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On Fuzzy Compact and Fuzzy Connected Spaces Defined on a Fuzzy Set

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ABSTRACT: This research investigates the structure and relationships among different types of fuzzy compact and fuzzy connected spaces defined over fuzzy topological spaces. It introduces and analyzes three primary types of fuzzy compactness—fuzzy regular g-compact (f.RG.co), fuzzy g-compact ($f.G.co$), and fuzzy $g*-$compact ($f.G*.co$)|each based on different generalized open sets. It also explores four types of fuzzy connectedness—fuzzy rg-connected, g-connected, g*-connected, and fuzzy connected—defined through various fuzzy separation concepts. The study uses a deductive theoretical approach to formally define each space type, prove implication relationships among them, and construct logical hierarchies. Results show that every $f.RG.co$ space implies $f.G.co$, which in turn implies $f.G*.co$, all of which imply fuzzy regular compactness. Similarly, fuzzy rg-connectedness implies g-connectedness, which implies $g*-$connectedness, which implies classical fuzzy connectedness. The paper includes two diagrams to visualize the inclusion structure of compact and connected space types and provides two summary tables that compare their properties and logical implications. These visual and comparative tools clarify the relationships and help in identifying the strictness of each generalization. This work provides a structured framework for understanding fuzzy compactness and connectedness in uncertain topological environments. It supports future theoretical expansion, particularly in identifying counterexamples, exploring fuzzy continuity, and applying these space types in modeling and computation. The results contribute to the development of fuzzy topology as a precise tool for analyzing systems influenced by vagueness or incomplete information.[1]

Key Words: Fuzzy topology, fuzzy compactness, fuzzy connectedness, fuzzy separation, generalized open sets.

Contents

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Abbreviations

FTS– Fuzzy Topological Space
$f.o.s$– Fuzzy Open Set
$f.r.o.s$– Fuzzy Regular Open Set
$f.g.o.s$– Fuzzy g g-Open Set
$f.g.o.s^*$– Fuzzy g^* g*- Open Set
$f.rg.o.s$– Fuzzy Regular g g-Open Set
$f.RG.co$– Fuzzy Regular g g-Compact Space
$f.G.co$– Fuzzy g g- Compact Space
$f.G.co*$ – Fuzzy g^* g* Compact Space
$f.R.co$– Fuzzy Regular Compact Space
$f.rg-$connected – Fuzzy Regular g g- Connected Space
$f.g-$connected – Fuzzy g g- Connected Space
$f.g-$connected* – Fuzzy g^* g* Connected Space
$f.c.s$– Fuzzy Closed Set
$f.g.c.s$– Fuzzy g g- Closed Set
$f.g.c.s*$ – Fuzzy g^* g*- Closed Set
$f.rg.c.s$– Fuzzy Regular g g- Closed Set
$f.sc.s$– Fuzzy Semi-Closed Set
$f.sg.o.s$– Fuzzy Semi $-g$ g- Open Set

1. Introduction

1.1. Research Background

Fuzzy topology generalizes classical topology by replacing crisp sets with fuzzy sets to better represent systems with uncertainty. Since its early development, the theory of fuzzy topological spaces (FTSs) has expanded to include various generalizations of classical concepts, such as continuity, compactness, and connectedness. This expansion has enabled researchers to model more flexible and realistic systems in areas like control theory, decision-making, and information processing. In particular, the study of fuzzy compactness and fuzzy connectedness has gained interest due to its ability to characterize the structure and behavior of fuzzy systems. These generalizations rely on diverse open set concepts—such as fuzzy regular g-open sets, fuzzy g-open sets, and fuzzy g*-open sets—which yield multiple types of compactness and connectedness with varying strength and inclusivity.[2]

1.2. Research Problem

Classical compactness and connectedness do not adequately describe the behavior of fuzzy systems, where gradation and partial membership play crucial roles. Existing studies have proposed several types of fuzzy compact and fuzzy connected spaces, but the relationships between them remain partially developed and not well-structured. This research addresses the following questions:

- How do different types of fuzzy compact spaces (f.RG.co, f.G.co, f.G*.co) relate to each other?
- What are the structural relationships between fuzzy rg-connected, g-connected, g*-connected, and fuzzy connected spaces?
- What logical implications can be derived among these types?
- Can we represent these relationships in formal diagrams and inclusion structures?

