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Dynamical Analysis and Optimal Control Strategies of a General Reaction-Diffusion Waterborne Pathogen Model

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ABSTRACT: This paper investigates the global stability analysis and optimal control strategies of a reaction–diffusion waterborne pathogen model with a general incidence rate, incorporating both direct and indirect transmission pathways. First, we establish the well-posedness of solutions for the model. Then, we analyze the threshold dynamics in terms of the basic reproduction number R_0: the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, while the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. The model is subsequently extended by introducing control intervention strategies such as vaccination, treatment, and water purification, with the aim of minimizing disease spread at the lowest possible cost. We further prove the existence of an optimal control. Finally, we derive the first-order necessary conditions for optimality and characterize the optimal controls in terms of the state and adjoint variables. Numerical simulations are performed to confirm and illustrate the different theoretical results.

Key Words: Waterbone pathogen, reaction-diffusion equation, strong solution, optimal control, nonlinear incidences.

Contents

1. Introduction

According to the WHO, about 1.1 billion people lack reliable water access, and nearly 700,000 children die annually from diarrheal diseases linked to poor sanitation. Waterborne illnesses such as cholera, hepatitis A and E, giardiasis, cryptosporidiosis, and rotavirus remain significant health threats, particularly in developing regions where clean water and sanitation are inadequate. These diseases are primarily driven by unsafe water sources, poor hygiene, and limited access to basic sanitation.

While interventions like vaccination, water purification, and treatment can effectively reduce transmission, affordability and resource limitations often hinder implementation in vulnerable communities. Optimal control theory provides a framework for identifying strategies that minimize both disease spread and intervention costs.

Mathematical models are widely used to study the spread and control of waterborne diseases [7,8,11]. Tien and Earn [26] extended the SIR model by adding a compartment $W(t)$ to represent pathogen concentration, capturing both direct and waterborne transmission. Since then, several recent studies have explored these dynamics using ordinary differential equation models [2,24,25]. Taking into account

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the influence of spatial factors, the dynamical properties of reaction–diffusion models with direct and indirect transmission have attracted some interests, see [2,26,28,29]. In 2018 and motivating by the works of [28,29], J. Zhou et al. [30] investigate the following generalized model:

$$\begin{cases} \partial_t S &= d_1\Delta S + \Lambda - \mu_1 S - Sf(W) - Sg(I), & \text{in } Q_T, \\ \partial_t I &= d_3\Delta I - (\mu_3+\gamma)I + Sf(W) + Sg(I), & \text{in } Q_T, \\ \partial_t W &= d_4\Delta W + \alpha I - \xi W, & \text{in } Q_T, \\ \partial_t R &= d_5\Delta R - \mu_4 R + \gamma I, & \text{in } Q_T, \end{cases} \tag{1.1}$$

with the homogeneous Neumann boundary conditions (no flux of the quantity across the boundary) and assuming some geneneralized under some realistic assumptions (A_1)-(A_2), see section 2.1.
In actuality, diseases have a latent phase in which persons who are exposed to an infectious person get infected but are unable to transmit the disease. The SEIR model, first presented by Hethcote [13], is created by adding a latent compartment (E) in this context, see, for example, [1,4,14,15,16]. Taking all these factors into account and inspired by previous studies, this paper considers a generalized pathogen model SEIWR, with five compartments, to investigate the dynamics and control of waterborne diseases, see (2.1). In this paper, we establish the well-posedness of (2.1) and we establish its dynamical behavior of the solution, such as the existence of equilibria and the global stability with respect to the basic reproduction number R_0.

Recently, considering the spatial factor, the optimal control problem of spatiotemporal models is addressed in [3] and [5] as well as [10] for tumor-immune interactions governed by a reaction-diffusion system and the references cited therein.

In this study, three control strategies are explored for the SEIWR model, see (2.4): one concern the susceptible individuals by involving a control u_1 using vaccination, social distancing, or quarantine measures to limit contact between susceptible and infected individuals, another based on a treatment protocol u_2 for the infected individuals I to enhance recovery, and the third as a water purification or a pathogen-suppressing drugs u_3 to reduce the pathogen concentration W from the water to make it safe for drinking or other uses, which aimed to minimize the density of susceptible and infected individuals while minimizing the costs associated with the control strategies.
This paper's structure is set up as follows. Section 2 is devoted to the dynamic analysis of the investigated model. We outline the issue formulation of the dynamic model in Section 2.1 The purpose of subsection 2.2 is to show the well-posedness of the investigated system (2.1). In subsection 2.3, we study the dynamical behavior of (2.1), such as the existence of equilibria and global stability of the disease free equilibrium and the endemic equilibrium. Finally, in subsection 2.4, we present some numerical simulations to support the stability results.
In Section 3, optimal control strategies are proposed. We establish some uniform estimates for the strong solution for the controlled system (2.4) in subsection 3.1. Furthermore in Section 3.2, the existence of an optimal control is proven and the required first-order optimality conditions are derived. Finally, Section 3.3 provides numerical simulations to confirm the proposed optimal control strategies.

2. Dynamical model

2.1. Problem formulation

In this work, let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, be a bounded domain with smooth boundary $\partial\Omega$. We define $Q_T := (0, T) \times \Omega$ for any fixed $T \in (0, \infty)$ and $\Sigma_T := (0, T) \times \partial\Omega$. We denote by ∂_ν the directional derivative along the outward unit normal vector ν on $\partial\Omega$. Building on our previous discussions, We consider temporal and spatial factors, latency, as well as nonlinear incidence rates. We consider the reaction-diffusion $SEWIR$ model governed by the following five equations:

$$\begin{cases} \partial_t S &= d_1\Delta S + \Lambda - \mu_1 S - Sf(W) - Sg(I), & \text{in } Q_T, \\ \partial_t E &= d_2\Delta E - (\mu_2+\sigma)E + Sf(W) + Sg(I), & \text{in } Q_T, \\ \partial_t I &= d_3\Delta I - (\mu_3+\delta+\gamma)I + \sigma E, & \text{in } Q_T, \\ \partial_t W &= d_4\Delta W - \xi W + \alpha I, & \text{in } Q_T, \\ \partial_t R &= d_5\Delta R - \mu_4 R + \gamma I, & \text{in } Q_T, \end{cases} \tag{2.1}$$

subject to the homogeneous Neumann boundary conditions:

$$\partial_\nu S = \partial_\nu E = \partial_\nu I = \partial_\nu R = 0, \text{ on } \Sigma_T, \tag{2.2}$$

and for $x \in \Omega$

$$S(0,x) = S_0(x),\ E(0,x) = E_0(x),\ I(0,x) = I_0(x),\ R(0,x) = R_0(x). \tag{2.3}$$

Moreover, based on biological reasons, we assume that the initial conditions of the system (2.4) satisfy

$$S_0(x) \geq 0\,, E_0(x) \geq 0\,, I_0(x) \geq 0\,, R_0(x) \geq 0, \text{ on } \Omega.$$

In this framework, the densities corresponding to susceptible, exposed, infected, recovered individuals and pathogen concentration at a specific time t and spatial position x are represented by $S(t,x)$, $E(t,x)$, $I(t,x)$, $R(t,x)$ and $W(t,x)$, respectively.
The parameter Λ represents the recruitment rate of the population, σ describes the transition rate from the latent phase E to the infected phase I. The pathogen shedding rate from infected individuals into the water compartment is α. Finally, parameter ξ describes the decay rate of pathogen in the water. δ refers to the death rate induced by the disease, and γ signifies the natural recovery rate after infection. Moreover, for $k = 1,\ldots,4$, μ_k signifies the natural death rates for the categories of susceptible, exposed, infected and recovered individuals, while for $k = 1,\ldots,5$, d_k is the diffusion rate correspond to any compartment. Lastly, $Sf(W)$ and $Sg(I)$ are defined as the nonlinear incidence rates.
In [30], the diffusive SIWR model is presented and the well-posedness is demonstrated and the analytic stability is established, under the following realistic assumptions (A_1)-(A_2) :
(A_1): Functions f and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are continuously differentiable, $f(0) = g(0) = 0$ and $f(W), g(I) > 0$, for $W, I > 0$;
(A_2): $f'(W), g'(I) > 0$ and $f''(W), g''(I) \leq 0$, for $W, I \geq 0$.
Biologically, assumptions (A_1) and (A_2) mean that the disease transmission rates are monotonically increasing, but subject to saturation effects. It is clear that this form of functions includes the common incidence functions such as $f(W) = \beta_1 \frac{W}{1+KW}$ and $g(I) = \beta_2 I$, where $\beta_1, \beta_2, K > 0$.
In this work, we implement optimal control strategies while aiming to minimize the resources allocated to disease control. A key aspect of this paper is that it does not concentrate on any particular disease. Instead, it introduces a general approach to tackling this category of optimization problems. Furthermore, we did not introduce a control law in the compartment of exposed individuals due to the difficulty in identifying these individuals in reality, as the symptoms of the disease in this group are not visible. The dynamics of the controlled system are represented by

$$\begin{cases} \partial_t S &= d_1\Delta S + \Lambda - (\mu_1+u_1)S - Sf(W) - Sg(I), & \text{in } Q_T, \\ \partial_t E &= d_2\Delta E - (\mu_2+\sigma)E + Sf(W) + Sg(I), & \text{in } Q_T, \\ \partial_t I &= d_3\Delta I - (\mu_3+\delta+\gamma+u_2)I + \sigma E, & \text{in } Q_T, \\ \partial_t W &= d_4\Delta W - (\xi+u_3)W + \alpha I, & \text{in } Q_T, \\ \partial_t R &= d_5\Delta R - \mu_4 R + u_1 S + (\gamma+u_2)I, & \text{in } Q_T, \end{cases} \tag{2.4}$$

subject to the same homogeneous Neumann boundary conditions (2.2) and initial conditions (2.3), with $\partial_\nu W = 0$ and $W(0,x) = W_0(x) \geq 0$.
Our objective is to minimize the density of susceptible and infected individuals as well as the costs associated with the control strategies. This can be represented as the optimization of the objective of the following weighted objective (or cost) functional

$$J(u)=\int_0^T\int_\Omega k_1S(t,x)+k_2I(t,x)\,dx\,dt+l_1u_1^2(t,x)\,dx+l_2u_2^2(t,x)+l_3u_3^2(t,x)\,dx, \tag{2.5}$$

where the positive constants k_1, k_2, l_1, l_2 and l_3 serve as weights that indicate the relative importance of each term in the objective functional, (S,E,I,W,R) is the solution to (2.4) satisfying the corresponding initial and Neumann boundary conditions, and (u_1,u_2) lies within the set of admissible controls

$$\mathcal{U}_{ad}=\left\{u=(u_1,u_2,u_3)\in\left[L^2(Q_T)\right]^3 : 0\le u_i\le u_i^{max}\le 1,\text{ a.e. in } Q_T,\ \forall i=1,3\right\}. \tag{2.6}$$

In this study, we consider a bounded domain $\Omega\subset\mathbb{R}^n$, with a boundary $\partial\Omega$ that is sufficiently smooth. Moreover, to simplify the notation, we will use C to represent a general positive constant, which may vary from one expression to another. We define $F(t,y):=(F_1(t,y),F_2(t,y),F_3(t,y),F_4(t,y),F_5(t,y))^\top$ in Q_T by

$$\begin{cases} F_1(t,y)=\Lambda-(\mu_1+u_1)y_1-y_1f(y_4)-y_1g(y_3),\\ F_2(t,y)=-(\mu_2+\sigma)y_2+y_1f(y_4)+y_1g(y_3),\\ F_3(t,y)=-(\mu_3+\delta+\gamma+u_2)y_3+\sigma y_2,\\ F_4(t,y)=-(\xi+u_3)y_4+\alpha y_3,\\ F_5(t,y)=-\mu_4y_5+u_1y_1+(\gamma+u_2)y_3.\end{cases} \tag{2.7}$$

For the uncontrolled model (2.1), that is when $u=0$, we denote to $F(t,y)$ by $F^0(t,y)$.
Taking into account the latent compartment E, who have exposed to the infection but not yet capable of transmitting the diseases. Our framework will be the same as [30] by considering the same hypothesis (A_1)-(A_2) and using the same arguments. Next, we show that our generalized SEIWR model of the SIWR model admits a unique global non-negative solution.

