Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.77882

Comparison of Analytical Methods for Solving Fractional Biological Model Via The (G'/G)-Expansion Method

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ABSTRACT: This study explores exact traveling wave solutions for a time fractional biological population model by applying the (G'/G)-expansion method. The model, which incorporates nonlinear diffusion and memory effects via fractional derivatives, captures the dynamics of population distribution in a spatially extended biological system. By applying a systematic wave transformation, we reduce the governing partial differential equation to an ordinary differential form and construct a broad class of analytical solutions, including hyperbolic, trigonometric, and rational wave structures. The obtained results not only generalize known solutions from prior literature but also yield novel solution families such as kink, lump, and peakon type waves. Comprehensive symbolic computations using `Maple` validate the derived expressions, and graphical illustrations demonstrate the physical relevance and diversity of the wave phenomena. The findings highlight the robustness and versatility of the (G'/G)-expansion method for solving complex nonlinear fractional PDEs in biological and ecological modeling.

Key Words: Time fractional biological population model; The (G'/G)-expansion technique; Caputo's derivative; Exact solutions;fractional calculus;

Contents

1. Introduction

Nonlinear partial differential equations (PDEs), especially of fractional order, serve as powerful tools in modeling various complex physical, biological, and engineering phenomena characterized by memory effects and spatial heterogeneity [1,2,3,4]. Over the past decades, the growing interest in exact and approximate solutions to these models has led to the development of numerous analytical methods that enhance our understanding of wave propagation, diffusion, and reaction mechanisms [5,8,7]. This study focuses on applying the (G'/G)-expansion method to a time-fractional biological population model described by the following equation:

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}(w^2) + \frac{\partial^2}{\partial y^2}(w^2) + f(w)$$

Here, $w = w(x, y, t)$ represents the population density, and $f(w)$ denotes the net population source due to birth and death processes. This equation captures the essential nonlinear diffusion behavior in two spatial dimensions, with a memory effect introduced through the fractional time derivative of order α.Mathematical models like this provide deep insight into population dynamics by simulating nonlinear interactions in biological systems. The application of such models has proven valuable for predicting and analyzing ecological and epidemiological patterns. This method and its variants, including

2020 *Mathematics Subject Classification:* 35R11.
Submitted July 16, 2025. Published January 20, 2026

the improved and generalized (G'/G)-expansion techniques, have been successfully applied to numerous nonlinear models arising in mathematical physics among these, the (G'/G)-expansion method has gained considerable attention due to its simplicity, efficiency, and ability to uncover various wave structures, including solitary, periodic, and kink type solutions [6,9,10,11]. This method has been effectively applied to fractional order models such as the Klein Gordon equation, gas dynamics, and biological population equations, providing closed form solutions that capture essential physical behaviors [12,13,14].Fractional differential equations extend classical models by incorporating derivatives of non integer order, thereby capturing long term memory and hereditary properties in natural systems [15,16]. These have proven vital in studying wave patterns in biological media, nonlinear optics, and fluid dynamics, where standard integer order models fall short [17,18,19,20].To address these systems, researchers have proposed several enhanced analytical techniques. The exp-function method [21], tanh–coth method [22], sine–cosine method [23], and extended rational function approaches [24] offer alternative frameworks to derive novel solutions. Each of these methods contributes to enlarging the set of solvable nonlinear models with precise wave representations [25,26,27].In parallel, Lie symmetry analysis and transformation techniques have been instrumental in reducing PDEs to ordinary differential equations (ODEs), aiding in symmetry based classification and conservation law derivation [28,29,30]. These symmetry techniques, in conjunction with mapping methods [31].The synergy between these mathematical tools applied to fractional biological population models, nonlinear optical systems, and generalized evolution equations has facilitated the discovery of a wide range of exact solutions, validating both the physical insight and computational effectiveness of each method [32]. This study aims to apply the (G'/G)-expansion method to derive new solitary wave solutions of a fractional biological population model. The approach provides a systematic framework to explore the dynamic behavior of population systems influenced by memory effects, which are effectively captured through fractional calculus. By obtaining exact wave structures, the study contributes to a deeper understanding of nonlinear biological processes governed by fractional order dynamics.

2. Description of the (G'/G)Expansion Method

Consider the fractional nonlinear partial differential equation (NLPDE) given by:

$$\phi\left(\omega, D_t^{\alpha}\omega, D_x^{\beta}\omega, D_y^{\gamma}\omega, D_t^{2\alpha}\omega, D_x^{2\beta}\omega, D_y^{2\gamma}\omega, \dots\right) = 0,$$

where $t > 0$, $x \in \mathbb{R}$, and $0 \le \alpha \le 1$. Here, ϕ is a polynomial in $\omega = \omega(x, y, t)$, the unknown function. **Step 1**Apply the wave transformation:

$$\omega(x, y, t) = \omega(\eta), \quad \text{where } \xi = lx + ry + \frac{a_0 t^{\alpha}}{\Gamma(\alpha+1)}$$

Using this transformation, the original NLPDE is reduced to an ordinary differential equation:

$$P\left(Y, kY', \mu Y', k^2 Y'', k\mu Y'', \dots\right) = 0,$$

where primes denote derivatives with respect to η. We assume the solution has the form:

$$Y(\eta) = \sum_{i=1}^{m} a_i \left(\frac{G'}{G}\right)^i + a_0, \quad \text{with } a_m \neq 0,$$

where a_0 and a_i are constants to be determined. The function $G(\eta)$ satisfies
Step 2 According to the $\left(\frac{G'}{G}\right)$-expansion method, the solution to Equation (1.2) is assumed to take the form of a polynomial in $\left(\frac{G'}{G}\right)$:

$$w(\zeta) = \sum_{i=1}^{s} b_i \left(\frac{G'}{G}\right)^i + b_0,$$

In this expression, b_0 and b_i (for $i = 1, 2, \dots, s$) are unknown constants that will be calculated later. The function $G = G(\zeta)$ follows a second-order linear ordinary differential equation:

$$\frac{d^2 G(\zeta)}{d\zeta^2} + \varsigma \frac{dG(\zeta)}{d\zeta} + \sigma G(\zeta) = 0,$$

where ς and σ are real constants.
Step 3 The value of the positive integer s is determined by balancing the highest-order derivative with the most nonlinear term in the equation.
Step 4 After obtaining the constants l, r, n_0, a_0, ς, σ, and b_i, we substitute them into Equation (1.4) to solve for the function $G(\zeta)$. The resulting expression for $G(\zeta)$ is then substituted into Equation (1.3) to obtain the solution to Equation (1.1). The type of solution depends on the sign of the discriminant $\varsigma^2 - 4\sigma$.
Case 1: $\varsigma^2 - 4\sigma > 0$, and $\sigma \neq 0$