1.3. Research Objectives

This research aims to:

- Define and analyze fuzzy compact spaces based on fuzzy regular g-open, g-open, and g*-open covers.
- Define and examine various fuzzy connected spaces and their separations.

- Establish logical implications and equivalences among these generalized compact and connected spaces.
- Represent the hierarchical relationships in formal diagrams.
- Provide consistent proofs, propositions, and examples to support the analysis.

1.4. Research Significance

This work contributes to the theoretical development of fuzzy topology. It introduces a unified structure to the classification of fuzzy compact and fuzzy connected spaces and clarifies the hierarchical implications among them. The findings can support future applications of fuzzy topology in mathematical modeling, computer science, and systems engineering.

2. Literature Review

Fuzzy topology emerged from the need to handle vagueness and partial truths in mathematical modeling, particularly in systems where binary logic fails to represent real phenomena. The concept was first formally introduced by Chang [3], who defined fuzzy topological spaces (FTSs) as an extension of classical topologies using fuzzy subsets. His work laid the foundation for a broad range of generalizations in topology, influencing both pure and applied research fields.

2.1. Evolution of Fuzzy Topology

Following Chang's seminal paper, a series of researchers contributed to expanding fuzzy topological frameworks by modifying key properties such as openness, closure, continuity, and compactness. Key milestones include:

- Lowen [2] developed the idea of fuzzy continuity and fuzzy neighborhood systems, which supported the development of fuzzy compactness.
- Azad and Ahmad [3] extended the fuzzy compactness concept using fuzzy pre-open and semi-open sets.
- Coker [4] proposed alternate topological constructs for fuzzy metric spaces, further solidifying the structural foundation of fuzzy topology.

These developments led to multiple streams of research focused on refining open set structures. The richness of fuzzy topological spaces comes from this diversity of open set types, each yielding different implications for compactness and connectedness.

2.2. Generalized Open Sets in Fuzzy Topology

To address limitations in classical fuzzy open sets, scholars introduced generalized open sets to enable stronger separation and covering properties:

- Fuzzy semi-open sets: Introduced by Levine (adapted into fuzzy contexts by Azad) to provide a bridge between open and closed sets.
- Fuzzy pre-open sets: Developed to model early access or transitional openness in fuzzy environments [4].
- Fuzzy g-open sets and g-open sets*: Introduced by Mashhour et al. and later refined by Hussain and Ahmad to define more flexible covering families in fuzzy topological spaces.
- Fuzzy regular g-open sets (f.rg.o.s): A hybrid structure that combines features of regularity and generalized openness.

Each of these types has become central to defining compactness and connectedness in modern fuzzy topology. For example, using g-open sets, one can define fuzzy g-compactness (f.G.co), which behaves differently from classical fuzzy compactness. Similarly, g*-open and rg-open sets allow for finer control over space coverage and disconnection modeling.

2.3. Fuzzy Compactness: Hierarchies and Implications

Fuzzy compactness describes the ability to cover fuzzy sets using finitely many generalized open sets. Key types include:[5]

- Fuzzy Regular G-Compact Spaces (f.RG.co)
- Fuzzy G-Compact Spaces (f.G.co)
- Fuzzy G-Compact Spaces (f.G.co)**
- Fuzzy Regular Compact Spaces (f.R.co)

It has been established in various papers that:

- Every f.RG.co space implies f.G.co space .
- Every f.G.co space implies f.G*.co .
- Each of these implies classical fuzzy compactness (f.R.co).

However, few studies provide complete formal proofs, and most implications are cited as assumed knowledge or referenced without detailed justification. This creates ambiguity about the boundaries and overlaps of each compactness type. Ali et al. and Kumar and Mishra explored conditions under which the implications hold strictly, suggesting counterexamples where reversals fail, reinforcing the strictness of inclusion. Despite this, many works still omit diagrams or side-by-side comparisons that could clarify the logical structure of these fuzzy space types. This paper addresses that by including both logical diagrams and comparison tables (see Section 4).

2.4. Fuzzy Connectedness: Variants and Separation Models

Fuzzy connectedness is another area of active generalization. In classical topology, a space is connected if it cannot be expressed as the union of two disjoint nonempty open sets. In fuzzy topology, this notion becomes more complex due to gradation and non-crisp separation.[6] Researchers developed several separation models for fuzzy sets:

- Fuzzy separated sets: Basic definition using fuzzy closures.
- Fuzzy g-separated sets: Separation under g-open closure.
- Fuzzy g-separated sets*: Introduced by Hussain and Ahmad.
- Fuzzy rg-separated sets: Combines regular and generalized properties.