2.2. Well-posedness of the epidemic model

In this section, we will show that the uncontrolled system (2.1) with the conditions (2.2) $-$ (2.3) is well posed in a biological sence, that is the existence and uniqueness of a non negativity solution. For this purpose, we introduce the following notations.
Let $\mathbf{X}:=C\left(\bar{\Omega},\mathbb{R}^4\right)$ be the Banach space with the supremum form $\|\cdot\|_{\mathbf{X}}$ and define $\mathbf{X}_+:=C\left(\bar{\Omega},\mathbb{R}^4_+\right)$. Assume that $T_i(t):C(\bar{\Omega},\mathbb{R})\to C(\bar{\Omega},\mathbb{R})$ are the C_0 is the semigroups associated with $d_i\Delta-\alpha_i$, subject to the Neumann boundary condition, respectively for $i=1,\dots,5$, with $\alpha_1=\mu_1$, $\alpha_2=(\mu_2+\sigma)$, $\alpha_3=(\mu_3+\delta+\gamma)$, $\alpha_4=\xi$ and $\alpha_5=\mu_4$. It is obvious that for any $\varphi\in C(\bar{\Omega},\mathbb{R}), t\ge 0$, one has for $i=1,\dots,5$

$$T_i(t)\varphi(x)=e^{-\alpha_i t}\int_\Omega\Gamma_i(t,x,s)\varphi(s)ds$$

where Γ_i is the Green functions associated with $d_i\Delta$ subject to the Neumann boundary condition and it is well known that $T_i(t):C(\bar{\Omega},\mathbb{R})\to C(\bar{\Omega},\mathbb{R})$ is compact and strongly positive, see [19], [18, Cor.4] and [23]. Reformulating the system (2.1), the solution $y(x,t)=(S(x,t),E(x,t),I(x,t),W(x,t),R(x,t))$ can be written as

$$y(x,t)=T(t)\varphi(x)+\int_0^t T(t-s)F^0(y(x,s))ds,$$

where $T(t)=\operatorname{diag}(T_1(t),T_2(t),T_3(t),T_4(t),T_5(t))$. With these notations and discussion, we can prove the following existence result.

Theorem 2.1 *For any initial value $\varphi\in X_+$, the model (2.1) admits a unique non-negative global solution $y=(S,E,I,W,R)$.*

Proof: By a standard argument. For any $\varphi\in\mathbf{X}_+$ and $h>0$, we have

$$\varphi(x)+hF^0(\varphi)(x)=\begin{cases}\varphi_1(x)+h(\Lambda-\varphi_1(x)f(\varphi_4)(x)-\varphi_1(x)g(\varphi_3)(x))\\ \varphi_2(x)+h(\varphi_1(x)f(\varphi_4)(x)+\varphi_1(x)g(\varphi_3)(x))\geq 0\\ \varphi_3(x)+h\sigma\varphi_2(x)\geq 0\\ \varphi_4(x)+h\alpha\varphi_4(x)\geq 0\\ \varphi_5(x)+h\gamma\varphi_5(x)\geq 0\end{cases}$$

Using that $\varphi\in\mathbf{X}_+$ and the continuity of the functions f and g, it is easy to see that for h small enough

$$\lim_{h\to 0^+}\operatorname{dist}\left(\varphi+hF^0(\varphi),\mathbf{X}_+\right)=0$$

From [18, Cor.4], we know that system (2.1) has a unique mild solution $y(\cdot,t,\varphi)$ on $[0,\tau)$ with $y(\cdot,0,\varphi)=\varphi$ and $y(\cdot,t,\varphi)\in\mathbf{X}_+$ for $t\in[0,\tau)$, for some $\tau\leq+\infty$.
For the globality of the solution, it is sufficient to establish boundedness of $y(\cdot,t,\varphi)$ on $[0,\tau)$. In fact, we show that is uniformly bounded with resp to τ.
To this end, it is well known that $\|T_i(t)\|\leq M_ie^{-\alpha_i t}$, $t\geq 0$, for $i=1,\dots,5$, see [17] and [20], see also [23]. From the first equation of (2.1), we get

$$\partial_t S\leq d_1\Delta S+\Lambda-\mu_1 S,\text{ in } Q_\tau.$$

By the comparaison principle,

$$S(t,x)\leq S_1(t,x),$$

where $S_1(t,x)$ is the system's solution

$$\begin{cases}\partial_t S_1=d_1\Delta S_1-\mu_1S_1+\Lambda,\text{ in } Q_\tau,\\ S_1(0,.)=S^0,\\ \partial_\nu S_1=0,\text{ on }\partial\Omega.\end{cases}$$

That is

$$S_1(t,x)=T_1(t)S^0+\int_0^t T_1(t-s)\Lambda\,ds.$$

Then

$$\begin{aligned}\|S_1(t,x)\|_{L^\infty(\Omega)}&\leq M_1e^{-\mu_1 t}\|S^0\|_{L^\infty(\Omega)}+\Lambda M_1\int_0^t e^{-\mu_1(t-s)}\,ds\\ &=M_1\|S^0\|_{L^\infty(\Omega)}+\frac{\Lambda M_1}{\mu_1}:=M_1'\end{aligned}$$

Therefore

$$S(t,x)\leq M_1',\text{ in } Q_\tau, \tag{2.8}$$

Now define,

$$N(t)=\int_\Omega(S(t,x)+E(t,x)+I(t,x)+W(t,x))\,dx$$

By differentiating $N(t)$ and applying the Neumann boundary conditions, we obtain

$$\begin{aligned}\frac{\partial N}{\partial t}&\leq\int_\Omega(\Lambda-\mu_1S-\mu_2E-(\mu_3+\gamma)I)\,dx\\ &\leq\Lambda|\Omega|-aN(t),\end{aligned}$$

where $|\Omega|$ is the measure of Ω and $a=\min\{\mu_1,\mu_2,\mu_3+\gamma\}$. The comparison principle implies

$$N(t)\leq N(0)e^{-at}+\frac{\Lambda|\Omega|}{a}(1-e^{-at}),\quad\forall t\in[0,\tau].$$

This proves that $N(t)$ is bounded, and as a result, there exists $M_2' > 0$ such that

$$\int_\Omega I(t,x)\,dx \le M_2' \quad \text{and} \quad \int_\Omega E(t,x)\,dx \le M_2', \quad \forall t \in [0,\tau]. \tag{2.9}$$

Similarly, from the second equation of (2.1), one can write

$$E(t,x) = T_2(t)E^0(x) + \int_0^t T_2(t-s)[Sf(W) + Sg(I)]\,ds, \text{ in } Q_\tau,$$

where $\{T_2(t)\}_{t\ge 0}$ is the semigroup generated by $B = d_2\Delta - \alpha_2$, with the homogeneous Neumann Boundary conditions and it is well known that it possess a green function $\Gamma_2(t,x,y)$ satisfies,

$$\Gamma_2(t,x,y) = \sum_{n=1}^{\infty} e^{\lambda_n t}\varphi_n(x)\varphi_n(y),$$

where λ_n is the eigenvalue corresponding to the eigenfunction $\varphi_n(x)$ uniformly bounded in $\bar{\Omega}$ with $\lambda_1 = -\alpha_2$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_n > \ldots$ and λ_n decreases like $-n^2$, see [12]. That is

$$\Gamma_2(t,x,y) = \kappa_1 \sum_{n=1}^{\infty} e^{\lambda_n t} \le \kappa_2 e^{-\alpha_2 t}, \quad t > 0, x, y \in \bar{\Omega},$$

where κ_1 and κ_2 are positive constants. Then

$$\begin{aligned} E(t,x) &= T_2(t)E^0(x) + \int_0^t \int_\Omega \Gamma_2(t-s,x,y)[Sf(W) + Sg(I)](s,y)\,dyds \\ &\le M_2 e^{-\alpha_2 t}\|E^0\|_{L^\infty(\Omega)} + \kappa_2 M_1' f'(0) \int_0^t e^{-\alpha_2(t-s)} \int_\Omega W(s,y)\,dyds \\ &+ \kappa_2 M_1' g'(0) \int_0^t e^{-\alpha_2(t-s)} \int_\Omega I(s,y)\,dyds \\ &\le M_2 \|E^0\|_{L^\infty(\Omega)} + \frac{\kappa_2 M_1' M_2'(f'(0) + g'(0))}{\alpha_2} := M_2''. \end{aligned}$$

Here we use (2.8), (2.9) and the Assumptions (A_1)-(A_2) and . Using a similar argument as for S, and based on the third equation in (2.1) along with the maximum principle, we get for any $x \in \Omega$ and $t \in [0,\tau]$

$$\begin{aligned} I(t,x) &\le M_3 e^{-\alpha_3 t}\|I^0\|_{L^\infty(\Omega)} + M_3 \int_0^t e^{-\alpha_3(t-s)}\sigma E(x,s)\,ds \\ &\le M_3\|I^0\|_{L^\infty(\Omega)} + \frac{\sigma M_3 M_2''}{\alpha_3} =: M_3', \end{aligned}$$

where $\alpha_3 := \mu_3 + \gamma > 0$.
Similarly as in above, from the fourth equation in (2.1) and denote $\alpha_4 := -\xi$, we get also for any $x \in \Omega$ and $t \in [0,\tau]$

$$\begin{aligned} W(t,x) &\le M_4 e^{-\alpha_4 t}\|W^0\|_{L^\infty(\Omega)} + M_4 \int_0^t e^{-\alpha_4(t-s)}\alpha I(x,s))\,ds \\ &\le M_4\|W^0\|_{L^\infty(\Omega)} + M_4\frac{\alpha M_3'}{\alpha_4} =: M_4'. \end{aligned}$$

Finally, similarly from the fifth equation, we obtain there exists a constant $M_5 >$ such that for any $x \in \Omega$ and $t \in [0,\tau]$

$$R(t,x) \le M_5'.$$

Consequently, the local solution $y(\cdot,t,\varphi)$ is uniformly bounded in Q_τ with respect to τ. □

2.3. Dynamical behavior of the epidemic model

2.3.1. Existence of equilibrium points. In this subsection, we will discuss the existence of spatially homogeneous steady for the uncontrolled system (2.1)–(2.3). These steady states correspond to solutions that do not depend on time, and thus must satisfy the following system of coupled partial differential equations:

$$\begin{cases} -d_1\Delta S(x) &=& \Lambda - \mu_1 S(x) - S(x)f(W(x)) - S(x)g(I(x)), \\ -d_2\Delta E(x) &=& -(\mu_2+\sigma)E(x) + S(x)f(W(x)) + S(x)g(I(x)), \\ -d_3\Delta I(x) &=& -(\mu_3+\delta+\gamma)I(x) + \sigma E(x) \\ -d_4\Delta W(x) &=& -\xi W + \alpha I, \\ -d_5\Delta R(x) &=& -\mu_4 R + \gamma I, \\ \partial_\nu S = \partial_\nu E &=& \partial_\nu I = \partial_\nu W = \partial_\nu R = 0, \text{ on } \partial\Omega. \end{cases} \tag{2.10}$$

In this work, we limit our analysis to spatially homogeneous steady states. Under this assumption, any non-negative solution of the following system of coupled algebraic equations represents a spatially homogeneous steady state $P^* = (S^*, E^*, I^*, W^*, R^*)$ of the uncontrolled system described by equations (2.1)-(2.3):

$$\begin{cases} \Lambda - \mu_1 S^* &=& S^* f(W^*) + S^* g(I^*), \\ S^* f(W^*) + S^* g(I^*) &=& (\mu_2+\sigma)E^*, \\ (\mu_3+\delta+\gamma)I^* &=& \sigma E^* \\ \xi W^* &=& \alpha I^*, \\ \mu_4 R^* &=& \gamma I^*, \end{cases} \tag{2.11}$$

By analyzing the system of algebraic equations presented above and defining the basic reproduction number

$$\mathcal{R}_0 = \frac{\Lambda}{\mu_1} \cdot \frac{\sigma}{(\mu_2+\sigma)(\mu_3+\delta+\gamma)} \cdot \left(f'(0) \cdot \frac{\alpha}{\xi} + g'(0) \right),$$

we obtain the following result.