When the discriminant is positive, the solution involves hyperbolic functions. The corresponding hyperbolic traveling wave solution is given by:

$$S_1 = \frac{\sqrt{\varsigma^2 - 4\sigma}}{2}\left(\frac{B_1 \sinh\left(\frac{1}{2}\text{Trans}\sqrt{\varsigma^2 - 4\sigma}\right) + B_2 \cosh\left(\frac{1}{2}\text{Trans}\sqrt{\varsigma^2 - 4\sigma}\right)}{B_1 \cosh\left(\frac{1}{2}\text{Trans}\sqrt{\varsigma^2 - 4\sigma}\right) + B_2 \sinh\left(\frac{1}{2}\text{Trans}\sqrt{\varsigma^2 - 4\sigma}\right)}\right) - \frac{\varsigma}{2}.$$

Case 2: $\varsigma^2 - 4\sigma < 0$, and $\sigma \neq 0$ When the discriminant is negative, the solution involves trigonometric functions and takes the form:

$$S_2 = \frac{\sqrt{-(\varsigma^2 - 4\sigma)}}{2}\left(\frac{-B_1 \sin\left(\frac{1}{2}\text{Trans}\sqrt{-(\varsigma^2 - 4\sigma)}\right) + B_2 \cos\left(\frac{1}{2}\text{Trans}\sqrt{-(\varsigma^2 - 4\sigma)}\right)}{B_1 \cos\left(\frac{1}{2}\text{Trans}\sqrt{-(\varsigma^2 - 4\sigma)}\right) + B_2 \sin\left(\frac{1}{2}\text{Trans}\sqrt{-(\varsigma^2 - 4\sigma)}\right)}\right) - \frac{\varsigma}{2}.$$

Case 3: $\varsigma^2 - 4\sigma = 0$, $\varsigma \neq 0$, $\sigma \neq 0$

When the discriminant is zero, a rational function solution arises:

$$S_3 = \frac{1}{2}\sqrt{\varsigma^2 - 4\sigma}\left(\frac{B_2}{B_1 + B_2 \cdot \text{Trans}}\right) - \frac{\varsigma}{2}.$$

3. Solution Method

The fractional biological population model

This model, which describes how a biological population evolves over time with memory effects, is expressed using the following mathematical equation:

$$D_t^{\alpha} w - w_x - w_y - bw + cw^2 = 0, \qquad 0 < \alpha \leq 1 \tag{1}$$

The application of transformation employed on the above partial differential equation (PDE) is:

$$w = w(\xi), \quad \text{where } \xi = lx + ry + \frac{a_0 t^{\alpha}}{\Gamma(\alpha + 1)} \tag{2}$$

By applying the above transformation to equation (1), ODE is obtained, i.e.,

$$(w')v - 2(w')^2 - 2(w'')^2 - 2vw''q^2 - bw + cw^2 = 0 \tag{3}$$

The derivatives are taken with respect to the variable "ξ". Upon doing the integration of the ordinary differential equation (ODE) with respect to the variable ξ, the resulting expression is as follows:

$$\int \left[(w')v - 2(w')^2 - 2(w'')^2 - 2vw''q^2 - bw + cw^2\right] d\xi \tag{4}$$

Assuming the solution of eq. (4) is:

$$w(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i \tag{5}$$

Where $G = G(\xi)$ is a function that satisfies the linear ordinary differential equation of the second order in the form:

$$G'' + \lambda G' + \delta G = 0 \tag{6}$$

By utilizing equations (5) and (6), it can be readily inferred that:

$$w'' = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^{i+2} \cdots \tag{7}$$

$$w' = -na_n \left(\frac{G'}{G}\right)^{n+1} \tag{8}$$

$$w' = n(n+1)a_n \left(\frac{G'}{G}\right)^{n+1} \cdots$$

$$I = \sum_{i=0}^{n} a_i (w(G'))^i \tag{9}$$

Now, by applying the idea of coefficient balancing, it can be observed that $N = 2$. Therefore, the solution to equation (9) is given as:

$$I = b_0 + b_1 \left(\frac{G'}{G}\right) + b_2 \left(\frac{G'}{G}\right)^2 \tag{10}$$

Where the constants b_0, b_1, and b_2 are arbitrary and non-zero. By substituting equation (10) into equation (3), we obtain:

$$\begin{aligned} -4\zeta\Bigg(& 5b_2^2(q^2+l^2)y^6 + 6b_2\left(\frac{3b_2\varsigma}{2} + b_1\right)(q^2+l^2)y^5 \\ &+ \Bigg(\left((4q^2+4l^2)\varsigma^2 + (8q^2+8l^2)\sigma - \frac{c}{4}\right)b_2^2 \\ &+ 3(q^2+l^2)\left(\frac{7b_1\lambda}{2} + b_0\right)b_2 + \frac{3b_1^2(q^2+l^2)}{2}\Bigg)y^4 \\ &- \frac{b_0(b_0c-b)}{4}\Bigg) = 0 \end{aligned} \tag{11}$$

