v_4\}$ is a nonsplit pendant dominating set in $\overline{T}$. Thus $\gamma_{nsp}(\overline{T}) = 2 \, or \, 3$.

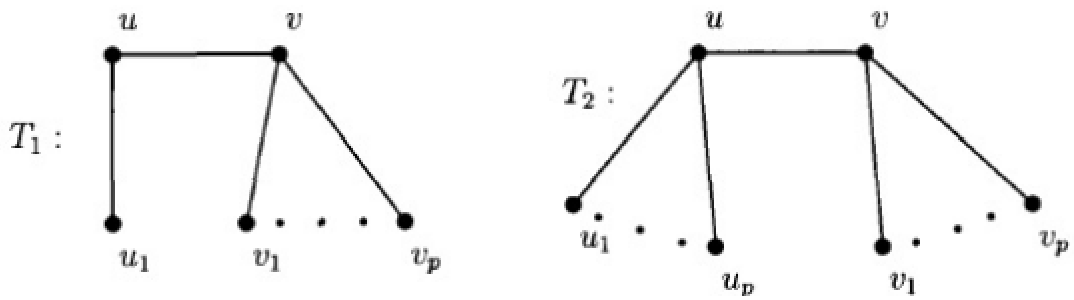

Figure 2: A Simple Graph

Theorem 3.13 *If G is a connected graph with $diam(G) \geq 5$ such that $\overline{G}$ is connected the $\gamma_{nsp}(G) + \gamma_{nsp}(\overline{G}) \leq n+1$ and the bound is sharp.*

Proof: Since $diam(G) \geq 5$, there exist 2 vertices u and v such that $d(u,v) = diam(G) \geq 5$. Let $P = \{u, v_1, ..., v_{n-1}, v\}$ be a diametrical path in G. Every vertex in $V - \{v_1, v_{n-1}\}$ is adjacent to at least one of $\{v_1, v_{n-1}\}$ and at least one of $\{u, v\}$ in $\overline{G}$ so that $\{v_1, v_{n-1}\}$ is a minimum nonsplit pendant dominating set of $\overline{G}$. Thus $\gamma_{nsp}(\overline{G}) = 2$. Also we have $\gamma_{nsp}(G) \leq n-1$. Hence $\gamma_{nsp}(G) + \gamma_{nsp}(\overline{G}) \leq n+1$. If $G \cong C_n$ $(n \geq 4)$, $\gamma_{nsp}(G) = n-2$, $\gamma_{nsp}(\overline{G}) = 3$ so that $\gamma_{nsp}(G) + \gamma_{nsp}(\overline{G}) = n+1$.

Proposition 3.3 *For any tree T,*

$$\gamma_{nsp}(T) = \begin{cases} n-1, & \text{if } diam(T) = 3, \\ n-2, & \text{if } diam(T) \geq 4. \end{cases}$$

Theorem 3.14 *Let T be any tree such that $\overline{T}$ is connected. Then*

$$\gamma_{nsp}(T) + \gamma_{nsp}(\overline{T}) = \begin{cases} n+2, & \text{if and only if } T \cong P_4, T_1, \\ n, & \text{if otherwise.} \end{cases}$$

Proof: By Proposition 3.3, $\gamma_{nsp}(T) = n-1$ if $diam(T) = 3$. Since $\overline{T}$ is connected.
Case(i): $diam(T) = 3$.
If $T \cong P_4$ then $\gamma_{nsp}(\overline{T}) = 3$ and so $\gamma_{nsp}(T) + \gamma_{nsp}(\overline{T}) = n+2$. If $T \cong T_1$ then $\{v, u_1, v_i\}(1 \leq i \leq n)$ is a minimum nonsplit pendant dominating set of $\overline{T_1}$ and so $\gamma_{nsp}(\overline{T_1}) = 3$. Thus $\gamma_{nsp}(T) + \gamma_{nsp}(\overline{T}) = n+2$. If $T \cong T_2$ then $\{u, u_i, v_j\}$ $(1 \leq i \leq k, 1 \leq j \leq m)$ is a minimum nonsplit pendant dominating set in $\overline{T}$ and so $\gamma_{nsp}(\overline{T}) = 2$. Now $\gamma_{nsp}(T) + \gamma_{nsp}(\overline{T}) = n+1$.
Case(ii): $diam(T) = 4$
Let $(v_1, v_2, v_3, v_4, v_5)$ be the diametrical path $\{v_2, v_5\}$ is a minimum nonsplit pendant dominating set of $\overline{T}$ and so $\gamma_{nsp}(\overline{T}) = 2$. Hence $\gamma_{nsp}(T) + \gamma_{nsp}(\overline{T}) = n$.
Case(iii): $diam(T) \geq 5$.
By the Theorem, $\gamma_{nsp}(\overline{T}) = 2$. Hence $\gamma_{nsp}(T) + \gamma_{nsp}(\overline{T}) = n$. Converse is obvious.

Proposition 3.4 *If $G \cong C_n \circ K_1$ $(n \geq 4)$, then $\gamma_{nsp}(G) = n+2$.*

Proposition 3.5 *For any connected graph G, $\gamma_{nsp}(G) + diam(G) \leq 2n-2$ and equality holds if and only if G is a path with 4 vertices.*

Proof: For any connected graph G, $\gamma_{nsp}(G) \leq n-1$ and $diam(G) \leq n-1$ so that $\gamma_{nsp}(G) + diam(G) \leq 2n-2$. Moreover, $diam(G) = n-1$ if and only if G is a path and hence the theorem follows.

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