Theorem 2.2 *The uncontrolled system (2.1)-(2.3) has always the disease-free equilibriuma unique disease-free equilibrium $P_0 = (S_0, 0, 0, 0, 0)$, where $S_0 = \frac{\Lambda}{\mu}$. If the basic reproduction number $\mathcal{R}_0 > 1$, the uncontrolled system (2.1)-(2.3) admits a unique endemic equilibrium, denoted by $P^* = (S^*, E^*, I^*, W^*, R^*)$.*

Proof: It is Obvious that the model (5.1) has always the disease-free equilibrium $P_0 = (S_0, 0, 0, 0, 0)$. From the first, third, fourth, and fifth equations, we have:

$$S^* = \frac{\Lambda}{\mu_1 + f(W^*) + g(I^*)},\ E^* = \frac{(\mu_3+\delta+\gamma)}{\sigma} I^*,\ W^* = \frac{\alpha}{\xi} I^* \text{ and } R^* = \frac{\gamma}{\mu_4} I^*.$$

By substituting these into the second equation, we obtain $h(I) = 0$, where

$$h(I) := \frac{\Lambda}{\mu_1 + f\left(\frac{\alpha}{\xi} I\right) + g(I)} \left(f\left(\frac{\alpha}{\xi} I\right) + g(I) \right) - (\mu_2+\sigma)\frac{(\mu_3+\delta+\gamma)}{\sigma} I.$$

It is clear that $h(+\infty) = -\infty$ and $h(0) = 0$. Moreover, given that $h'(0) > 0$, it follows that the equation $h(I) = 0$ has at least one positive solution, which we denote by I^*, where

$$\begin{aligned} h'(0) &= \frac{\Lambda}{\mu_1} \left(f'(0) \cdot \frac{\alpha}{\xi} + g'(0) \right) - (\mu_2+\sigma) \cdot \frac{\mu_3+\delta+\gamma}{\sigma} \\ &= (\mu_2+\sigma) \cdot \frac{\mu_3+\delta+\gamma}{\sigma} \cdot (\mathcal{R}_0 - 1) > 0. \end{aligned}$$

This leads to $\mathcal{R}_0 > 1$. Consequently, equation (2.11) admits at least one positive solution satisfying the following relations

$$S^* = \frac{\Lambda}{\mu_1 + f\left(\frac{\alpha}{\xi} I^*\right) + g(I^*)},\ E^* = \frac{(\mu_3+\delta+\gamma)}{\sigma} I^*,\ W^* = \frac{\alpha}{\xi} I^* \text{ and } R^* = \frac{\gamma}{\mu_4} I^*.$$

We now proceed to demonstrate the uniqueness of the endemic equilibrium. Note that

$$h'(I) = \Lambda \cdot \frac{\mu_1 \left(f'\left(\frac{\alpha}{\xi} I\right) \cdot \frac{\alpha}{\xi} + g'(I) \right)}{\left(\mu_1 + f\left(\frac{\alpha}{\xi} I\right) + g(I) \right)^2} - (\mu_2 + \sigma) \cdot \frac{\mu_3 + \delta + \gamma}{\sigma}.$$

and

$$h''(I) = \Lambda \cdot \mu_1 \cdot \frac{f''\left(\frac{\alpha}{\xi} I\right)\left(\frac{\alpha}{\xi}\right)^2 + g''(I)}{\left(\mu_1 + f\left(\frac{\alpha}{\xi} I\right) + g(I) \right)^2} - 2\Lambda \cdot \mu_1 \cdot \frac{\left(f'\left(\frac{\alpha}{\xi} I\right) \cdot \frac{\alpha}{\xi} + g'(I) \right)^2}{\left(\mu_1 + f\left(\frac{\alpha}{\xi} I\right) + g(I) \right)^3}.$$

From hypothesis (A_2), it follows that $h''(I) < 0$ for every $I > 0$, which implies that the function h is strictly concave on the positive real axis. Let us suppose, for the sake of contradiction, that the equation $h(I) = 0$ admits at least two distinct positive solutions. this would imply the existence of a point $I^* > 0$ such that satisfies $h''(I^*) = 0$, which contradicts the strict concavity of h on the positive real axis. Therefore, the endemic equilibrium is unique. □

2.3.2. Global stability. In this subsection, we investigate the global asymptotic stability of two equilibria based on the value of the basic reproduction number R_0. By constructing suitable Lyapunov functions, we show that the disease-free equilibrium is globally asymptotically stable when $R_0 \leq 1$, while the endemic equilibrium becomes globally asymptotically stable when $R_0 > 1$.

Theorem 2.3 *If $R_0 \leq 1$, then the disease-free equilibrium P_0 of the uncontrolled system (2.1)-(2.3) is globally asymptotically stable.*

Proof: We consider the following Lyapunov function

$$L(t) = \int_\Omega \left[\left(S - S_0 - S_0 \ln\left(\frac{S}{S_0}\right) \right) + E + \frac{(\mu_2 + \sigma)}{\sigma} I + \frac{f'(0) S_0}{\xi \mathcal{R}_0} W \right] dx \tag{2.12}$$

The time derivative of L_1 along the trajectories of the uncontrollable system (2.1)-(2.3) is given by:

$$\begin{aligned}
\frac{d}{dt} L(t) = \int_\Omega \Bigg[& \left(1 - \frac{S_0}{S}\right) (d_1 \Delta S + \Lambda - \mu_1 S - S f(W) - S g(I)) \\
& + (d_2 \Delta E - (\mu_2 + \sigma) E + S f(W) + S g(I)) \\
& + \frac{\mu_2 + \sigma}{\sigma} (d_3 \Delta I - (\mu_3 + \delta + \gamma) I + \sigma E) \\
& + \frac{f'(0) S_0}{\xi \mathcal{R}_0} (d_4 \Delta W - \xi W + \alpha I) \Bigg] dx
\end{aligned} \tag{2.13}$$

Leveraging the divergence theorem and the Neumann boundary conditions, we can deduce:

$$\begin{aligned}
\frac{dL(t)}{dt} &= \int_\Omega \Bigg\{ - d_1 S_0 \frac{\|\nabla S\|^2}{S^2} - d_2 \|\nabla E\|^2 - \frac{d_3(\mu_2 + \sigma)}{\sigma} \|\nabla I\|^2 - \frac{d_4 f'(0) S_0}{\xi \mathcal{R}_0} \|\nabla W\|^2 \\
&\quad - \mu_1 \frac{(S - S_0)^2}{S} + S_0 (f(W) + g(I)) - \frac{(\mu_2 + \gamma)(\mu_3 + \delta + \gamma)}{\sigma} I + \frac{f'(0) S_0}{\xi \mathcal{R}_0} (\alpha I - \xi W) \Bigg\} dx \\
&= \int_\Omega \Bigg\{ - d_1 S_0 \frac{\|\nabla S\|^2}{S^2} - d_2 \|\nabla E\|^2 - \frac{d_3(\mu_2 + \sigma)}{\sigma} \|\nabla I\|^2 - \frac{d_4 f'(0) S_0}{\xi \mathcal{R}_0} \|\nabla W\|^2 \\
&\quad - \mu_1 \frac{(S - S_0)^2}{S} + S_0 (f(W) + g(I)) - \frac{(f'(0)\frac{\alpha}{\xi} + g'(0)) S_0}{\mathcal{R}_0} I + \frac{f'(0) S_0}{\xi \mathcal{R}_0} (\alpha I - \xi W) \Bigg\} dx \\
&= \int_\Omega \Bigg\{ - d_1 S_0 \frac{\|\nabla S\|^2}{S^2} - d_2 \|\nabla E\|^2 - \frac{d_3(\mu_2 + \sigma)}{\sigma} \|\nabla I\|^2 - \frac{d_4 f'(0) S_0}{\xi \mathcal{R}_0} \|\nabla W\|^2 \\
&\quad - \mu_1 \frac{(S - S_0)^2}{S} + S_0 I \left(\frac{g(I)}{I} - \frac{g'(0)}{\mathcal{R}_0} \right) + S_0 W \left(\frac{f(W)}{W} - \frac{f'(0)}{\mathcal{R}_0} \right) \Bigg\} dx.
\end{aligned} \tag{2.14}$$

Using the assumptions (A_1)-(A_2), the functions

$$W \mapsto \frac{f(W)}{W} \quad \text{and} \quad I \mapsto \frac{g(I)}{I}$$

are decreasing on $(0, +\infty)$. As a consequence, we obtain:

$$\frac{f(W)}{W} \leq f'(0) \quad \text{and} \quad \frac{g(I)}{I} \leq g'(0), \quad \text{for all } W, I > 0.$$

Therefore, we have

$$\begin{aligned} \frac{dL(t)}{dt} &= \int_\Omega \Big\{ - d_1 S_0 \frac{\|\nabla S\|^2}{S^2} - d_2 \|\nabla E\|^2 - \frac{d_3(\mu_2+\sigma)}{\sigma}\|\nabla I\|^2 - \frac{d_4 f'(0) S_0}{\xi \mathcal{R}_0}\|\nabla W\|^2 \\ &\quad -\mu_1 \frac{(S-S_0)^2}{S} + S_0 g'(0) I \left(1 - \frac{1}{\mathcal{R}_0}\right) + S_0 f'(0) W \left(1 - \frac{1}{\mathcal{R}_0}\right) \Big\} dx. \end{aligned} \tag{2.15}$$

Since $\mathcal{R}_0 \leq 1$, it follows that $\frac{dL(t)}{dt} \leq 0$, for all $t \geq 0$. Moreover, from the previous expression, we have

$$\frac{dL(t)}{dt} = 0 \quad \text{if and only if} \quad S(t) = S_0, \quad E(t) = 0, \quad I(t) = 0, \quad \text{and} \quad W(t) = 0.$$

Therefore, the largest invariant set contained in

$$\left\{ (S, E, I, W, R) \middle| \, \frac{dL}{dt} = 0 \right\}$$

the set

$$\mathcal{D}_1 = \Big\{ (S, E, I, W, R) \Big| S = S_0, \, E = 0, \, I = 0, \, W = 0 \Big\}.$$

Therefore, by LaSalle's Invariance Principle and using the restriction of the last equation to $\mathcal{D}_1$, we conclude that $P_0 = (S_0, 0, 0, 0, 0)$ is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.

□

Theorem 2.4 *If $\mathcal{R}_0 > 1$, the endemic equilibrium P^* of the uncontrolled system (2.1)-(2.3) is globally asymptotically stable.*

Proof: Let us consider the following Lyapunov function

$$L(t) = L_1(t) + L_2(t),$$

where

$$L_1(t) = \int_\Omega \left[S - S^* - S^* \ln\left(\frac{S}{S^*}\right) + E - E^* - E^* \ln\left(\frac{E}{E^*}\right) \right] dx$$

and

$$L_2(t) = \int_\Omega \left[\frac{(\mu_2+\sigma)}{\sigma} \left(I - I^* - I^* \ln\left(\frac{I}{I^*}\right)\right) + \frac{S^* f(W^*)}{\alpha I^*} \left(W - W^* - W^* \ln\left(\frac{W}{W^*}\right)\right) \right] dx$$

By calculating the time derivatives of L_1 and L_2 along the solution of the system, we obtain

$$\begin{aligned} \frac{dL_1(t)}{dt} &= \int_\Omega \Big[\left(1 - \frac{S^*}{S}\right) (d_1 \Delta S + \Lambda - \mu_1 S - S f(W) - S g(I)) \\ &\quad + \left(1 - \frac{E^*}{E}\right) (d_2 \Delta E - (\mu_2+\sigma) E + S f(W) + S g(I)) \Big] dx \end{aligned}$$

and

$$\begin{aligned} \frac{dL_2(t)}{dt} &= \int_\Omega \Big[\Big(\frac{\mu_2+\sigma}{\sigma}\Big) \left(1 - \frac{I^*}{I}\right) (d_3 \Delta I - (\mu_3+\delta+\gamma) I + \sigma E) \\ &\quad + \frac{S^* f(W^*)}{\alpha I^*} \left(1 - \frac{W^*}{W}\right) (d_4 \Delta W - \xi W + \alpha I) \Big] dx \end{aligned}$$