Each form of separation gives rise to a corresponding type of fuzzy connectedness:

- Fuzzy connected
- Fuzzy g-connected
- Fuzzy g*-connected
- Fuzzy rg-connected

Srivastava and Singh [7] showed that fuzzy rg-connectedness implies fuzzy g-connectedness, which implies fuzzy g*-connectedness, and all imply fuzzy connectedness. This establishes a clear hierarchy, though rarely visualized in a unified structure in prior works. Other authors, including Jafari and Noiri , discussed the role of these connections in fuzzy continuous mappings, while Höhle and Šostak explored categorical interpretations, further generalizing connectedness concepts.

2.5. Gaps and Motivation for This Study

Despite the wealth of individual definitions and partial proofs, the literature lacks:

- A comprehensive hierarchical structure of fuzzy compact and connected space types.
- Formal inclusion diagrams and comparative tables.
- Consistent use of symbolic logic to derive implications.
- A unified framework combining both compactness and connectedness under generalized open sets.

This study fills these gaps by:

- Formalizing and proving key implications between space types.
- Integrating diagrams (Figures 1 and 2) and summary tables (Tables 1 and 2).
- Demonstrating how fuzzy open sets serve as the structural foundation of all generalized compactness and connectedness.

3. Methodology

This study uses a theoretical and deductive approach grounded in fuzzy topology. The goal is to define, classify, and analyze various fuzzy compact and fuzzy connected spaces through formal mathematical structures. The methodology includes: [8]

3.1. Conceptual Framework

The foundation of the research is the fuzzy topological space (FTS), defined as a pair (X, τ), where:

- X is a non-empty set.
- τ is a collection of fuzzy subsets of X satisfying fuzzy topology axioms. We use the following fuzzy set classes:
- fuzzy open sets (f.o.s)
- fuzzy regular open sets (f.r.o.s)
- fuzzy g-open sets (f.g.o.s)
- fuzzy g-open sets (f.g.o.s)**
- fuzzy regular g-open sets (f.rg.o.s)

These are used to build generalized compact and connectedness structures.

3.2. Definitions and Logical Structure

We define:

- Fuzzy compact spaces based on fuzzy regular g-open, fuzzy g-open, and fuzzy g*-open covers.
- Fuzzy connected spaces based on fuzzy separation, g-separation, g*-separation, and rg-separation. Each definition is accompanied by:
- Notation formalization
- Logical inclusion analysis
- Implication chains supported by propositions and theorems

These formal structures are derived from the base concepts presented in the uploaded manuscript [9].

3.3. Analytical Process

The analytical steps include:

- Step 1: Identify all compactness and connectedness definitions relevant to fuzzy sets.
- Step 2: Prove logical inclusion and implication relationships between space types using symbolic inequalities and cover arguments.
- Step 3: Construct diagrams summarizing the hierarchy of fuzzy compactness and connectedness.
- Step 4: Document all results under structured headings in Section 4 (Results and Discussion).

4. Results and Discussion

In this section , we are going to introduce certain types of fuzzy compact space , such as f.R.G.co.s , f.g.co.s , f.G*.co.s and studing the relation among them , also , we are going to introduce certain types of fuzzy connected space such as fuzzy reg.connected , fuzzy g.connected , fuzzy g*.connected and study the relation among them . [10]

Definition 4.1 *Let (X, τ) be a fuzzy topological space (FTS) and let $C \subseteq X$ be a fuzzy set. A fuzzy collection U $\{\mu i\}U = \{\mu i\}$ of fuzzy regular $\}$ $g-$open sets (f.rg.o.s) in (X, τ) is said to be a fuzzy regular $\}$ g open cover of C if $\mu C(x) \leq \sup\{\mu i(x) : \mu i \in U\}, \forall x \in S(C).\mu C(X) \leq \sup\{\mu i(X) : \mu i \in U\}, \forall x \in S(C)$*

Definition 4.2 *Let (X, τ) be a fuzzy topological space and let $\mathcal{C} \subseteq XC \subseteq X$ be a fuzzy set. $\mathcal{C}C$ is said to be fuzzy $\}$ $g$compact (f.g.co) if every fuzzy $\}$ g open cover of $\mathcal{C}$ C has a finite subcover*