By performing the substitution of equation (11) into equation (3) and subsequently gathering all terms with the same power of $\left(\frac{G'}{G}\right)$, a system of simultaneous algebraic equations can be derived. This system involves the coefficients b_0, b_1, b_2, c, b, G and E. To obtain this system, each coefficient of the resulting polynomial is set to zero. As follows:

$$E_1 = -4\Bigg(3\sigma\left(\sigma(q^2+l^2)b_1 + b_0(q^2+l^2)\varsigma + \frac{\nu}{6}\right)b_2 + b_1\left(\frac{3\sigma\varsigma(q^2+l^2)b_1}{2} + \frac{b_0(q^2+l^2)^2\varsigma^2}{2} + \frac{\nu\varsigma}{4} + b_0(q^2+l^2)\sigma + \frac{b}{4} - \frac{b_0 c}{2}\right)\Bigg)\zeta = 0 \tag{12}$$

$$E_2 = -4\Bigg(3\sigma^2(q^2+l^2)b_2^2 + \left(\frac{15\sigma\varsigma(q^2+l^2)b_1}{2} + 2b_0(q^2+l^2)^2\varsigma^2 + \frac{\nu\varsigma}{2} + 4b_0(q^2+l^2)\sigma + \frac{b}{4} - \frac{b_0 c}{2}\right)b_2 + \frac{3b_1}{2}\left(\left(\left(\frac{2q^2}{3}+\frac{2l^2}{3}\right)\varsigma^2 + \left(\frac{4q^2}{3}+\frac{4l^2}{3}\right)\sigma - \frac{c}{6}\right)b_1 + b_0(q^2+l^2)\varsigma + \frac{\nu}{6}\right)\Bigg)\zeta = 0 \tag{13}$$

$$E_3 = -4\Bigg(7\sigma\varsigma(q^2+l^2)b_2^2 + \left(\left(\left(\frac{9q^2}{2}+\frac{9l^2}{2}\right)\varsigma^2 + (9q^2+9l^2)\sigma - \frac{c}{2}\right)b_1 + 5b_0(q^2+l^2)\varsigma + \frac{\nu}{2}\right)b_2 + \left(\frac{5b_1\varsigma}{2} + b_0\right)b_1(q^2+l^2)\Bigg)\zeta = 0 \tag{14}$$

$$E_4 = -4\Bigg(\left((4q^2+4l^2)\varsigma^2 + (8q^2+8l^2)\sigma - \frac{c}{4}\right)b_2^2 + 3\left(\frac{7b_1\varsigma}{2} + b_0\right)(q^2+l^2)b_2 + \frac{3b_1^2(q^2+l^2)}{2}\Bigg)\zeta = 0 \tag{15}$$

$$E_5 = -24\left(\frac{3b_2\varsigma}{2} + b_1\right)(q^2+l^2)b_2\zeta = 0 \tag{16}$$

$$E_6 = -20b_2^2(q^2+l^2)\zeta = 0 \tag{17}$$

By employing the software Maple to solve the given algebraic equations, we are able to get the solution sets. Furthermore, by selecting particular instances, we achieve the subsequent outcomes for the following three cases that are:

- **Case 1**: When $\varsigma^2 - 4\sigma > 0$ and $\sigma \neq 0$, the solution takes the form of hyperbolic functions.
- **Case 2**: When $\varsigma^2 - 4\sigma < 0$ and $\sigma \neq 0$, the solution involves trigonometric functions.
- **Case 3**: When $\varsigma^2 - 4\sigma = 0$, with $\varsigma \neq 0$ and $\sigma \neq 0$,the solution becomes a rational function.

Solution 1

$$[b = b,\ c = c,\ b_1 = 0,\ b_0 = b_0,\ b_2 = 0,\ \varsigma = \varsigma,\ q = q,\ l = l,\ \nu = \nu,\ \sigma = \sigma]$$

Case 1:

$$w_1 = \frac{\left(\begin{array}{c}\left(\sqrt{\varsigma^2-4\sigma}\,B_1 - \varsigma B_2\right)\sinh\left(\frac{(qy+lx+\nu t)\sqrt{\varsigma^2-4\sigma}}{2}\right)\\ +\left(\sqrt{\varsigma^2-4\sigma}\,B_2 - \lambda B_1\right)\cosh\left(\frac{1}{2}(qy+lx+\nu t)\sqrt{\varsigma^2-4\sigma}\right)\end{array}\right)}{2B_2\sinh\left(\frac{1}{2}(qy+lx+\nu t)\sqrt{\varsigma^2-4\sigma}\right) + 2B_1\cosh\left(\frac{1}{2}(qy+lx+\nu t)\sqrt{\varsigma^2-4\sigma}\right)} \tag{18}$$

$$v_1 = \frac{\left(\begin{array}{c} \left(\sqrt{\varsigma^2-4\sigma}\, B_1 - \varsigma B_2\right) \sinh\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2} \right) \\ + \left(\sqrt{\varsigma^2-4\sigma}\, B_2 - \nu B_1\right) \cosh\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2} \right) \end{array} \right)}{2B_2 \sinh\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2} \right) + 2B_1 \cosh\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2} \right)} \tag{19}$$

Case 2:

$$w_2 = \frac{\left(\begin{array}{c} \left(-\sqrt{-\varsigma^2+4\sigma}\, B_1 - \varsigma B_2\right) \sin\left(\frac{(qy+lx+\nu t)\sqrt{-\varsigma^2+4\sigma}}{2} \right) \\ + \left(-\sqrt{-\varsigma^2+4\sigma}\, B_2 - \varsigma B_1\right) \cos\left(\frac{1}{2}(qy+lx+\nu t)\sqrt{-\varsigma^2+4\sigma} \right) \end{array} \right)}{2B_2 \sin\left(\frac{1}{2}(qy+lx+\nu t)\sqrt{-\varsigma^2+4\sigma} \right) + 2B_1 \cos\left(\frac{1}{2}(qy+lx+\nu t)\sqrt{-\varsigma^2+4\sigma} \right)} \tag{20}$$

$$v_2 = \frac{\left(\begin{array}{c} \left(-\sqrt{-\varsigma^2+4\sigma}\, B_1 - \varsigma B_2\right) \sin\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2} \right) \\ + \left(-\sqrt{-\varsigma^2+4\sigma}\, B_2 - \varsigma B_1\right) \cos\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2} \right) \end{array} \right)}{2B_2 \sin\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2} \right) + 2B_1 \cos\left(\frac{\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2} \right)} \tag{21}$$

Case 3:

$$w_3 = \frac{\sqrt{\varsigma^2-4\sigma}\, B_2 - ((qy+lx+\nu t)B_2 + B_1)\,\varsigma}{(2qy+2lx+2\nu t)B_2 + 2B_1} \tag{22}$$

$$v_3 = \frac{\sqrt{\varsigma^2-4\sigma}\, B_2 - \left(\left(qy+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)B_2 + B_1\right)\varsigma}{\left(2qy+2lx+\frac{2\nu t^\alpha}{\Gamma(\alpha+1)}\right)B_2 + 2B_1} \tag{23}$$