By applying the divergence theorem, the Neumann boundary conditions, and (2.11), we can derive

$$\begin{aligned}
\frac{dL(t)}{dt} &= \int_{\Omega}\Bigg\{ -d_1S^*\frac{\|\nabla S\|^2}{S^2} - d_2E^*\frac{\|\nabla E\|^2}{E^2} - \frac{d_3(\mu_2+\sigma)}{\sigma}I^*\frac{\|\nabla I\|^2}{I^2} - \frac{d_4S^*f(W^*)}{\alpha I^*}W^*\frac{\|\nabla W\|^2}{W^2}\\
&+\left[\left(1-\frac{S^*}{S}\right)\left(\mu_1S^*\left(1-\frac{S}{S^*}\right)+S^*f(W^*)\left(1-\frac{Sf(W)}{S^*f(W^*)}\right)+S^*g(I^*)\left(1-\frac{Sg(I)}{S^*g(I^*)}\right)\right)\right.\\
&+\left(1-\frac{E^*}{E}\right)\left(S^*f(W^*)\left(\frac{Sf(W)}{S^*f(W^*)}-\frac{E}{E^*}\right)+S^*g(I^*)\left(\frac{Sg(I)}{S^*g(I^*)}-\frac{E}{E^*}\right)\right)\\
&+S^*f(W^*)\left[\left(1-\frac{I}{I^*}\right)\left(\frac{E}{E^*}-\frac{I}{I^*}\right)+\left(1-\frac{W}{W^*}\right)\left(\frac{I}{I^*}-\frac{W}{W^*}\right)\right]+S^*g(I^*)\left(1-\frac{I}{I^*}\right)\left(\frac{E}{E^*}-\frac{I}{I^*}\right)\Bigg]\Bigg\}dx\\
&= \int_{\Omega}\Bigg\{ -d_1S^*\frac{\|\nabla S\|^2}{S^2} - d_2E^*\frac{\|\nabla E\|^2}{E^2} - \frac{d_3(\mu_2+\sigma)}{\sigma}I^*\frac{\|\nabla I\|^2}{I^2} - \frac{d_4S^*f(W^*)}{\alpha I^*}W^*\frac{\|\nabla W\|^2}{W^2}\\
&+\mu_1S^*\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right)+S^*f(W^*)\left[\left(1-\frac{f(W^*)}{f(W)}\right)\left(\frac{f(W)}{f(W^*)}-\frac{W}{W^*}\right)+5-\frac{S^*}{S}-\frac{E^*Sf(W)}{ES^*f(W^*)}\right.\\
&\left.-\frac{I^*E}{IE^*}-\frac{W^*I}{WI^*}-\frac{Wf(W^*)}{W^*f(W)}\right]+S^*g(I^*)\left[\left(1-\frac{g(I^*)}{g(I)}\right)\left(\frac{g(I)}{g(I^*)}-\frac{I}{I^*}\right)+4-\frac{S^*}{S}-\frac{E^*Sg(I)}{ES^*g(I^*)}\right.\\
&\left.-\frac{I^*E}{IE^*}-\frac{Ig(I^*)}{I^*g(I)}\right]\Bigg\}dx.
\end{aligned}$$

Assuming (A_1)-(A_2). If $\mathcal{R}_0 > 1$, we obtain that

$$\frac{dL(t)}{dt} \leq 0, \quad \text{for all } t \geq 0.$$

Moreover, from the previous expression, we have

$$\frac{dL(t)}{dt} = 0 \quad \text{if and only if} \quad S(t) = S^*, \quad E(t) = E^*, \quad I(t) = I^*, \quad \text{and} \quad W(t) = W^*.$$

Therefore, the largest invariant set contained in

$$\left\{(S,E,I,W,R)\,\middle|\, \frac{dL}{dt} = 0\right\}$$

the set

$$\mathcal{D}_2 = \left\{(S,E,I,W,R)\,\middle|\, S = S^*,\ E = E^*,\ I = I^*,\ W = W^*\right\}.$$

Therefore, by LaSalle's Invariance Principle and using the restriction of the last equation to $\mathcal{D}_2$, we conclude that $P^* = (S^*, E^*, I^*, W^*, R^*)$ is globally asymptotically stable when $\mathcal{R}_0 > 1$.

□

2.4. Numerical simulations illustrating the stability results

In this subsection, we present a set of numerical simulations that aim to validate and illustrate the theoretical findings. We adopt the simplifying assumptions $f(W) = \beta_W W$ and $g(I) = \beta_I I$. Under these conditions, the system of equations (2.1)–(2.3) takes the following reduced form:

$$\begin{cases}
\partial_t S = d_1\Delta S + \Lambda - \mu_1 S - \beta_W SW - \beta_I SI, & \text{in } Q_T,\\
\partial_t E = d_2\Delta E - (\mu_2+\sigma)E + \beta_W SW + \beta_I SI, & \text{in } Q_T,\\
\partial_t I = d_3\Delta I - (\mu_3+\delta+\gamma)I + \sigma E, & \text{in } Q_T,\\
\partial_t W = d_4\Delta W - \xi W + \alpha I, & \text{in } Q_T,\\
\partial_t R = d_5\Delta R - \mu_4 R + \gamma I, & \text{in } Q_T,\\
\partial_\nu S = \partial_\nu E = \partial_\nu I = \partial_\nu W = \partial_\nu R = 0, & \text{on } \Sigma_T,\\
S(0,x,y) = S_0(x,y), E(0,x,y) = E_0(x,y), & \\
I(0,x,y,0) = I_0(x,y), W(0,x,y) = W_0(x,y), R(0,x,y) = R_0(x,y), & \text{on } \Omega.
\end{cases} \tag{2.16}$$

For the numerical implementation, we consider a two-dimensional spatial domain $\Omega = [0,1] \times [0,1]$ and set $d_1 = d_2 = d_3 = d_4 = d_5 = 0.2$. Under these assumptions, system (2.16) is supplemented with the following initial conditions:

$$\begin{aligned} S(0,x,y) &= |\cos(3\pi x)\cos(3\pi y)| \geq 0, & I(0,x,y) &= |\sin(2\pi x)\sin(2\pi y)| \geq 0, \\ W(0,x,y) &= |\sin(2\pi x)\sin(2\pi)| \geq 0, & R(0,x,y) &= |\sin(2\pi x)\sin(2\pi y)| \geq 0, \end{aligned} \quad (x,y) \in [0,1]\times[0,1]. \tag{2.17}$$

To numerically solve system (2.16), we discretize each of its equations using a finite difference approach. In particular, we employ the Crank–Nicolson method [9], which is widely used for solving parabolic partial differential equations.This scheme is second-order accurate in both time and space, and it is unconditionally stable.

In what follows, we briefly outline the implementation of the Crank–Nicolson method as applied to our model. To this end, we begin by discretizing the spatial domain $\Omega = [0,1]\times[0,1]$ and the temporal interval $[0,t_f]$ using uniform grids, as described below.

$$\begin{aligned} x_i &= (i-1)\,\Delta x, & i &= 1,2,\ldots,N_x+1, & \text{where } \Delta x &:= \frac{1}{N_x}, \\ y_j &= (j-1)\,\Delta y, & j &= 1,2,\ldots,N_y+1, & \text{where } \Delta y &:= \frac{1}{N_y}, \\ t_k &= (k-1)\,\Delta t, & k &= 1,2,\ldots,N_t+1, & \text{where } \Delta t &:= \frac{t_f}{N_t}. \end{aligned}$$

Therefore, using discretization, we can describe $S(x,y,t)$ as $S^k_{i,j}(i=1,\cdots,N_x+1,\ j=1,\cdots,N_y+1,,\ j=1,\cdots,N_t+1)$, $E(x,y,t)$ as $E^k_{i,j}(i=1,\cdots,N_x+1,\ j=1,\cdots,N_y+1,,\ j=1,\cdots,N_t+1)$, $I(x,y,t)$ as $I^k_{i,j}(i=1,\cdots,N_x+1,\ j=1,\cdots,N_y+1,,\ j=1,\cdots,N_t+1)$,$W(x,y,t)$ as $W^k_{i,j}(i=1,\cdots,N_x+1,\ j=1,\cdots,N_y+1,,\ j=1,\cdots,N_t+1)$ and $R(x,y,t)$ as $R^k_{i,j}(i=1,\cdots,N_x+1,\ j=1,\cdots,N_y+1, k=1,\cdots,N_t+1)$, respectively. In addition, we can discretize the system (2.16) as follows:

$$\begin{aligned}
\frac{S^{k+1}_{i,j}-S^k_{i,j}}{\Delta t} &= \frac{d_1}{2}\left\{\frac{S^{k+1}_{i+1,j}-2S^{k+1}_{i,j}+S^{k+1}_{i-1,j}}{\Delta x^2}+\frac{S^k_{i+1,j}-2S^k_{i,j}+S^k_{i-1,j}}{\Delta x^2}\right\} \\
&\quad +\frac{d_1}{2}\left\{\frac{S^{k+1}_{i,j+1}-2S^{k+1}_{i,j}+S^{k+1}_{i,j-1}}{\Delta y^2}+\frac{S^k_{i,j+1}-2S^k_{i,j}+S^k_{i,j-1}}{\Delta y^2}\right\} \\
&\quad +\Lambda-\mu_1 S^k_{i,j}-\beta_W S^k_{i,j}W^k_{i,j}-\beta_I S^k_{i,j}I^k_{i,j}, \\
\frac{E^{k+1}_{i,j}-E^k_{i,j}}{\Delta t} &= \frac{d_2}{2}\left\{\frac{E^{k+1}_{i+1,j}-2E^{k+1}_{i,j}+E^{k+1}_{i-1,j}}{\Delta x^2}+\frac{E^k_{i+1,j}-2E^k_{i,j}+E^k_{i-1,j}}{\Delta x^2}\right\} \\
&\quad +\frac{d_2}{2}\left\{\frac{E^{k+1}_{i,j+1}-2E^{k+1}_{i,j}+E^{k+1}_{i,j-1}}{\Delta y^2}+\frac{E^k_{i,j+1}-2E^k_{i,j}+E^k_{i,j-1}}{\Delta y^2}\right\} \\
&\quad +\beta_W S^k_{i,j}W^k_{i,j}+\beta_I S^k_{i,j}I^k_{i,j}-(\mu_2+\sigma)E^k_{i,j} \\
\frac{I^{k+1}_{i,j}-I^k_{i,j}}{\Delta t} &= \frac{d_3}{2}\left\{\frac{I^{k+1}_{i+1,j}-2I^{k+1}_{i,j}+I^{k+1}_{i-1,j}}{\Delta x^2}+\frac{I^k_{i+1,j}-2I^k_{i,j}+I^k_{i-1,j}}{\Delta x^2}\right\} \\
&\quad +\frac{d_3}{2}\left\{\frac{I^{k+1}_{i,j+1}-2I^{k+1}_{i,j}+I^{k+1}_{i,j-1}}{\Delta y^2}+\frac{I^k_{i,j+1}-2I^k_{i,j}+I^k_{i,j-1}}{\Delta y^2}\right\} \\
&\quad +\sigma E^k_{i,j}-(\mu_3+\delta+\gamma)I^k_{i,j}, \\
\frac{W^{k+1}_{i,j}-W^k_{i,j}}{\Delta t} &= \frac{d_4}{2}\left\{\frac{W^{k+1}_{i+1,j}-2W^{k+1}_{i,j}+W^{k+1}_{i-1,j}}{\Delta x^2}+\frac{W^k_{i+1,j}-2W^k_{i,j}+W^k_{i-1,j}}{\Delta x^2}\right\} \\
&\quad +\frac{d_4}{2}\left\{\frac{W^{k+1}_{i,j+1}-2W^{k+1}_{i,j}+W^{k+1}_{i,j-1}}{\Delta y^2}+\frac{W^k_{i,j+1}-2W^k_{i,j}+W^k_{i,j-1}}{\Delta y^2}\right\} \\
&\quad +\alpha I^k_{i,j}-\xi W^k_{i,j} \\
\frac{R^{k+1}_{i,j}-R^k_{i,j}}{\Delta t} &= \frac{d_5}{2}\left\{\frac{R^{k+1}_{i+1,j}-2R^{k+1}_{i,j}+R^{k+1}_{i-1,j}}{\Delta x^2}+\frac{R^k_{i+1,j}-2R^k_{i,j}+R^k_{i-1,j}}{\Delta x^2}\right\} \\
&\quad +\frac{d_5}{2}\left\{\frac{R^{k+1}_{i,j+1}-2R^{k+1}_{i,j}+R^{k+1}_{i,j-1}}{\Delta y^2}+\frac{R^k_{i,j+1}-2R^k_{i,j}+R^k_{i,j-1}}{\Delta y^2}\right\} \\
&\quad +\gamma I^k_{i,j}-\mu_4 R^k_{i,j}.
\end{aligned} \tag{2.18}$$

To incorporate the Neumann boundary conditions from problem (2.16) into the numerical scheme, we use the central difference method. Applying this approximation at the boundaries, along with the use of

equations (2.16), leads to a numerically stable recursive scheme.
In the following examples, all parameter values are taken from practical studies [26,21], which model the spread of cholera and other waterborne diseases. Furthermore, To numerically illustrate the global stability of equilibrium points, we modify the initial conditions of system (2.16). We perturb the initial conditions of system (2.16). Specifically, we begin with the analysis of the disease-free equilibrium P_0. By setting the parameters as follows: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.02$, $\Lambda = 0.02$, $\alpha = 0.04$, $\beta_W = \beta_I = 0.1072$, $\sigma = 0.4$, $\xi = 0.0333$, $\gamma = 0.25$, and $\delta = 0.25$, the basic reproduction number is computed as $R_0 = 0.4322$, which satisfies $R_0 \leq 1$. Based on Theorem 2.3, the disease-free equilibrium point $P_0 = (1, 0, 0, 0, 0)$ is globally asymptotically stable. As a consequence, regardless of the initial distributions of the susceptible, exposed, infected, recovered populations, and the concentration of pathogens in the environment, the system evolves toward a state in which only susceptible individuals persist. This result confirms that, under the given parameters, the infection cannot sustain itself and eventually disappears from the population (see Figure 1).