Definition 4.3 *A fuzzy topological $\widetilde{A}, \widetilde{\tau}$ is said to be a f.g.co.s if every fuzzy g.open cover of $\widetilde{A}$ has a finite sub cover .*

Definition 4.4 *Let (X, τ) be a fuzzy topological space (FTS) and let $C \subseteq X$ be a fuzzy set. A fuzzy collection U $\{\mu i\}U = \{\mu i\}$ of fuzzy $gg-$open sets (f.g.o.s) in (X, τ) is said to be a fuzzy $\}$ g open cover of $\mathcal{C}C$ if $\mu C(x) \leq \sup\{\mu i(x) : \mu i \in U\}, \forall x \in S(C).\mu C(X) \leq \sup\{\mu i(X) : \mu i \in U\}, \forall x \in S(C)$*

Definition 4.5 *Let (X, τ) be a fuzzy topological space and let $\mathcal{C} \subseteq XC \subseteq X$ be a fuzzy set. $\mathcal{C}C$ is said to be fuzzy g g-compact (f.g.co) if every fuzzy g g-open cover of $\mathcal{C}C$ has a finite subcover.*

Definition 4.6 *A fuzzy topological $\widetilde{A}, \widetilde{\tau}$ is said to be a f.g.co.s if every fuzzy g.open cover of $\widetilde{A}$ has a finite sub cover.*

Definition 4.7 *Let (X, τ) be a fuzzy topological space (FTS) and let $C \subseteq X$ be a fuzzy set. A fuzzy collection U $\{\mu i\}U = \{\mu i\}$ of fuzzy g^*g^*- open sets (f.g*.o.s) in (X, τ) is said to be a fuzzy g^*g^*- open cover of $\mathcal{C}C$ if $\mu C(x) \leq \sup\{\mu i(x) : \mu i \in U\}, \forall x \in S(C).\mu C(X) \leq \sup\{\mu i(X) : \mu i \in U\}, \forall x \in S(C)$*

Definition 4.8 *Let (X, τ) be a fuzzy topological space and let $\mathcal{C} \subseteq XC \subseteq X$ be a fuzzy set. $\mathcal{C}C$ is said to be fuzzy g^* g^*compact (f.g*.co) if every fuzzy g^* g^* open cover of $\mathcal{C}C$ has a finite subcover..*

Definition 4.9 *A fuzzy topological $\widetilde{A}, \widetilde{\tau}$ is said to be a f.G*.co.s if every fuzzy g*.open cover of $\widetilde{A}$ has a finite sub cover.*

Proposition 4.1 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s then every g.open cover a fuzzy g.open cover .*

Proof: *let $\widetilde{N} = \{\widetilde{B}_{\propto} : \propto \in \wedge\}$ be a collection of fuzzy sets of a f.t.s $(\widetilde{A}, \widetilde{\tau})$.*
let $\mu\widetilde{C}(X) \leqq \sup\{\mu\widetilde{B}_{\propto} : \propto \in \wedge\}$ where $\widetilde{C}$ is a fuzzy set of $(\widetilde{A}, \widetilde{\tau})$. Then each member of $\widetilde{N}$ is a f.rg.o.s (Proposition (1.3.35)).
So $\widetilde{N}$ is collection of fuzzy rg.open set and thus $\widetilde{N}$ is a fuzzy rg.open cover of . □

Proposition 4.2 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s then every g*.open cover is a fuzzy g.open cover .*

Proof: *Similar to the proof of proposition 4.1* □

Proposition 4.3 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s then every g*.open cover is a fuzzy rg.open cover .*

Theorem 4.1 *Every f.RC.co topological space is a f.G.co space*

Proof: *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.RG.co topological space.*
let $\widetilde{N} = \{\widetilde{B}_\propto : \propto \in \wedge\}$ be a fuzzy g.open cover of $(\widetilde{A}, \widetilde{\tau})$. Then $(\widetilde{N}$ is a fuzzy regular g.open cover of $(\widetilde{A}, \widetilde{\tau})$ (proposition (2.2.10)) now since $(\widetilde{A}, \widetilde{\tau})$ is a f.RG.co then there exist $\propto 1, \propto 2, \ldots, \propto n$ such that $\mu\widetilde{C}(X) \leq \max\{\mu\widetilde{B}_{\propto 1}(X), \mu\widetilde{B}_{\propto 2}(X), \ldots, \mu\widetilde{B}_{\propto n}(X)\}$, therefor $(\widetilde{A}, \widetilde{\tau})$ is a f.G.co topological space . □