Solution 2

$$\begin{aligned} b &= -\frac{12(q^2\sigma^2 b_2^2 + l^2\sigma^2 b_2^2 - 4b_2 q^2\sigma b_0 - 4b_2 l^2\sigma b_0 - 2q^2 b_0^2 - 2l^2 b_0^2)}{b_2}, \\ c &= \frac{4(8q^2\sigma b_2 + 8l^2\sigma b_2 + 3q^2 b_0 + 3l^2 b_0)}{b_2}, \\ b_1 &= 0, \quad b_0 = b_0, \quad b_2 = b_2, \quad \varsigma = 0, \quad q = q, \quad l = l, \quad \nu = 0, \quad \sigma = \sigma \end{aligned}$$

Case 1:

$$w_4 = \frac{\sqrt{-\sigma}\left(B_1 \sinh\left((qy+lx)\sqrt{-\sigma}\right) + B_2 \cosh\left((qy+lx)\sqrt{-\sigma}\right)\right)}{B_1 \cosh\left((qy+lx)\sqrt{-\sigma}\right) + B_2 \sinh\left((qy+lx)\sqrt{-\sigma}\right)} \tag{24}$$

$$v_4 = \frac{\sqrt{-\sigma}\left(B_1 \sinh\left((qy+lx)\sqrt{-\sigma}\right) + B_2 \cosh\left((qy+lx)\sqrt{-\sigma}\right)\right)}{B_1 \cosh\left((qy+lx)\sqrt{-\sigma}\right) + B_2 \sinh\left((qy+lx)\sqrt{-\sigma}\right)} \tag{25}$$

Case 2:

$$w_5 = -\frac{\sqrt{\sigma}\,(B_1 \sin((qy+lx)\sqrt{\sigma}) - B_2 \cos((qy+lx)\sqrt{\sigma}))}{B_1 \cos((qy+lx)\sqrt{\sigma}) + B_2 \sin((qy+lx)\sqrt{\sigma})} \tag{26}$$

$$v_5 = -\frac{\sqrt{\sigma}\,(B_1 \sin((qy+lx)\sqrt{\sigma}) - B_2 \cos((qy+lx)\sqrt{\sigma}))}{B_1 \cos((qy+lx)\sqrt{\sigma}) + B_2 \sin((qy+lx)\sqrt{\sigma})} \tag{27}$$

Case 3:

$$w_6 = \frac{\sqrt{-\sigma}\, B_2}{qy\, B_2 + lx\, B_2 + B_1} \tag{28}$$

$$v_6 = \frac{\sqrt{-\sigma}\, B_2}{qy\, B_2 + lx\, B_2 + B_1} \tag{29}$$

Solution 3

$$\begin{aligned}
b &= \frac{16}{\left(12\sigma b_2^2 + 7b_1^2\right)^2 b_2^3}\big(1026q^2\sigma^4 b_2^8 + 1026l^2\sigma^4 b_2^8 - 81q^2\sigma^3 b_1^2 b_2^6 \\
&\quad - 81l^2\sigma^3 b_1^2 b_2^6 - 414q^2\sigma^2 b_1^4 b_2^4 - 414l^2\sigma^2 b_1^4 b_2^4 \\
&\quad + 29q^2\sigma b_1^6 b_2^2 + 29l^2\sigma b_1^6 b_2^2 + 40q^2 b_1^8 + 40l^2 b_1^8\big), \\
c &= -\frac{2(540q^2\sigma^2 b_2^4 + 540l^2\sigma^2 b_2^4 - 150q^2\sigma b_1^2 b_2^2 - 150l^2\sigma b_1^2 b_2^2 - 89q^2 b_1^4 - 89l^2 b_1^4)}{9(12\sigma b_2^2 + 7b_1^2) b_2^2}, \\
b_1 &= b_1, \quad b_0 = -\frac{126\sigma^2 b_2^4 + 3\sigma b_1^2 b_2^2 - 31b_1^4}{3(12\sigma b_2^2 + 7b_1^2) b_2}, \quad b_2 = b_2, \\
\varsigma &= -\frac{2b_1}{3b_2}, \quad q = q, \quad l = l, \\
\nu &= -\frac{10b_1(12q^2\sigma^4 b_2^4 + 126l^2\sigma^4 b_2^4 + 12q^2\sigma b_1^2 b_2^2 + 12l^2\sigma b_1^2 b_2^2 - 19q^2 b_1^4 - 19l^2 b_1^4)}{3(12\sigma b_2^2 + 7b_1^2) b_2^2}, \\
\mu &= \mu
\end{aligned}$$

Case 1:

$$w_7 = \frac{\left(\frac{\left(-b_2\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}B_2 - B_1 b_1\right)\cosh\left(1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19tb_1^5(q^2+l^2)}{126}+\frac{2t\sigma b_2^2(q^2+l^2)b_1^3}{21}\right)\right)}{108\sigma b_2^4+63b_1^2b_2^2} + \frac{\sinh\left(1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19tb_1^5(q^2+l^2)}{126}+\frac{2t\sigma b_2^2(q^2+l^2)b_1^3}{21}\right)\right)\left(b_2\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}B_1 + B_2 b_1\right)}{108\sigma b_2^4+63b_1^2b_2^2}\right)}{3\left(-B_1\cosh\left(\frac{1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19tb_1^5(q^2+l^2)}{126}+\frac{2t\sigma b_2^2(q^2+l^2)b_1^3}{21}\right)}{108\sigma b_2^4+63b_1^2b_2^2}\right) + \frac{B_2\sinh\left(1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19tb_1^5(q^2+l^2)}{126}+\frac{2t\sigma b_2^2(q^2+l^2)b_1^3}{21}\right)\right)}{108\sigma b_2^4+63b_1^2b_2^2}\right)b_2} \tag{30}$$