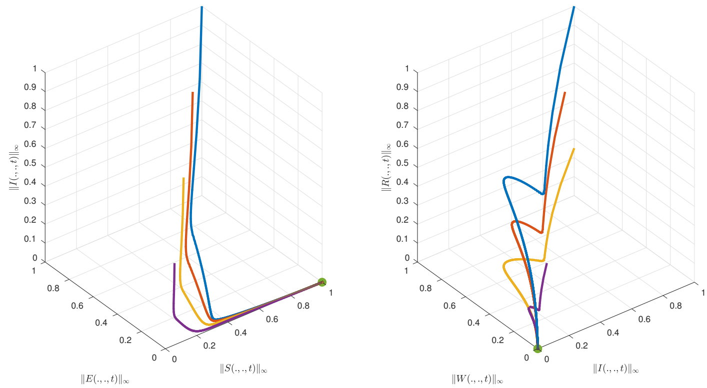

Figure 1: The spatiotemporal solution was obtained by numerically integrating system (2.1) subject to the boundary conditions (2.2) and initial conditions (2.3), with a basic reproduction number $R_0 = 0.4322 \leq 1$.

By setting $\Lambda = 0.2$ and $\xi = 0.333$, while keeping the other parameters fixed, we obtain $\mathcal{R}_0 = 2.1992$. According to the theorem 2.4, the equilibrium point $P^* = (4.5469,\ 0.2595,\ 0.1997,\ 0.0240, 2.4967)$ is asymptotically stable. This result indicates that the disease becomes endemic in the population (see Figure 2).

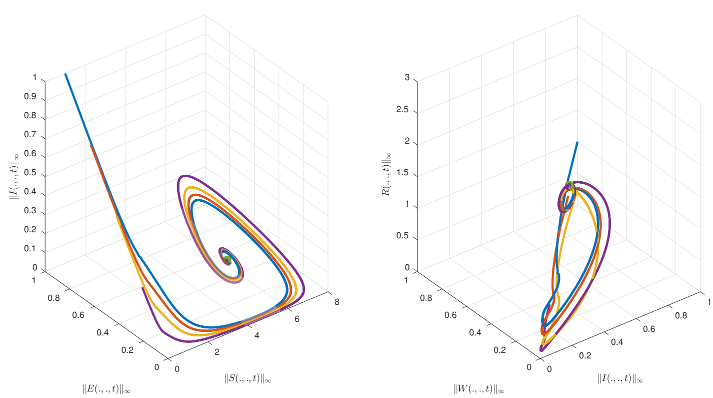

Figure 2: The spatiotemporal solution was obtained by numerically integrating system (2.1) subject to the boundary conditions (2.2) and initial conditions (2.3), with a basic reproduction number $R_0 = 2.1992 \geq 1$.

3. Optimal control strategies

For the Hilbert space $H = [L^2(\Omega)]^5$, let the initial value of the state variable $y = (y_1, y_2, y_3, y_4, y_5)^\top = (S, E, I, W, R)^\top$ be $y_0 = (S_0, E_0, I_0, , W_0, R_0)$ and let the linear operator $A : \mathcal{D}(A) \subset H \to H$ satisfy

$$Ay = (d_1\Delta y_1, d_2\Delta y_2, d_3\Delta y_3, d_4\Delta y_4, d_5\Delta y_5)^\top, \quad \forall y \in \mathcal{D}(A) \tag{3.1}$$

with

$$\mathcal{D}(A) = \{y \in [H^2(\Omega)]^5 : \partial_\nu y_1 = \partial_\nu y_2 = \partial_\nu y_3 = \partial_\nu y_4 = \partial_\nu y_5 = 0 \quad \text{on}\, \partial\Omega\}. \tag{3.2}$$

Recall that A is a dissipative operator in the Hilbert space H, since for each $y \in \mathcal{D}(A)$, the following holds:

$$\langle y, Ay\rangle = -d_1\|\nabla y_1\|^2_{L^2(\Omega)} - d_2\|\nabla y_2\|^2_{L^2(\Omega)} - d_3\|\nabla y_3\|^2_{L^2(\Omega)} - d_4\|\nabla y_4\|^2_{L^2(\Omega)} - d_5\|\nabla y_2\|^2_{L^2(\Omega)},$$

which is clear non-negative.
It is well known that A generates a C_0-semigroup of contractions in the Hilbert space H, see [27]. Moreover from [27, Lemma 1.6.1], , we can infer that operator A is self-adjoint in the Hilbert space H. Accordingly, the initial-boundary value problem (2.4) can be rewritten as the following abstract Cauchy problem

$$(\mathcal{P}) \begin{cases} \partial_t y = Ay + F(t, y), \text{ in}\, Q_T \\ y(0) = y_0, \quad \text{in}\, \Omega. \end{cases}$$

3.1. Existence and uniform estimates on the solution

In order to demonstrate the existence and uniqueness of the strong solution to the problem $(\mathcal{P})$, let's review the following result, see [6, Prop.1.2, p.175], see also [19,27]. We denote by $W^{1,2}([0, T], H)$ the space of all absolutely continuous functions $y : [0, T] \to H$ satisfying $\partial_t y \in L^2([0, T], H)$.

Theorem 3.1 *[[6], Proposition 1.2, Chapter IV], see also [19,27]*
Let X be a Banach space, $A : \mathcal{D}(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ in X, and $F : [0, T] \times X \to X$ be a function measurable with respect to t and Lipschitz continuous with respect to $y \in X$ uniformly for $t \in [0, T]$

i. If $y_0 \in X$, then the problem $(\mathcal{P})$ admits a unique mild (global) solution $y \in C([0, T]; X)$ and

$$y(t) = S(t)y_0 + \int_0^t S(t-s)F(s, y(s))\, ds, \quad \forall t \in [0, T].$$

ii. If X is a Hilbert space, A is dissipative and self-adjoint in X and $y_0 \in \mathcal{D}(A)$, then the mild solution is in fact a strong solution. Moreover, $y \in W^{1,2}(0, T, X) \cap L^2(0, T, \mathcal{D}(A))$.

Theorem 3.2 (Existence of the solution of the controlled system) *Let $\Omega \subset \mathbb{R}^n$, for $n = 1, 2, 3$, be a bounded domain with a sufficiently smooth boundary $\partial\Omega$, and let $T \in (0, +\infty)$ be fixed. Suppose that the assumptions (A_1) and (A_2) hold, and that the initial condition $y_0 = (S_0, E_0, I_0, W_0, R_0) \in \mathcal{D}(A)$, defined by (3.2), satisfies $S_0, E_0, I_0, W_0, R_0 > 0$ in Ω. For each control $u \in \mathcal{U}_{ad}$, the state system (2.4) has a unique strong solution $y = (S, E, I, W, R) \in \big[W^{1,2}(0, T, L^2(\Omega)) \cap L^2(Q_T) \cap L^2(0, T, H^2(\Omega))$ $\cap L^\infty(0, T, H^1(\Omega)) \cap L^\infty(Q_T)\big]^5$, where $S, E, I, W, R \geq 0$ in Ω. Additionally, there exists a constant $C > 0$, independent of both the control u and the corresponding state variable y, such that*

$$\|\partial_t y_i\|_{L^\infty(Q_T)} + \|y_i\|_{L^2(0,T,H^2(\Omega))} + \|y_i\|_{L^\infty(0,T,H^1(\Omega))} + \|y_i\|_{L^\infty(Q_T)} \leq C, \tag{3.3}$$

for $i = 1, \ldots, 5$.

Proof: The main idea here, is to apply Thm.3.1, we have see in section 3 that A is dissipative, self adjoint and generates a C_0-semigroup of contractions in H. However, it is clear that the Lipschitz continuity condition with respect to $y \in H$ uniformly with respect to $t \in [0, T]$ does not necessarily hold for the

function F. But with the use of a truncation function technique, Thm.3.1, can be applied. We divided the proof into the subsequent tree steeps.
Steep 1: Solutions of the truncated problem $(\mathcal{P}^M)$.
Following an idea of [43], for a sufficiently large constant $M > 0$, we define

$$y_1^M(t,x) := \begin{cases} M, & \text{if} \quad y_1(t,x) > M, \\ y_1(t,x), & \text{if} \quad |y_1(t,x)| \le M, \\ -M, & \text{if} \quad y_1(t,x)| < -M. \end{cases} \tag{3.4}$$

for $(t,x) \in Q_T$. Analogously, we define $F^M(t,x)$ and $y_i^M(t,x)$ for $i = 1,\dots,5$. Therefore the problem (2.4) can be expressed as the truncated abstract Cauchy problem can be transformed into the following truncated problem

$$\begin{cases} \partial_t y^M = Ay^M + F^M(t,y^M),\ t \in [0,T] \\ y^M(0) = y^0, \text{ in } \Omega. \end{cases} \tag{$\mathcal{P}^{\mathcal{M}}$}$$

Clearly, function $F^M(t,y^M)$ turns into Lipschitz continuous in y with respect to $t \in [0,T]$. Consequently, Thm.3.1 guarantees that the problem $(\mathcal{P}^{\mathcal{M}})$ admits a unique strong solution

$$y^M \in [W^{1,2}(0,T,L^2(\Omega)) \cap L^2(0,T,H^2(\Omega)]^5. \tag{3.5}$$

Step 2: Bounedness of y^M in Q_T. In order to show that $y^M \in [L^\infty(Q_T)]^5$, we denote $G^M := \max\{\|F(.,y^M)\|_{[L^\infty(Q_T)]^5}, \|y^0\|_{[L^2([0,T],H^2(\Omega))]^5}\}$ and $Y_1^M := y_1^M - G_M t - \|y_1^0\|_{L^\infty(\Omega)}$ for $(t,x) \in Q_T$. It is clear that Y_1^M satisfies the system

$$\begin{cases} \partial_t Y_1^M = d_1 \Delta Y_1^M + F_1^M(t,Y_1^M) - G^M,\ t \in [0,T] \\ Y_1^M(0) = y^0 - \|y_1^0\|_{L^\infty(\Omega)}, \text{ in } \Omega. \end{cases} \tag{3.6}$$

Using Thm.3.1, we conclude that (3.6) possesses a unique strong solution

$$Y_1^M(t) = S_1(t)(Y_1^0 - \|y_1^0\|_{L^\infty(\Omega)}) + \int_0^t S_1(t-s)[F_1^M(s,Y_1^M) - G^M]\, ds. \tag{3.7}$$