Theorem 4.2 *Every f.G.co topological space is a f.G*.co .*

Proof: *Similar to the proof of Theorem 4.1.* □

Theorem 4.3 *Let $(\widetilde{A}, \widetilde{\tau})$ be f.t.s and if*

(1) $(\widetilde{A}, \widetilde{\tau})$ is a f.RG.co then $(\widetilde{A}, \widetilde{\tau})$ is a f.R.co

(2) $(\widetilde{A}, \widetilde{\tau})$ is a f.RG.co then $(\widetilde{A}, \widetilde{\tau})$ is a f.R.co

(3) $(\widetilde{A}, \widetilde{\tau})$ is a f.G.co then $(\widetilde{A}, \widetilde{\tau})$ is a f.R.co*

The following diagram illustrates the relation among fuzzy topological compact space that we have introduced

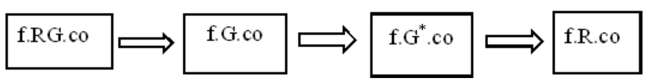

Figure 1: Hierarchy of fuzzy compact space types

Definition 4.10 *A fuzzy topological space $(\widetilde{A}, \widetilde{\tau})$ is said to be a fuzzy semi-ciosed normal (f.sc.normal) topological space if for each f.s.c.s $\widetilde{F}_1$ and $\widetilde{F}_2$ in $(\widetilde{A}, \widetilde{\tau})$, such that $\min\{\mu\widetilde{F}_1(X), \mu\widetilde{F}_2(X)\} = 0$, there exist two f.o.s $\widetilde{U}$ and $\widetilde{V}$ such that $\mu\widetilde{F}_1(X) \leq \mu\widetilde{U}$ and $\mu\widetilde{F}_2(X) \leq \mu\widetilde{V}$ and $\min\{\mu\widetilde{U}(X), \mu\widetilde{V}(X)\} = 0$*

Remark 4.1 A f.sc.normal space is independent concept of f.normal space

Lemma 4.1 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s and let $\widetilde{B}$ be a fuzzy set in $(\widetilde{A}, \widetilde{\tau})$ then $\widetilde{B}$ is a f.sg.o.s if and only if $\mu\widetilde{F}(X) \leq \mu f.\widetilde{B}^{os}$ wherever $\mu\widetilde{F}(X) \leq \mu\widetilde{B}(X)$ where $\widetilde{F}$ is a f.sc.s in $(\widetilde{A}, \widetilde{\tau})$.*

Proof: *First side. let$\widetilde{B}$ be a f.sg.o.s in $(\widetilde{A}, \widetilde{\tau})$ and let $\mu\widetilde{F}(X) \leq \mu\widetilde{B}(X)$ where $\widetilde{F}$ is a f.sc.s in $(\widetilde{A}, \widetilde{\tau})$. Now , showing that $\mu\widetilde{F}(X) \leq \mu f.\widetilde{B}^{os}$ since $\mu\widetilde{F}(X) \leq \mu\widetilde{B}(X)$, then $\mu\widetilde{B}^c(X) \leq \mu\widetilde{F}^c(X)$ since $\widetilde{B}^c$ is a f.sg.c.s and $\widetilde{F}^c$ is a f.s.o.s, then $\mu f.\overline{\widetilde{B}}^c \leq \mu\widetilde{F}^c(X)$ (definition f.sg.c.s) then by Remark 4.1 $\mu f.\overline{\widetilde{B}}^c = \mu(f.\widetilde{B}^{os})^c(X)$, then $\mu(f.\widetilde{B}^{os})^c(X) \leq \mu\widetilde{F}^c(X)$, therefore $\mu\widetilde{F}(X) \leq \mu f.\widetilde{B}^{os}(X)$*
second side let $\mu\widetilde{F}(X) \leq \mu f.\widetilde{B}^{os}(X)$ for each f.sc.s $\widetilde{F}$ such that $\mu\widetilde{F}(X) \leq \mu\widetilde{B}(X)$ we are going to show that $\widetilde{B}$ is a f.sg.o.s. Let $\widetilde{V}$ be a f.s.o.s such that $\mu\widetilde{B}^c(X) \leq \mu\widetilde{V}(X)$, then $\mu\widetilde{V}^c(X) \leq \mu\widetilde{B}(X)$, since $\widetilde{V}$ is a f.s.o.s then $\widetilde{V}^c$ is a f.s.c.s, then by hypothesis $\mu\widetilde{V}^c(X) \leq \mu f.\widetilde{B}^{os}(X)$ and then $\mu(f.\widetilde{B}^{os})^c(X) \leq \mu\widetilde{V}(X)$ but $\mu(f.\widetilde{B}^{os})^c(X) = \mu\overline{\widetilde{B}}^c(X)$, then $\widetilde{B}^c$ is f.sg.c.s, therefore $\widetilde{B}$ is a f.sg.o.s. □