$$v_7 = \frac{\left(\frac{\left(-b_2\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}B_2 - B_1 b_1\right)\cosh\left(1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19t^\alpha b_1^5(q^2+l^2)}{126\,\Gamma(\alpha+1)}+\frac{2t^\alpha\sigma b_2^2(q^2+l^2)b_1^3}{21\,\Gamma(\alpha+1)}\right)\right)}{108\sigma b_2^4+63b_1^2b_2^2} + \frac{\sinh\left(1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19t^\alpha b_1^5(q^2+l^2)}{126\,\Gamma(\alpha+1)}+\frac{2t^\alpha\sigma b_2^2(q^2+l^2)b_1^3}{21\,\Gamma(\alpha+1)}\right)\right)\left(b_2\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}B_1 + B_2 b_1\right)}{108\sigma b_2^4+63b_1^2b_2^2}\right)}{3\left(-B_1\cosh\left(\frac{1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19t^\alpha b_1^5(q^2+l^2)}{126\,\Gamma(\alpha+1)}+\frac{2t^\alpha\sigma b_2^2(q^2+l^2)b_1^3}{21\,\Gamma(\alpha+1)}\right)}{108\sigma b_2^4+63b_1^2b_2^2}\right) + \frac{B_2\sinh\left(1260\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}}\left(-\frac{19t^\alpha b_1^5(q^2+l^2)}{126\,\Gamma(\alpha+1)}+\frac{2t^\alpha\sigma b_2^2(q^2+l^2)b_1^3}{21\,\Gamma(\alpha+1)}\right)\right)}{108\sigma b_2^4+63b_1^2b_2^2}\right)b_2} \tag{31}$$

Case 2:

$$w_8 = \frac{\left(\frac{\left(-\sqrt{\frac{9\sigma b_2^2+b_1^2}{b_2^2}} \sin\left(1260\left(t\mu(q^2+l^2)b_1 - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}\right)B_1 b_2}{108\sigma b_2^4+63b_1^2b_2^2} - \frac{\left(\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}} \cos\left(1260\left(t\sigma(q^2+l^2)b_1 - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}\right)B_2 b_2}{108\sigma b_2^4+63b_1^2b_2^2} \right)}{\left(\frac{B_2 \sin\left(1260\left(t\sigma(q^2+l^2)b_1 - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}B_1 b_1}{108\sigma b_2^4+63b_1^2b_2^2} \frac{-B_1 \cos\left(1260\left(t\mu(q^2+l^2)b_1 - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}b_2}{108\sigma b_2^4+63b_1^2b_2^2} \right)} \tag{32}$$

$$v_8 = \frac{\left(\frac{\left(-\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}} \sin\left(1260\left(\frac{t^\alpha\sigma(q^2+l^2)b_1}{\Gamma(\alpha+1)} - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}\right)B_1 b_2}{108\sigma b_2^4+63b_1^2b_2^2} - \frac{\left(\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}} \cos\left(1260\left(\frac{t^\alpha\sigma(q^2+l^2)b_1}{\Gamma(\alpha+1)} - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}\right)B_2 b_2}{108\sigma b_2^4+63b_1^2b_2^2} \right)}{\left(\frac{B_2 \sin\left(1260\left(\frac{t^\alpha\sigma(q^2+l^2)b_1}{\Gamma(\alpha+1)} - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}B_1 b_1}{108\sigma b_2^4+63b_1^2b_2^2} \frac{-B_1 \cos\left(1260\left(\frac{t^\alpha\sigma(q^2+l^2)b_1}{\Gamma(\alpha+1)} - \frac{qy}{35} - \frac{lx}{35}\right)\sigma b_2^4\right)\sqrt{\frac{9\sigma b_2^2-b_1^2}{b_2^2}}b_2}{108\sigma b_2^4+63b_1^2b_2^2} \right)} \tag{33}$$

Case 3:

$$w_9 = \frac{\left(-36\sigma B_2 b_2^5 - 21B_2 b_1^2 b_2^3\right)\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}} + 1260\left(\left(t\sigma(q^2+l^2)b_1 - \frac{qy}{35} - \frac{lx}{35}\right)B_2 - \frac{B_1}{35}\right)\sigma b_2^4 b_1}{3780\left(\left(t\sigma(q^2+l^2)b_1 - \frac{qy}{35} - \frac{lx}{35}\right)B_2 - \frac{B_1}{35}\right)\sigma b_2^4 b_2} \tag{34}$$

$$v_9 = \frac{\left(-36\sigma B_2 b_2^5 - 21B_2 b_1^2 b_2^3\right)\sqrt{\frac{-9\sigma b_2^2+b_1^2}{b_2^2}} + 1260\left(\left(\frac{t^\alpha\sigma(q^2+l^2)b_1}{\Gamma(\alpha+1)} - \frac{qy}{35} - \frac{lx}{35}\right)B_2 - \frac{B_1}{35}\right)\sigma b_2^4 b_1}{3780\left(\left(\frac{t^\alpha\sigma(q^2+l^2)b_1}{\Gamma(\alpha+1)} - \frac{qy}{35} - \frac{lx}{35}\right)B_2 - \frac{B_1}{35}\right)\sigma b_2^4 b_2} \tag{35}$$

Solution 4

$$b = -\frac{\nu(b_1\varsigma - 2b_0)}{b_1}, \quad c = \frac{\nu}{b_1}, \quad b_1 = b_1, \quad b_0 = b_0, \quad b_2 = 0, \tag{3.1}$$

$$\varsigma = \varsigma, \quad q = \sqrt{z^2+1}\,l, \quad l = l, \quad \nu = \nu, \quad \sigma = \sigma \tag{3.2}$$

Case 1:

$$w_{10} = \frac{\left(\left(\sqrt{\varsigma^2-4\sigma}\,B_2 - \varsigma B_1\right)\cosh\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{\varsigma^2-4\sigma}}{2}\right) + \sinh\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{\varsigma^2-4\sigma}}{2}\right)\left(\sqrt{\varsigma^2-4\sigma}\,B_1 - \varsigma B_2\right) \right)}{2B_2\sinh\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{\varsigma^2-4\sigma}}{2}\right) + 2B_1\cosh\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{\varsigma^2-4\sigma}}{2}\right)} \tag{36}$$

$$v_{10} = \frac{\left(\left(\sqrt{\varsigma^2-4\sigma}\,B_2 - \varsigma B_1\right)\cosh\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2}\right) + \sinh\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2}\right)\left(\sqrt{\varsigma^2-4\sigma}\,B_1 - \varsigma B_2\right) \right)}{2B_2\sinh\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2}\right) + 2B_1\cosh\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{\varsigma^2-4\sigma}}{2}\right)} \tag{37}$$