Here, $\{S_1(t)\}_{t\ge 0}$ refers to the C_0-semigroup that is generated by the operator $A_1 : D(A_1) \subset L^2(\Omega) \to L^2(\Omega)$ which is defined by

$$A_1 y_1 = d_1 \Delta y_1, \quad \mathcal{D}(A) = \{y_1 \in H^2(\Omega) : \partial_\nu y_1 = 0 \text{ on } \partial\Omega\}. \tag{3.8}$$

Since $Y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \le 0$ and $F_1^M(.,Y_1^M) - G^M \le 0$, then (3.7) gives $Y_1^M(t,x) \le 0$, for $(t,x) \in Q_T$. That is

$$y_1^M(t,x) \le G_M t + \|y_1^0\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q_T \tag{3.9}$$

Similarly, let $Z_1^M := y_1^M + G_M t + \|y_1^0\|_{L^\infty(\Omega)}$, for $(t,x) \in Q_T$ and we define the problem

$$\begin{cases} \partial_t Z_1^M = d_1 \Delta Z_1^M + F_1^M(t,Z_1^M) + G^M,\ t \in [0,T] \\ Z_1^M(0) = y^0 - \|y_1^0\|_{L^\infty(\Omega)}, \text{ in } \Omega. \end{cases} \tag{3.10}$$

Obviously, as for Y_1^M, we obtain that (3.10) possess a unique strong solution $Z_1^M(t,x) \ge 0$ for $(t,x) \in Q_T$. This means that

$$y_1^M(t,x) \ge -(G_M t + \|y_1^0\|_{L^\infty(\Omega)}), \quad \forall (t,x) \in Q_T. \tag{3.11}$$

Together (3.9) with (3.11) leads to

$$|y_1^M(t,x)| \le G_M t + \|y_1^0\|_{L^\infty(\Omega)} \le C(M), \quad \forall (t,x) \in Q_T. \tag{3.12}$$

where $C(M)$ is a constant that does not depend on the control u. This implies that $y_1^M \in L^\infty(Q_T)$. Like the proof of (3.12), we can infer from the equations of y_2^M, y_3^M, y_4^M and y_5^M that

$$|y_i^M(t,x)| \le G_M t + \|y_i^0\|_{L^\infty(\Omega)} \le C(M), \quad \forall (t,x) \in Q_T. \tag{3.13}$$

for all $i = 2, \dots, 4$. Consequently,

$$y^M \in [L^\infty(Q_T)]^5. \tag{3.14}$$

Step 3: Global solvablity, Validity of $y \in L^\infty(0, T, H^1(\Omega))$ and the estimate (3.3).
First of all, we show that y^M is a local solution of (2.4) defined in some Q_τ. For fixed $M > 2\|y^0\|_{[L^\infty(\Omega)]^5}$, there exists $\tau \in]0, T[$ such that

$$G_M \tau + \|y^0\|_{[L^\infty(\Omega)]^5} < M. \tag{3.15}$$

This together with (3.12) and (3.13) implies that

$$|y^M(t, x)| \leq M, \quad \forall (t, x) \in Q_\tau. \tag{3.16}$$

From the troncature form, we get that $y^M = y$ in Q_τ. That is y^M is a local solution of (2.4) in Q_τ. Furthermore, we have shown the non-negativity of the solutions, see Thm.2.1, we can deduce immediately the nonnegativity of y^M.
To infer the globality of the solution and (3.3), boundedness of y^M in Q_τ with respect to τ.
By following the same proof as it done for the globality in Thm. 2.1, the uniform boundedness of the local solution y is clear.
In the end, we show that $y \in L^\infty(0, T, H^1(\Omega))$ and the estimate (3.3). Using the green formula, for $i = 1, \dots, 5$, we derive from (2.4) and (2.7) that

$$\int_0^t \int_\Omega |\partial_t y_i|^2 \, dxds + d_i \int_0^t |\Delta y_i|^2 dxds + d_i \int_0^t |\nabla y_i|^2 \, dxds = \int_0^t \int_\Omega F_i^2(s, y) \, dxds + d_i \int_0^t |\nabla y_i^0|^2 \, dxds. \tag{3.17}$$

On account of $y^0 \in H^2(\Omega)$, $y_i \in L^2(0, T, H^2(\Omega)) \cap L^\infty(Q_T)$ and the uniform boundeness of y_i as well as the assumptions $(A_1) - (A_2)$, we obtain from (3.17) that $y_i \in L^\infty(0, T, H^1(\Omega))$ and

$$\|\partial_t y_i\|_{L^\infty(Q_T)} + \|y_i\|_{L^2(0,T,H^2(\Omega))} + \|y_i\|_{L^\infty(0,T,H^1(\Omega))} + \|y_i\|_{L^\infty(Q_T)} \leq C, \tag{3.18}$$

for $i = 1, \dots, 5$, where the constant $C > 0$ is independent of the control u. □

3.2. Existence of optimal control and optimality conditions

Theorem 3.3 (Existence of the optimal control) *Under the assumptions of Theorem 3.2. The functional (2.5) possesses at least one optimal control $u^* \in \mathcal{U}_{ad}$. That is a solution of the control problem*

$$J(u^*) = \inf_{u \in \mathcal{U}_{ad}} J(u) \tag{3.19}$$

Proof: From Thm.3.2, we know that for every $u \in \mathcal{U}_{ad}$ (which is clearly non empty), a unique non-negative strong solution to the system (2.4) exists, then $\inf_{u \in \mathcal{U}_{ad}} J(u)$ is finite. Let $\{u_n\}_n$ be a minimizing sequence such that

$$\lim_{n \to +\infty} J(u^n) = \inf_{u \in \mathcal{U}_{ad}} J(u),$$

where $y^n = (y_1^n, y_2^n, \dots, y_5^n)$ is the solution of system (2.4) corresponding to the control $u^n = (u_1^n, u_2^n, u_3^n)$ for $n = 1, 2, \dots$
That is

$$\begin{aligned}
\partial_t S &= d_1 \Delta y_1^n + \Lambda - (\mu_1 + u_1^n) y_1^n - y_1^n f(y_4^n) - y_1^n g(y_3^n), && \text{in } Q_T,, && (3.20a)\\
\partial_t y_2^n &= d_2 \Delta y_2^n - (\mu_2 + \sigma) y_2^n + y_1^n f(y_4^n) + y_1^n g(y_3^n), && \text{in } Q_T, && (3.20b)\\
\partial_t y_3^n &= d_3 \Delta y_3^n - (\mu_3 + \delta + \gamma + u_2^n) y_3^n + \sigma y_2^n, && \text{in } Q_T, && (3.20c)\\
\partial_t y_4^n &= d_4 \Delta y_4^n - (\xi + u_3^n) y_4^n + \alpha y_3^n, && \text{in } Q_T, && (3.20d)\\
\partial_t y_5^n &= d_5 \Delta y_5^n - \mu_4 y_5^n + u_1^n y_1^n + (\gamma + u_2^n) I, && \text{in } Q_T, && (3.20e)\\
\partial_\nu y_1^n &= \partial_\nu y_2^n = \partial_\nu y_3^n = \partial_\nu y_4^n = \partial_\nu y_5^n = 0, && \text{on } \Sigma_T, && (3.20f)\\
y_1^n(0, x) &= y_2^n(0, x) = y_3^n(0, x) = y_4^n(0, x) = y_5^n(0, x) = 0, && \text{on } \Omega. && (3.20g)
\end{aligned}$$

Again by Thm.3.2, we have the estimate (3.3) which yields there exists $C > 0$ such that

$$\|\partial_t y_i^n\|_{L^\infty(Q_T)} + \|y_i^n\|_{L^2(0,T,H^2(\Omega)} + \|y_i^n\|_{L^\infty(0,T,H^1(\Omega))} + \|y_i^n\|_{L^\infty(Q_T)} \leq C, \tag{3.21}$$

for $i = 1, \ldots, 5$. Then, a subsequence can be extracted and similarly denoted by a y_i^n such that

$$\partial_t y_i^n \to \partial_t y_i^* \text{ weakly in } L^2(Q_T) \quad \text{and} \quad \Delta y_i^n \to \Delta y_i^* \text{ weakly in } L^2(Q_T) \tag{3.22}$$

and

$$y_i^n \to y_i^* \quad \text{weakly star in} \quad L^\infty(Q_T) \tag{3.23}$$

for $i = 1, \ldots, 5$. Furthermore, considering the embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow L^2(\Omega)$ and taking account of the compact embedding arguments, see [22], Cor.4], we get

$$y_i^n \to y_i^* \text{ strongly in } C([0,T], L^2(\Omega)) \quad \text{and} \quad \text{strongly in } L^2(Q_T), \tag{3.24}$$

for $i = 1, \ldots, 5$. Furthermore, we can obtain the following convergence results

$$y_1^n f(y_4^n) \to y_1^* f(y_4^*) \text{ strongly in } L^2(Q_T) \quad \text{and} \quad y_1^n g(y_3^n) \to y_1^*) g(y_3^*) \text{ strongly in } L^2(Q_T). \tag{3.25}$$

Indeed, we decompose

$$y_1^n f(y_4^n) - y_1^* f(y_4^*) = [y_1^n - y_1^*] f(y_4^n) + y_1^* [f(y_4^n) - f(y_4^*)].$$

We make use the continuously differentiability of the function f given by $(A_1) - (A_2)$, the uniform convergence (3.24), the uniform boundness of y_4^n in $L^\infty(Q_T)$ and that $y_4^* \in L^\infty(Q_T)$ given by (3.21) and (3.23). Which implies in one hand $\|f(y_4^n) - f(y_4^*)\|_{L^2(Q_T)} \to 0$ as $n \to +\infty$ and we use $y_1^* \in L^\infty(Q_T)$. In the other hand give the uniform boundness of $f(y_4^n)$ in $L^\infty(Q_T)$ and then $\|(y_1^n - y_1^*) f(y_4^n)\|_{L^2(Q_T)} \to 0$ as $n \to +\infty$. Following a similar argument as above, we can immediately derive the second convergence for (3.25).

Furthermore, since $\{u^n\}_{n\geq 1} \subset \mathcal{U}_{ad}$ which is clear bounded in $L^2(Q_T)$. We can then extract a subsequence from $\{u^n\}_{n\geq 1}$, still denoted by u_n, such that $u_n \rightharpoonup u^*$ weakly in $L^2(Q_T)$, which along with the weak closedness (closed and convex) of $\mathcal{U}_{ad}$ entails $u^* \in \mathcal{U}_{ad}$. By writing

$$u_1^n y_1^n - u_1^* y_1^* = u_1^n (y_1^n - y_1^*) + (u_1^n - u_1^*) y_1^*.$$

Making use of the convergence (3.24) and the uniform boundness of $u^n \in L^\infty(Q_T)$ (since $u^n \in \mathcal{U}_{ad}$), we obtain the weakly convergence of $u_1^n y_1^n$ to $u_1^* y_1^*$. Analogously, we derive the weakly convergence of $u_2^n y_3^n$ to $u_2^* y_3^*$ and $u_3^n y_4^n$ to $u_3^* y_4^*$

Due to (3.24), by taking $n \to +\infty$ in (3.20) and using the existence and uniqueness of the solution to (3.20), we conclude that y^* is the solution of (2.4) associated to $u^* \in \mathcal{U}_{ad}$ and therefore

$$\begin{aligned}
J(u^*) &= \int_0^T \int_\Omega k_1 S^*(t,x) + k_2 I^*(t,x) + l_1 (u_1^*)^2(t,x) + l_2 (u_2^*)^2(t,x) + l_3 (u_3^*)^2(t,x)\, dx\, dt \\
&\leq \int_0^T \int_\Omega k_1 S^*(t,x) + k_2 I^*(t,x)\, dx\, dt + \liminf_{n\to+\infty} \int_0^T \int_\Omega l_1 (u_1^n)^2(t,x) + l_2 (u_2^n)^2(t,x) + l_3 (u_3^*)^2(t,x)\, dx \\
&= \liminf_{n\to+\infty} J(u^n) \\
&= \inf_{u\in\mathcal{U}_{ad}} J(u),
\end{aligned}$$

thanks to the weak l.s.c. of the convex functions Cv^2. Thus, the proof of Thm.3.3 is complete □

To derive the first-order optimality conditions, firstly, we need to show the Gateaux-Differentiability of the control-to-state mapping.