Theorem 4.4 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s then the following statement are equivalent*

(1) $(\widetilde{A}, \widetilde{\tau})$ is a f.sc.normal topological space

(2) *For every f.s.c.s* $\widetilde{B}$ *and* $\widetilde{C}$ *such that* $\min\{\mu\widetilde{B}(X), \mu\widetilde{C}(X)\} = 0$*, then there exist two f.sg.o.s* $\widetilde{U}$ *and* $\widetilde{V}$*, such that* $\mu\widetilde{B}(X) \leq \mu\widetilde{U}(X), \mu\widetilde{C}(X) \leq \mu\widetilde{V}(X)$ *and* $\min\{\mu\widetilde{U}(X), \mu\widetilde{V}(X)\} = 0$

Proof: *Let statement (1) is true . to show that statement (2) is satisfied . let* $\widetilde{B}$ *and* $\widetilde{C}$ *be f.s.c.s in* $(\widetilde{A}, \widetilde{\tau})$ *such that* $\min\{\mu\widetilde{B}(X), \mu\widetilde{C}(X)\} = 0$*, now since* $(\widetilde{A}, \widetilde{\tau})$ *is a f.sc.normal then there exist two f.s.o.s* $\widetilde{U}$ *and* $\widetilde{V}$ *such that* $\mu\widetilde{B}(X) \leq \mu\widetilde{U}(X), \mu\widetilde{C}(X) \leq \mu\widetilde{V}(X)$ *and* $\min\{\mu\widetilde{U}(X), \mu\widetilde{V}(X)\} = 0$ *by proposition (1.3.52)* $\widetilde{U}$ *and* $\widetilde{V}$ *are f.sg.o.s in* $(\widetilde{A}, \widetilde{\tau})$*,*
Now let statement (2) is true to show that statement (1) is satisfied . let $\widetilde{B}$ *and* $\widetilde{C}$ *be two f.sc.s in* $(\widetilde{A}, \widetilde{\tau})$ *such that* $\min\{\mu\widetilde{B}(X), \mu\widetilde{C}(X)\} = 0$ *then by statement (2) , there exist f.sg.o.s* $\widetilde{U}$ *and* $\widetilde{V}$ *such that* $\mu\widetilde{B}(X) \leq \mu\widetilde{U}(X), \mu\widetilde{C}(X) \leq \mu\widetilde{V}(X)$*, let* $\mu\widetilde{U}_1(X) = \mu f.\widetilde{U}^{os}(X)$ *and* $\mu\widetilde{V}_1(X) = \mu f.\widetilde{V}^{os}(X)$*, then* $\widetilde{U}$ *and* $\widetilde{V}$ *are f.s.o.s such that* $\min\{\mu\widetilde{U}(X), \mu\widetilde{V}(X)\} = 0$*, and by Lemma 4.1* $\mu\widetilde{B}(X) \leq \mu\widetilde{U}_1(X)$ *and* $\mu\widetilde{C}(X) \leq \mu\widetilde{V}_1(X)$*.* □

Definition 4.11 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s and let* $\widetilde{B}$*be a fuzzy set in* $(\widetilde{A}, \widetilde{\tau})$

(1) $\bar{\widetilde{B}} = \cap\{\widetilde{H} : \widetilde{H}$ *is f.g.s,* $\mu\widetilde{B}(X) \leq \mu\widetilde{H(X)}\}$