Case 2:

$$w_{11}=\frac{\begin{pmatrix}\left(-\sqrt{-\varsigma^2+4\sigma}\,B_1-\varsigma B_2\right)\sin\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\\+\left(\sqrt{-\varsigma^2+4\sigma}\,B_2-\lambda B_1\right)\cos\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\end{pmatrix}}{2B_2\sin\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{-\varsigma^2+4\sigma}}{2}\right)+2B_1\cos\left(\frac{\sqrt{z^2+1}(ly+lx+\nu t)\sqrt{-\varsigma^2+4\sigma}}{2}\right)} \tag{38}$$

$$v_{11}=\frac{\begin{pmatrix}\left(-\sqrt{-\varsigma^2+4\sigma}\,B_1-\varsigma B_2\right)\sin\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\\+\left(\sqrt{-\varsigma^2+4\sigma}\,B_2-\varsigma B_1\right)\cos\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\end{pmatrix}}{2B_2\sin\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2}\right)+2B_1\cos\left(\frac{\sqrt{z^2+1}\left(ly+lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)\sqrt{-\varsigma^2+4\sigma}}{2}\right)} \tag{39}$$

Case 3:

$$w_{12}=\frac{-B_2\left(\sqrt{z^2+1}\,lyB_2+(-lx-\nu t)B_2-B_1\right)\sqrt{\varsigma^2-4\sigma}-\varsigma\begin{pmatrix}\left((x^2+y^2)l^2+2tx\nu l+t^2\nu^2\right)B_2^2\\+2B_1(kx-\nu t)B_2+B_1^2\end{pmatrix}}{\left((2x^2+2y^2)l^2+4tx\nu l+2t^2\nu^2\right)B_2^2+4B_1(lx+\nu t)B_2+2B_1^2} \tag{40}$$

$$v_{12}=\frac{\begin{pmatrix}-B_2\left(\sqrt{z^2+1}\,lyB_2+\left(-lx-\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)B_2-B_1\right)\sqrt{\varsigma^2-4\sigma}\\-\varsigma\left(\left((x^2+y^2)l^2+\frac{2xl\nu t^\alpha}{\Gamma(\alpha+1)}+\frac{(t^\alpha)^2\nu^2}{\Gamma(\alpha+1)^2}\right)B_2^2+2B_1\left(lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)B_2+B_1^2\right)\end{pmatrix}}{\left((2x^2+2y^2)l^2+\frac{4xl\nu t^\alpha}{\Gamma(\alpha+1)}+\frac{2(t^\alpha)^2\nu^2}{\Gamma(\alpha+1)^2}\right)B_2^2+4B_1\left(lx+\frac{\nu t^\alpha}{\Gamma(\alpha+1)}\right)B_2+2B_1^2} \tag{41}$$

Solution 5

$$b=0,\quad c=0,\quad b_1=b_1,\quad b_0=b_0,\quad b_2=b_2,\quad \varsigma=\varsigma,$$
$$q=\sqrt{z^2+1}\,l,\quad l=l,\quad \nu=0,\quad \sigma=\sigma$$

Case 1:

$$w_{13}=\frac{\begin{pmatrix}\left(-\varsigma B_2+B_1\sqrt{\varsigma^2-4\sigma}\right)\sinh\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}}{2}\right)\\-\cosh\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}}{2}\right)\left(\varsigma B_1-B_2\sqrt{\varsigma^2-4\sigma}\right)\end{pmatrix}}{2B_2\sinh\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}}{2}\right)+2B_1\cosh\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}}{2}\right)} \tag{42}$$

$$v_{13}=\frac{\begin{pmatrix}\left(-\varsigma B_2+B_1\sqrt{\varsigma^2-4\sigma}\right)\sinh\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}}{2}\right)\\-\cosh\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}}{2}\right)\left(\varsigma B_1-B_2\sqrt{\varsigma^2-4\sigma}\right)\end{pmatrix}}{2B_1\cosh\left(\frac{1}{2}l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}\right)+2B_2\sinh\left(\frac{1}{2}l(\sqrt{z^2+1}\,y+x)\sqrt{\varsigma^2-4\sigma}\right)} \tag{43}$$

Case 2:

$$w_{14}=\frac{\begin{pmatrix}\left(-\varsigma B_2-B_1\sqrt{-\varsigma^2+4\sigma}\right)\sin\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\\-\cos\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\left(\varsigma B_1-B_2\sqrt{-\varsigma^2+4\sigma}\right)\end{pmatrix}}{2B_2\sin\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}}{2}\right)+2B_1\cos\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}}{2}\right)} \tag{44}$$

$$v_{14} = \frac{\begin{pmatrix} \left(-\varsigma B_2 - B_1\sqrt{-\varsigma^2+4\sigma}\right)\sin\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}}{2}\right) \\ -\cos\left(\frac{l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}}{2}\right)\left(\varsigma B_1 - B_2\sqrt{-\varsigma^2+4\sigma}\right)\end{pmatrix}}{2B_1\cos\left(\frac{1}{2}l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}\right)+2B_2\sin\left(\frac{1}{2}l(\sqrt{z^2+1}\,y+x)\sqrt{-\varsigma^2+4\sigma}\right)} \tag{45}$$

Case 3:

$$w_{15} = \frac{B_2\left(-lyB_2+lxB_2+B_1\right)\sqrt{1+z^2}\sqrt{\varsigma^2-4\sigma}-\varsigma\left((x^2+y^2)l^2B_2^2+2lxB_1B_2+B_1^2\right)}{(x^2+y^2)2l^2B_2^2+4lxB_1B_2+2B_1^2} \tag{46}$$

$$v_{15} = \frac{B_2\left(\sqrt{z^2+1}(-lyB_2)+lxB_2+B_1\right)\sqrt{\varsigma^2-4\sigma}-\varsigma\left((x^2+y^2)l^2B_2^2+2lxB_1B_2+B_1^2\right)}{(x^2+y^2)2l^2B_2^2+4lxB_1B_2+2B_1^2} \tag{47}$$

4. Graphical Representation

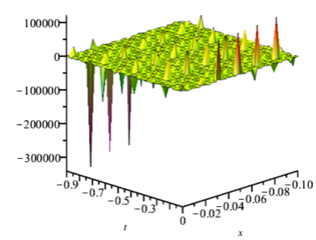

Figure 1:
Solution 5 shows solitary wave solution when
$B_2 = 2,\ B_1 = 2,\ \alpha = 0.01,\ b_1 = 0.0001,\ y = 0.1,\ \sigma = -8,\ b_2 = 5,\ l = 5,\ q = 77$