$$\begin{array}{rcl} G : \mathcal{U}_{ad} \subset [L^2(Q_T)]^2 & \longrightarrow & Y \subset [L^2(Q_T)]^5 \\ u & \longmapsto & y(u) \end{array} \tag{3.26}$$

where

$$Y = [W^{1,2} \cap L^2(Q_T) \cap L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^\infty(Q_T)]^5.$$

Proposition 3.4 *There exists a linear bounded operator $G'(u^*)$ such that for any $v \in [L^2(Q_T)]^2$:*

$$\lim_{\varepsilon\to 0} \|\frac{G(u^*+\varepsilon v)-G(u^*)}{\varepsilon} - G'(u^*)v\|_{[L^2(Q_T)]^5} = 0. \tag{3.27}$$

That is the mapping G is Gateaux-differentiable at u^*. Moreover, $G'(u^*)v := z = z(v) = (z_1(v), z_2(v), z_3(v), z_4(v))$ is the solution of the linearized system

$$\begin{cases} \partial_t z_1 & = d_1\Delta z_1 - [\mu_1 + u_1^* + f(y_4^*) + g(y_3^*)]z_1 - y_1^* f'(y_4^*)z_4 - y_1^* g'(y_3^*)z_3 - y_1^* v_1, \; in\, Q_T, \\ \partial_t z_2 & = d_2\Delta z_2 - (\mu_2+\sigma)z_2 + [f(y_4^*) + g(y_3^*)]z_1 + y_1^* f'(y_4^*)z_4 + y_1^* g'(y_3^*)z_3, \; in\, Q_T, \\ \partial_t z_3 & = d_3\Delta z_3 - (\mu_3+\gamma+u_2^*)z_3 + \sigma z_2 - y_3^* v_2, \; in\, Q_T, \\ \partial_t z_4 & = d_4\Delta z_4 + \alpha z_3 - (\xi + u_3^*)z_4 - y_4^* v_3, \; in\, Q_T, \\ \partial_t z_5 & = d_5\Delta z_5 + u_1^* z_1 + (\gamma + u_2^*)z_3 - \mu_4 z_5 + y_1^* v_1 + y_3^* v_2, \; in\, Q_T, \\ \partial_\nu z_1 & = \partial_\nu z_2 = \partial_\nu z_3 = \partial_\nu z_4 = \partial_\nu z_5 = 0, \; on\, \Sigma_T, \\ z_1(0,x) & = z_2(0,x) = z_3(0,x) = z_4(0,x) = z_5(0,x) = 0, \; on\, \Omega\,. \end{cases} \tag{3.28}$$

Proof: Setting

$$z^\varepsilon := \frac{G(u^\varepsilon)-G(u)}{\varepsilon} = (\frac{y_1^\varepsilon - y_1^*}{\varepsilon}, \ldots, \frac{y_5^\varepsilon - y_5^*}{\varepsilon})^\top = (z_1^\varepsilon, \ldots, z_5^\varepsilon)^\top,$$

with $y_i^\varepsilon := y_i(u^\varepsilon)$ and $u^\varepsilon := u^* + \varepsilon v$. To verify (3.27), we need to establish that $v \mapsto z$ is linear bounded from $[L^2(Q_T)]^2$ to $[L^2(Q_T)]^5$ and satisfies

$$\lim_{\varepsilon\to 0} \|z^\varepsilon - z\|_{[L^2(Q_T)]^5}. \tag{3.29}$$

On the one side, it is straightforward to verify that $z^\varepsilon = (z_1^\varepsilon, \ldots, z_5^\varepsilon)^\top$ solves the following equations

$$\begin{cases} \partial_t z_1^\varepsilon & = d_1\Delta z_1^\varepsilon - [\mu_1 + u_1^* + f(y_4^*) + g(y_3^*)]z_1^\varepsilon - y_1^* \frac{f(y_4^\varepsilon)-f(y_4^*)}{y_4^\varepsilon - y_4^*} z_4^\varepsilon - y_1^* \frac{g(y_3^\varepsilon)-g(y_3^*)}{y_3^\varepsilon - y_3^*}) z_3^\varepsilon - y_1^\varepsilon v_1, \text{ in } Q_T, \\ \partial_t z_2^\varepsilon & = d_2\Delta z_2^\varepsilon - (\mu_2+\sigma)z_2^\varepsilon + [f(y_4^*) + g(y_3^*)]z_1^\varepsilon + y_1^* \frac{f(y_4^\varepsilon)-f(y_4^*)}{y_4^\varepsilon - y_4^*} z_4 + y_1^* \frac{g(y_3^\varepsilon)-g(y_3^*)}{y_3^\varepsilon - y_3^*} z_3^\varepsilon, \text{ in } Q_T, \\ \partial_t z_3^\varepsilon & = d_3\Delta z_3^\varepsilon - (\mu_3+\gamma+u_2^*)z_3^\varepsilon + \sigma z_2^\varepsilon - y_3^\varepsilon v_2, \text{ in } Q_T, \\ \partial_t z_4^\varepsilon & = d_4\Delta z_4^\varepsilon + \alpha z_3^\varepsilon - (\xi+u_3^*)z_4^\varepsilon - y_4^\varepsilon v_3, \text{ in } Q_T, \\ \partial_t z_5^\varepsilon & = d_5\Delta z_5^\varepsilon + u_1^* z_1^\varepsilon + (\gamma+u_2^*)z_3^\varepsilon - \mu_4 z_5^\varepsilon + y_1^\varepsilon v_1 + y_3^\varepsilon v_2, \text{ in } Q_T, \\ \partial_\nu z_1^\varepsilon & = \partial_\nu z_2^\varepsilon = \partial_\nu z_3^\varepsilon = \partial_\nu z_4^\varepsilon = \partial_\nu z_5^\varepsilon = 0, \text{ on } \Sigma_T, \\ z_1^\varepsilon(0,x) & = z_2^\varepsilon(0,x) = z_3^\varepsilon(0,x) = z_4^\varepsilon(0,x) = z_5^\varepsilon(0,x) = 0, \text{ on } \Omega\,. \end{cases} \tag{3.30}$$

One can write the system (3.30) as the following abstract Cauchy problem

$$\begin{cases} \partial_t z^\varepsilon = Az^\varepsilon + H^\varepsilon z^\varepsilon + B^\varepsilon v, \\ z^\varepsilon(0) = 0 \end{cases} \tag{3.31}$$

where

$$A = \begin{bmatrix} d_1\Delta & 0 & 0 & 0 & 0 \\ 0 & d_2\Delta & 0 & 0 & 0 \\ 0 & 0 & d_3\Delta & 0 & 0 \\ 0 & 0 & 0 & d_4\Delta & 0 \\ 0 & 0 & 0 & 0 & d_5\Delta \end{bmatrix}, \; B^\varepsilon = \begin{bmatrix} -y_1^\varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -y_3^\varepsilon & 0 \\ 0 & 0 & -y_4^\varepsilon \\ y_1^\varepsilon & y_3^\varepsilon & 0 \end{bmatrix}, \; v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

and

$$H^\varepsilon = \begin{bmatrix} -(\mu_1 + u_1^* + \nu^\varepsilon) & 0 & -\rho_1^\varepsilon & -\rho_2^\varepsilon & 0 \\ \nu^\varepsilon & -(\mu_2+\sigma) & \rho_1^\varepsilon & \rho_2^\varepsilon & 0 \\ 0 & \sigma & -(\mu_3+\gamma+u_2^*) & 0 & 0 \\ 0 & 0 & \alpha & -(\xi+u_3^*) & 0 \\ u_1^* & 0 & \gamma+u_2^* & 0 & -\mu_4 \end{bmatrix}$$

with

$$\nu^\varepsilon = f(y_4^*) + g(y_3^*), \quad \rho_1^\varepsilon = y_1^* \frac{g(y_3^\varepsilon)-g(y_3^*)}{y_3^\varepsilon - y_3^*} \quad \text{and} \quad \rho_2^\varepsilon = y_1^* \frac{f(y_4^\varepsilon)-f(y_4^*)}{y_4^\varepsilon - y_4^*}.$$

Then from the semigroup arguments we conclude that the problem (3.30) possesses a unique strong solution defined by

$$z^\varepsilon(t) = \int_0^t S(t-s)H^\varepsilon(s)z^\varepsilon(s)\,ds + \int_0^t S(t-s)B^\varepsilon(s)v(s)\,ds, \quad \forall t \in [0,T], \tag{3.32}$$

where $\{S(t)\}_{t\geq 0}$ denotes the C_0-semigroup of contractions in H and A its generator operator. Furthermore, by the uniform boundness of y^ε in $[L^\infty(Q_T)]^5$ with respect to ε and that $y^* \in [L^\infty(Q_T)]^5$, see (3.3) and (3.23), together with the Continuity differentiability, see $(A_1)-(A_2)$, we deduce that all the coefficients of the matrix H^ε and B^ε are uniformly bounded with respect ε. Applying the Gronwall's inequality, we get there exists $C > 0$ which does not depend of ε such that

$$\|z^\varepsilon\|_{[L^2(Q_T)]^5} \leq C\|B^\varepsilon v\|_{[L^2(Q_T)]^5} \leq C\|v\|_{[L^2(Q_T)]^5}. \tag{3.33}$$

Then

$$\|y^\varepsilon - y^*\|_{[L^2(Q_T)]^5} = \varepsilon\|z^\varepsilon\|_{[L^2(Q_T)]^5} \leq \varepsilon C\|v\|_{[L^2(Q_T)]^5}.$$

Hence

$$y_i^\varepsilon \to y_i^* \quad \text{in}\, L^2(Q_T), \,\text{as}\, \varepsilon \to 0, \tag{3.34}$$

for all $i = 1, \ldots, 5$.
Analogously, the system (3.28) can be written in the following form

$$\begin{cases} \partial_t z = Az + Hz + Bv, \\ z(0) = 0 \end{cases} \tag{3.35}$$

where

$$B = \begin{bmatrix} -y_1^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -y_3^* & 0 \\ 0 & 0 & -y_4^* \\ y_1^* & y_3^* & 0 \end{bmatrix}$$

and

$$H = \begin{bmatrix} -(\mu_1 + u_1^* + \nu) & 0 & -\rho_1 & -\rho_2 & 0 \\ \nu & -(\mu_2+\sigma) & \rho_1 & \rho_2 & 0 \\ 0 & \sigma & -(\mu_3+\gamma+u_2^*) & 0 & 0 \\ 0 & 0 & \alpha & -(\xi+u_3^*) & 0 \\ u_1^* & 0 & \gamma+u_2^* & 0 & -\mu_4 \end{bmatrix}$$

with

$$\nu = f(y_4^*) + g(y_3^*), \quad \rho_1 = y_1^* g'(y_3^*) \quad \text{and} \quad \rho_2 = y_1^* f'(y_4^*).$$

Similarly, we deduce that the system (3.35) possesses a unique strong solution which can be expressed using the corresponding semigroup by

$$z(t) = \int_0^t S(t-s)H(s)z(s)\,ds + \int_0^t S(t-s)B(s)v(s)\,ds, \quad \forall t \in [0,T], \tag{3.36}$$

Taking the difference of (3.32) and (3.36), we get for any $t \in [0,T]$,

$$z^\varepsilon(t) - z(t) = \int_0^t S(t-s)[H^\varepsilon(s)-H(s)]z^\varepsilon(s)\,ds + \int_0^t S(t-s)H(s)[z^\varepsilon(s)-z(s)]\,ds + \int_0^t S(t-s)[B^\varepsilon(s)-B(s)]v(s)\,ds,$$

Thanks to the convergence in (3.34), the estimate (3.3) and the hypotheses $(A_1)-(A_2)$, all the coefficients of the matrix $H^\varepsilon(s)$ and B^ε tend or equal to the corresponding coefficients of the matrix H and B in $L^2(Q_T)$. By applying the Gronwall's inequality, we deduce

$$z^\varepsilon \to z \quad \text{in} \quad [L^2(Q_T)]^5 \quad \text{as}\, \varepsilon \to 0, \tag{3.37}$$

which means that the property (3.27) is satisfied.
On the other side, for any fixed $u \in \mathcal{U}_{ad}$, it is straightforward to see that the operator $G(u^*)v$ is linear bounded from $[L^2(Q_T)]^2$ to $[L^2(Q_T)]^5$. Indeed, the linearity is clear from equations (5.2). As to the boundedness, we infer from (3.33) that $\|z\|_{[L^2(Q_T)]^5} \leq C\|v\|_{[L^2(Q_T)]^2}$, we deduce that the mapping G is Gateaux differentiable at $u^* \in \mathcal{U}_{ad}$. □

Remark 3.1 By following the same reasoning as in the proof of Thm.3.2. One can obtain that the system (3.28) admits a unique strong solution in $Y = [W^{1,2} \cap L^2(Q_T) \cap L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^\infty(Q_T)]^5$; Moreover we show that

$$\|\partial_t z_i\|_{L^\infty(Q_T)} + \|z_i\|_{L^2(0,T,H^2(\Omega))} + \|z_i\|_{L^\infty(0,T,H^1(\Omega))} + \|z_i\|_{L^\infty(Q_T)} \leq C, \tag{3.38}$$

for $i = 1, \dots, 5$.
Now we introduce the following adjoint system as in [6],

$$\begin{aligned} -\partial_t p &= Ap + H^\top p + D^\top D\,\kappa, \\ p(T) &= 0, \end{aligned} \tag{3.39}$$

such that $H^\top$ and $D^\top$are the adjoint matrix associated respectively to H and D, where D and κ are defined by

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \kappa = \begin{bmatrix} k_1 \\ 0 \\ k_2 \\ 0 \\ 0 \end{bmatrix}.$$

We give the following result concerning the solution of the adjoint system (3.39).