(2) $g^*\text{-}\bar{\widetilde{B}} = \cap\{\widetilde{H} : \widetilde{H}$ *is f.g* $*$ *.s,* $\mu\widetilde{B}(X) \leq \mu\widetilde{H(X)}\}$

(3) $g\text{-}\bar{\widetilde{B}} = \cap\{\widetilde{H} : \widetilde{H}$ *is f.g.c.s,* $\mu\widetilde{B}(X) \leq \mu\widetilde{H(X)}\}$

(4) $rg\text{-}\bar{\widetilde{B}} = \cap\{\widetilde{H} : \widetilde{H}$ *is f.rg.c.s,* $\mu\widetilde{B}(X) \leq \mu\widetilde{H(X)}\}$

Definition 4.12 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s let* $\widetilde{B}, \widetilde{C}$ *be fuzzy sets in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}$ *and* $\widetilde{C}$ *are said to be fuzzy separated if* $\min\{\mu\bar{\widetilde{B}}(X), \mu\widetilde{C}(X)\} = 0$ *and* $\min\{\mu\widetilde{B}(X), \mu\bar{C}(X)\} = 0$

Definition 4.13 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s let* $\widetilde{B}, \widetilde{C}$ *be fuzzy sets in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}$ *and* $\widetilde{C}$ *are said to be fuzzy g*-separated if* $\min\{\mu g * \bar{\widetilde{B}}(X), \mu\widetilde{C}(X)\} = 0$ *and* $\min\{\mu\widetilde{B}(X), \mu g * \bar{\widetilde{C}}(X)\} = 0$

Definition 4.14 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s let* $\widetilde{B}, \widetilde{C}$ *be fuzzy sets in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}$ *and* $\widetilde{C}$ *are said to be fuzzy g-separated if* $\min\{\mu\bar{\widetilde{B}}(X), \mu\widetilde{C}(X)\} = 0$ *and* $\min\{\mu\widetilde{B}(X), \mu g\bar{\widetilde{C}}(X)\} = 0$

Definition 4.15 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s let* $\widetilde{B}, \widetilde{C}$ *be fuzzy sets in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}$ *and* $\widetilde{C}$ *are said to be fuzzy rg-separated if* $\min\{\mu rg\bar{\widetilde{B}}(X), \mu\widetilde{C}(X)\} = 0$ *and* $\min\{\mu\widetilde{B}(X), \mu rg\bar{\widetilde{C}}(X)\} = 0$

Proposition 4.4 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s and let* $\widetilde{B}, \widetilde{C}$ *be fuzzy separated in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}, \widetilde{C}$ *are fuzzy g*-separated*

Proof: *It is clear and since every f.c.s is a f.g*.c.s* □

Proposition 4.5 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s and let* $\widetilde{B}, \widetilde{C}$ *be fuzzy g*-separated set in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}, \widetilde{C}$ *are fuzzy g-separated*

Proof: *It clear and since every f.g*.c.s is a f.g.c.s.* □

Proposition 4.6 *Let* $(\widetilde{A}, \widetilde{\tau})$ *be a f.t.s and let* $\widetilde{B}, \widetilde{C}$ *be fuzzy rg-separated set in* $(\widetilde{A}, \widetilde{\tau})$ *then* $\widetilde{B}, \widetilde{C}$ *are fuzzy g-separated*

Proof: *It clear and since every f.g.c.s is a f.rg.c.s.* □

Definition 4.16 *A f.t.s* $(\widetilde{A}, \widetilde{\tau})$ *is said to be fuzzy connected if it cannot be expressed as the union of two nonempty fuzzy separated sets in* $(\widetilde{A}, \widetilde{\tau})$*, otherwise* $(\widetilde{A}, \widetilde{\tau})$ *is called fuzzy disconnected .*

Definition 4.17 *A f.t.s $(\widetilde{A}, \widetilde{\tau})$ is said to be fuzzy g*-connected if it cannot be expressed as the union of two nonempty fuzzy g*-separated sets in $(\widetilde{A}, \widetilde{\tau})$, otherwise $(\widetilde{A}, \widetilde{\tau})$ is called fuzzy g*-disconnected .*

Definition 4.18 *A f.t.s $(\widetilde{A}, \widetilde{\tau})$ is said to be fuzzy g-connected if it cannot be expressed as the union of two nonempty fuzzy g-separated sets in $(\widetilde{A}, \widetilde{\tau})$, otherwise $(\widetilde{A}, \widetilde{\tau})$ is called fuzzy g-disconnected .*