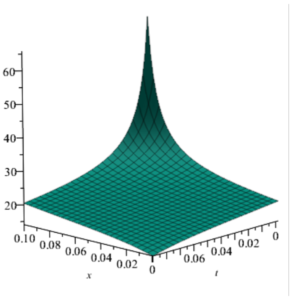

Figure 2:
Interprets the peakon wave in solution 4 when
$\alpha = 0.12,\ \varsigma = 205,\ B_2 = 1.2,\ B_1 = 2,\ \sigma = 900,\ y = 350,\ l = 4,\ \nu = 426,\ q = 10$

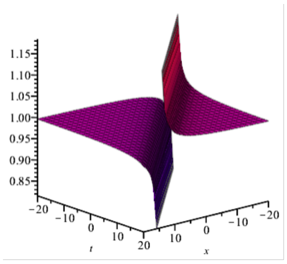

Figure 3:
Shows the singular kink profile of solution 5 when
$y = 10,\ \alpha = 0.04,\ B_2 = 2,\ B_1 = 2,\ b_1 = 0.5,\ \nu = -80,\ b_2 = 54,\ l = 3,\ q = 4$

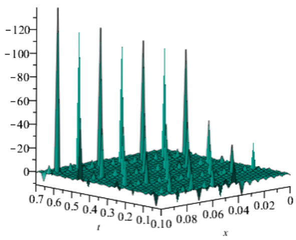

Figure 4:
Solution 2 displays the soliton wave solution when
$y = 0.01,\ \varsigma = 7,\ B_2 = 4,\ B_1 = 12,\ \sigma = 325,\ \alpha = 0.001,\ l = 4,\ \nu = 807,\ q = 3$

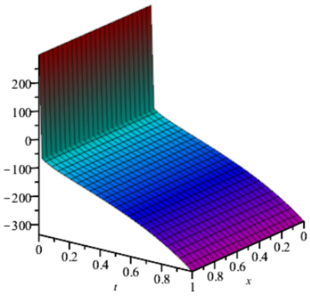

Figure 5:
Shows singular kink wave solution in solution 3 when
$\alpha = 0.0001,\ \varsigma = 31,\ B_2 = 2,\ B_1 = 2,\ \sigma = 288,\ y = 0.1,\ l = 5,\ \nu = 45,\ q = 5600$

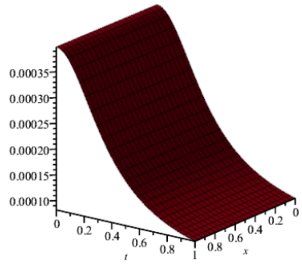

Figure 6:
Solution 3 shows a kink wave solution when
$\alpha = 0.9,\ B_2 = 2,\ B_1 = 2,\ y = 1,\ \varsigma = 1,\ b_1 = 0.01,\ \sigma = -60,\ b_2 = 52,\ l = 5,\ q = 5$

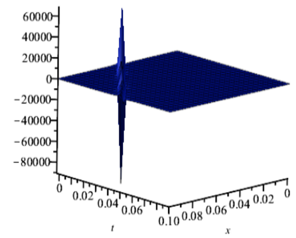

Figure 7:
Displays the lump wave with one bright and dark soliton in solution 4 when
$\alpha = 0.13,\ \varsigma = 207,\ B_2 = 19,\ B_1 = 2,\ \sigma = -85,\ y = 180,\ l = 3,\ \nu = 5,\ q = 12$

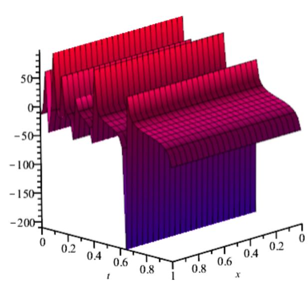

Figure 8:
In solution 1 shows singular solitary wave solution when
$\varsigma = 1,\ B_2 = 2,\ B_1 = 20,\ \sigma = 95,\ y = 0.1,\ \alpha = 0.001,\ l = 50,\ \nu = 700,\ q = 5$

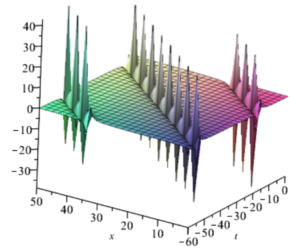

Figure 9:
Shows the singular periodic type of solution in solution 2 when
$y = 1$, $\alpha = 0.09$, $\varsigma = 1$, $B_2 = 2$, $B_1 = 2$, $b_1 = 0.01$, $\sigma = -60$, $b_2 = 52$, $l = 5$, $q = 5$

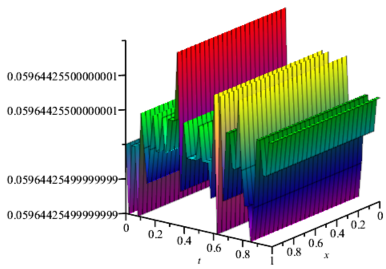

Figure 10:
Shows singular solitary wave solution in solution 1 when
$\varsigma = 10.0$, $B_2 = 2$, $B_1 = 2$, $\sigma = -0.6$, $y = 0.10$, $\alpha = 0.001$, $l = 2.5$, $\nu = 0.70$, $q = 4.0$