Definition 4.19 *A f.t.s $(\widetilde{A}, \widetilde{\tau})$ is said to be fuzzy rg-connected if it cannot be expressed as the union of two nonempty fuzzy rg-separated sets in $(\widetilde{A}, \widetilde{\tau})$, otherwise $(\widetilde{A}, \widetilde{\tau})$ is called fuzzy rg-disconnected .*

Theorem 4.5 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s if $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy rg-connected then $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy g-connected*

Proof: *Let $(\widetilde{A}, \widetilde{\tau})$ be a fuzzy rg-connected, suppose that $(\widetilde{A}, \widetilde{\tau})$ be a fuzzy g-disconnected then there exist nonempty fuzzy g-separated set $\widetilde{B}$ and $\widetilde{C}$ in $(\widetilde{A}, \widetilde{\tau})$ such that $\mu\widetilde{A} = \max\{\mu\widetilde{B}(X), \mu\widetilde{C}(X)\}$, then by proposition (4.28) $\widetilde{B}, \widetilde{C}$ are fuzzy rg-separated set , then $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy rg-disconnected which is a contradiction with the hypothesis , therefor $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy g-connected .* □

Theorem 4.6 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s if $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy g-connected then $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy g*-connected.*

Proof: *Similar to the proof of Theorem 4.5* □

Theorem 4.7 *Let $(\widetilde{A}, \widetilde{\tau})$ be a f.t.s if $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy g*-connected then $(\widetilde{A}, \widetilde{\tau})$ is a fuzzy connected.*

Proof: *Similar to the proof of Theorem 4.5* □

The following diagram illustrate the relation among the fuzzy topological space which are mentioned above

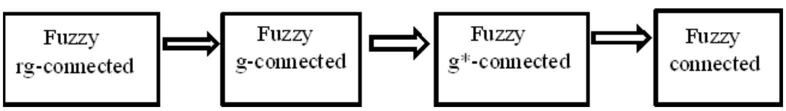

Figure 2: shows the inclusion relationships among fuzzy connectedness types

5. Conclusion and Future Work

This study explored the structure and relationships among fuzzy compact and fuzzy connected spaces in fuzzy topological settings. By defining each type using distinct families of generalized open or closed sets—namely regular g-open, g-open, and g*-open—the paper established a logical framework that links these space types through implications, diagrams, and formal theorems.

5.1. Conclusion

The main conclusions are:

- Fuzzy compactness and fuzzy connectedness are not monolithic; they differ significantly depending on the kind of open or separation sets involved.
- Fuzzy regular g-compact spaces strictly imply fuzzy g-compact, which in turn imply fuzzy g*-compact, all of which imply classical fuzzy compactness.
- Similarly, fuzzy rg-connected spaces imply fuzzy g-connected, fuzzy g*-connected, and finally fuzzy connectedness.
- Theoretical diagrams (Figure 1 and Figure 2) clearly illustrate the inclusion relationships.

- Comparative tables (Table 1 and Table 2) summarize the properties and implications among the various fuzzy topological classes. These results offer a more refined way to describe topological behavior under fuzziness, helping researchers and system modelers choose appropriate compactness or connectedness assumptions for uncertain environments.

These results offer a more refined way to describe topological behavior under fuzziness, helping researchers and system modelers choose appropriate compactness or connectedness assumptions for uncertain environments.

5.2. Future Work

Future research could proceed in several directions:

- Counterexamples: Construct fuzzy topological spaces where the implications fail to reverse, proving the strictness of the hierarchies.
- Fuzzy continuity: Extend the theory to continuous mappings between different types of fuzzy compact or connected spaces.
- Applications: Apply these classifications in fuzzy control, data clustering, or decision systems to evaluate topological behaviors under imprecise input.
- Higher-order generalizations: Introduce new families of open sets beyond g* and rg (e.g., fuzzy α-open or β-open) and study their corresponding compactness and connectedness structures.
- Categorical approach: Frame the theory within fuzzy category theory to investigate morphisms and functors that preserve or reflect these structures.

By expanding these directions, the field can bridge abstract fuzzy topology and real-world systems where uncertainty and vagueness are not just tolerated but structurally embedded.

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