5. Physical Interpretation

Figure 3 displays a kink profile of Solution 5 for $y = 10, \alpha = 0.04, B_2 = 2, B_1 = 2, b_1 = 0.5, \nu = -80, b_2 = 54, l = 3, q = 4$. The steep transition from one level to another models a rapid ecological shift or spatial migration front between two equilibrium population states.The periodic wave solution 2 shown in Figure 4 corresponds to $y = 0.01, \varsigma = 7, B_2 = 4, B_1 = 12, \sigma = 325, \alpha = 0.001, l = 4, \nu = 807, q = 3$. The smooth and repetitive oscillations suggest periodic ecological patterns, such as predator-prey or cyclical growth-decay interactions.In Figure 5, a singular kink wave solution is observed in Solution 3 for $\alpha = 0.0001, \varsigma = 31, B_2 = 2, B_1 = 2, \sigma = 288, y = 0.1, l = 5, \nu = 45, q = 5600$. The sharp front and high contrast suggest a nonlinear interface or boundary propagation in a spatially structured population. Figure 6 shows a kink wave solution with a gentler slope in Solution 3 for $\alpha = 0.9, B_2 = 2, B_1 = 2, y = 1, \varsigma = 1, b_1 = 0.01, \sigma = -60, b_2 = 52, l = 5, q = 5$. This gradual transition models slow but steady spatial diffusion in ecological systems, possibly influenced by memory effects and nonlinear damping. In Figure 7, the solution exhibits a lump wave structure with both bright and dark solitons 4 for $\alpha = 0.13, \varsigma = 207, B_2 = 19, B_1 = 2, \sigma = -85, y = 180, l = 3, \nu = 5, q = 12$. The symmetric bell-like shape represents localized clustering of species with stable amplitude, relevant to group formation and population localization. Figure 8 presents a singular solitary wave solution from Solution 1 for $\varsigma = 1, B_2 = 2, B_1 = 20, \sigma = 95, y = 0.1, \alpha = 0.001, l = 50, \nu = 700, q = 5$. The extremely sharp spike indicates a blow-up type behavior or an explosive population event due to strong nonlinear and fractional effects. Figure 9 shows a singular periodic wave solution in Solution 2 for the parameters $y = 1, \alpha = 0.09, \varsigma = 1, B_2 = 2, B_1 = 2, b_1 = 0.01, \sigma = -60, b_2 = 52, l = 5, q = 5$. The plot reveals a repeated series of sharp peaks and steep valleys, characteristic of a rogue wave train or breather-type solution. Such behavior models periodic surges or collapses in biological populations, driven by feedback

Table 1: Validation of our solutions by comparison with results from Alam, Aktar, and Tunc [13]

Our Derived Solutions	**Corresponding Solutions from Alam et al.**
i (i) For $B_2 = 0$, $l = 0$, and $x = 0$ values, our solution gives $v_{14} = -\frac{\varsigma}{2}$	(i) With $a = 45$, $b = 0$, $x = 1$, $y = 0$, $\mu = 1$, $S_2 = \frac{\varsigma}{2}$,and $\lambda = 0$ their solution become $W_9 = -\frac{\varsigma}{2}$
(ii) For $q = 0$, and $l = 0$ values, our solution gives $v_6 = \frac{\sqrt{-cB_2}}{B_1}$	(ii) With $x = 1$, $\alpha = 0$, $y = 0$, $y = 0$, $\lambda = 0$, $a = 45$, $S_2 = \frac{\sqrt{-cB_2}}{B_1}$, and $\mu = 1$ their solution becomes $W_{10} = \frac{\sqrt{-cB_2}}{B_1}$
(iii) For $l = 0$, $q = 45$, $\nu = 0$, $y = 1.1$, $\sigma = 1$, $\alpha = 45$, and $x = 1$ values, we obtained $v_2 = -\frac{B_1}{B_2}$	(iii) With $b = 0$, $\mu = 3$, $y = 0$, $\lambda = 4$, $a = -1$, $S_2 = \frac{B_1}{B_2}$, and $x = 1$ values, their solution becomes $W_8 = -\frac{B_1}{B_2}$

Table 2: Validation of our solutions by comparison with results from Mohyud-Din and Ali [2]

Our Derived Solutions	**Corresponding Solutions from Mohyud-Din and Ali**
(i) Using $q = 0$, $l = 0$, $v = 0$,and $\varsigma = 0$ values, our solution gives $v_2 = -\dfrac{\sqrt{4\sigma}}{2B_1}$	(i) With $x = 0$, $\alpha = 0$, $l = 0$, $a = -b$, $y = \dfrac{1}{2}$, $b = 2B_1$, $a_2 = 1$, $a_1 = 0$, and $a = -\sqrt{4\sigma}$, values, their solution becomes $u_4 = -\dfrac{\sqrt{4\sigma}}{2B_1}$
(ii) Using $x = 0$, $y = 0$, $v = 0$, and $\varsigma = 0$ values, our solution becomes $v_{11} = \dfrac{B_2\sqrt{4\sigma}}{2B_1}$	(ii) With $x = 0$, $y = 0$, $a = 0$, $a_1 = B_2\sqrt{4\sigma}$, and $b_0 = 2B_1$ values, then their solution becomes $u_1 = \dfrac{B_2\sqrt{4\sigma}}{2B_1}$
(iii) Using $\sigma = 1$, and $\varsigma = 2$ values, then our solution $v_3 = -1$	(iii) With $x = 0$, $\alpha = 0$, $l = 1$, $a = -y$, $b = 0$, $a_2 = -1$, $b_1 = 1$ values, their solution $u_3 = -1$
(iv) Using $x = 0$, $y = 0$, $v = 0$, and $\varsigma = 0$ values, our solution $v_{11} = \dfrac{B_2\sqrt{4\sigma}}{2B_1}$	(iv) With $x = 0$, $y = 0$, $a = 0$, $b = 1$, $a_0 = B_2\sqrt{4\sigma}$, and $b_2 = 2B_1$ values, their solution $u_2 = \dfrac{B_2\sqrt{4\sigma}}{2B_1}$

loops and nonlinear memory effects. These structures are important in understanding spatiotemporal pattern formation where energy or population density becomes highly concentrated and periodically redistributed. Figure 10 displays a singular solitary wave solution in Solution 1 for the parameter set $\varsigma = 10.0, B_2 = 2, B_1 = 2, \sigma = -0.6, y = 0.10, \alpha = 0.001, l = 2.5, \nu = 0.70, q = 4$. The graph presents a tall, narrow peak centered in space, representing a localized population spike with extreme amplitude and steep gradients. This blow-up-like behavior indicates a highly nonlinear dynamic, possibly triggered by memory-driven growth or diffusion-limited aggregation. Such singular solitary waves often arise in fractional models where time-memory effects amplify localized activity, reflecting intense population surges in confined spatial regions.

6. Results and Discussions

The (G'/G)-expansion method provides some new exact solutions that are not found in other literature. By comparing our results, we discovered that some of them are similar to the current literature, while others solutions are newly discovered and have not been explored elsewhere. As a result, we have taken specific values of the physical parameters, and some of our obtained solutions, v_2, v_3, v_6, v_{1_1} and

v_{1_4} coincide with some of the particular solutions obtained by other methods mentioned in the tables and the references [13,2]

Acknowledgments

We thanks the referee by your suggestion

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