Lemma 3.5 *Under the hypothesis of Thm.3.2, if (y^*, u^*) is an optimal pair, the adjoint system (3.39) admits a unique strong solution $p^* \in \left[W^{1,2}(0,T,L^2(\Omega)) \cap L^2(Q_T) \cap L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^\infty(Q_T)\right]^5$.*

Proof: Similar to Thm.3.2, by the change of the variable $s = T - t$ and the change of the functions $q_i(s,x) = p_i(T-s,x) = p_i(t,x)$, $i = 1, \dots, 5$. One can easily show the result. □

Now, We are ready to show the following optimality conditions of the control problem (3.19).

Theorem 3.6 (Optimality condition) *Given the assumptions of Theorem 3.2, let u^* denote an optimal control for the functional (2.5), and let $y^* = (S^*, E^*, I^*, W^*, R^*)$ be the corresponding variable state. Then, there exists $p^* = (p_1^*, p_2^*, p_3^*, p_4^*, p_5^*) \in \left[W^{1,2}(0,T,L^2(\Omega)) \cap L^2(Q_T) \cap L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^\infty(Q_T)\right]^5$ such that*

$$\left\{ \begin{array}{lcll} \partial_t S^* & = & d_1 \Delta S^* + \Lambda - (\mu_1 + u_1^*)S^* - S^* f(W^*) - S^* g(I^*), & \textit{in } Q_T, \\ \partial_t E^* & = & d_2 \Delta E - (\mu_2 + \sigma)E + S^* f(W^*) + S^* g(I^*), & \textit{in } Q_T, \\ \partial_t I^* & = & d_3 \Delta I^* - (\mu_3 + \delta + \gamma + u_2)I^* + \sigma E^*, & \textit{in } Q_T, \\ \partial_t W^* & = & d_4 \Delta W^* - (\xi + u_3^*)W^* + \alpha I^*, & \textit{in } Q_T, \\ \partial_t R^* & = & d_5 \Delta R^* - \mu_4 R^* + u_1^* S^* + (\gamma + u_2^*)I^*, & \textit{in } Q_T, \\ \partial_\nu S^* & = & \partial_\nu E^* = \partial_\nu I^* = \partial_\nu W^* = \partial_\nu R^* = 0, & \textit{on } \Sigma_T, \\ S^*(0,x) & = & S_0(x), E^*(0,x) = E_0(x), I^*(0,x) = I_0(x), W^*(0,x) = W_0(x), R^*(0,x) = R_0(x), & \textit{on } \Omega, \end{array} \right. \tag{3.40}$$

$$\left\{ \begin{array}{lcll} -\partial_t p_1^* & = & d_1 \Delta p_1^* - (\mu_1 + u_1^* + f(y_4^*) + g(y_3^*))p_1^* + (f(y_4^*) + g(y_3^*))p_2^* + u_1^* p_5^* + k_1, & \textit{in}\, Q_T, \\ -\partial_t p_2^* & = & d_2 \Delta p_2^* - (\mu_2 + \sigma)p_2^* + \sigma p_3^*, & \textit{in}\, Q_T, \\ -\partial_t p_3^* & = & d_3 \Delta p_3^* - y_1^* g'(y_3^*)p_1^* + y_1^* g'(y_3^*)p_2^* - (\mu_3 + \gamma + u_2^*)p_3^* + \alpha p_4^* + (\gamma + u_2^*)p_5^* + k_2, & \textit{in}\, Q_T, \\ -\partial_t p_4^* & = & d_4 \Delta p_4^* - y_1^* f'(y_4^*)p_1^* + y_1^* f'(y_4^*)p_2^* - (\xi + u_3^*)p_4^*, & \textit{in}\, Q_T, \\ -\partial_t p_5^* & = & d_5 \Delta p_5^* - \mu_5 p_5^*, & \textit{in}\, Q_T, \\ \partial_\nu p_1^* & = & \partial_\nu p_2^* = \partial_\nu p_3^* = \partial_\nu p_4^* = \partial_\nu p_5^* = 0, & \textit{on}\, \Sigma_T. \\ p_1^*(T,x) & = & 0, & \textit{on}\, \Omega. \end{array} \right. \tag{3.41}$$

Furthermore, the optimal control functions are given by

$$\begin{aligned} u_1^* &= \min\{u_1^{max}, \max(0, \frac{y_1^*(p_1 - p_5)}{2l_1})\}, \\ u_2^* &= \min\{u_2^{max}, \max(0, \frac{y_3^*(p_3 - p_5)}{2l_2})\}, \\ u_3^* &= \min\{u_3^{max}, \max(0, \frac{y_4^* p_4}{2l_3})\}, \end{aligned} \tag{3.42}$$

Proof: . Let u^* be an optimal control and y^* the corresponding optimal state. For any given $v \in [L^2(Q_T)]^3$, consider $u^\varepsilon := u^* + \varepsilon v \in \mathcal{U}_{ad}$ and the corresponding state $y^\varepsilon := y(u^\varepsilon)$.
One has

$$\begin{aligned} J'(u^*)(v) &= \lim_{\varepsilon \to 0} \frac{J(u^\varepsilon) - J(u^*)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \int_{Q_T} k_1 \frac{y_1^\varepsilon - y_1^*}{\varepsilon} + k_2 \frac{y_3^\varepsilon - y_3^*}{\varepsilon}\, dxdt \\ &+ \lim_{\varepsilon \to 0} \int_{Q_T} l_1 \frac{(u_1^\varepsilon)^2 - (u_1^*)^2}{\varepsilon} + l_2 \frac{(u_2^\varepsilon)^2 - (u_2^*)^2}{\varepsilon} + l_3 \frac{(u_3^\varepsilon)^2 - (u_3^*)^2}{\varepsilon}\, dxdt \\ &= \lim_{\varepsilon \to 0} \int_{Q_T} k_1 z_1^\varepsilon + k_2 z_3^\varepsilon\, dxdt + 2\int_{Q_T} l_1 u_1^* v_1 + l_2 u_2^* v_2 + l_3 u_3^* v_3\, dxdt \end{aligned}$$

Using (3.37), we know that $z^\varepsilon \to z$ in $[L^2(Q_T)]^5$, for $\varepsilon \to 0$, then it is clear that

$$\begin{aligned} J'(u^*)(v) &= \int_{Q_T} k_1 z_1 + k_2 z_3\, dxdt + 2\int_{Q_T} l_1 u_1^* v_1 + l_2 u_2^* v_2 + l_3 u_3^* v_3\, dxdt \\ &= \int_0^T \langle D\kappa, Dz\rangle_{[L^2(\Omega)]^5} + \langle Lu^*, v\rangle_{[L^2(\Omega)]^3}\, dt \\ &= \int_0^T \langle D^\top D\kappa, z\rangle_{[L^2(\Omega)]^5} + \langle Lu^*, v\rangle_{[L^2(\Omega)]^3}\, dt, \end{aligned}$$

where

$$L = \begin{bmatrix} 2l_1 & 0 & 0 \\ 0 & 2l_2 & 0 \\ 0 & 0 & 2l_3 \end{bmatrix}.$$

Using (3.35) and (3.39), we obtain

$$\begin{aligned} \int_0^T \langle D^\top D\kappa, z\rangle_{[L^2(\Omega)]^5} &= \int_0^T \langle -\partial_t p - Ap - H^\top p, z\rangle_{[L^2(\Omega)]^5} \\ &= \int_0^T \langle p, \partial_t z - Az - Hz\rangle_{[L^2(\Omega)]^5} \\ &= \int_0^T \langle p, Bv\rangle_{[L^2(\Omega)]^5} \\ &= \int_0^T \langle B^\top p, v\rangle_{[L^2(\Omega)]^3}. \end{aligned}$$

Therefore

$$J'(u^*)(v) = \int_0^T \langle B^\top p + Lu^*, v\rangle_{[L^2(\Omega)]^2}, \tag{3.43}$$

for all $v \in [L^2(Q_T)]^3$. Since J is Gateaux-differentiable at $u^* \in \mathcal{U}_{ad}$ and $\mathcal{U}_{ad}$ is convex, then

$$J'(u^*)(v - u^*) \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{3.44}$$

That is

$$\int_0^T \langle B^\top p + Lu^*, v - u^* \rangle_{L^2(\Omega)^3} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{3.45}$$

Using standard reasoning, varying v it yields $u^* = -L^{-1}B^\top p$. Since $u^* \in \mathcal{U}_{ad}$, we get the following expression

$$\begin{aligned} u_1^* &= \min\{u_1^{max}, \max(0, \frac{y_1^*(p_1 - p_5)}{2l_1})\}, \\ u_2^* &= \min\{u_2^{max}, \max(0, \frac{y_3^*(p_3 - p_5)}{2l_2})\}, \\ u_3^* &= \min\{u_3^{max}, \max(0, \frac{y_4^* p_4}{2l_3})\}, \end{aligned} \tag{3.46}$$

□

3.3. Numerical Results for the optimal control strategies results

In this section, we perform numerical simulations to demonstrate the practical application of the theoretical results derived in this paper. We present simulations related to the optimality system, encompassing the state system (3.40), the dual system (3.41), and the control characterization (3.42). This formulation treats the optimality system as a two-point boundary value problem, with initial conditions for the state variables and terminal conditions for the dual system. To solve this system, we developed a MATLAB code that uses an iterative discrete scheme, converging through a suitable test. Following this, the optimal control values are updated based on the state and adjoint variable results from the previous iterations. This iterative process continues until the differences between the current and previous values of the states, adjoints, and controls are within an acceptable margin of error.
We represent the population's habitat by a rectangular domain Ω of dimensions $45\,\text{km} \times 35\,\text{km}$. Within this spatial setting, we examine two distinct scenarios to simulate the initial emergence of the infection:

- In the first scenario, the infection begins in the region $\Omega_1 = \text{cell}(40, 30)$, located near the top-right boundary of Ω.

- In the second scenario, the infection originates in $\Omega_2 = \text{cell}(20, 20)$, corresponding to an area close to the center of the domain.

At time $t = 1$, the susceptible population is assumed to be evenly distributed, with 50 individuals in each $1\,\text{km} \times 1\,\text{km}$ cell, except in the subdomains Ω_i (for $i = 1, 2$), where 10 infected individuals are introduced, leaving 40 susceptibles in those areas.

Based on the initial conditions and parameters summarized in Table 1, we analyze the progression of the infection over a period of 150 days, comparing two scenarios: one without any intervention, and the other implementing our three control strategies—vaccination (u_1), treatment (u_2), and water disinfection (u_3). The primary objective of this section is to emphasize the effectiveness and relevance of these strategies in controlling the spread of the epidemic.

Table 1: Initial Conditions and Parameter Values for the Model.

Notations	Value	Description
$S_0(x,y)$	40 if $(x,y)\in\Omega_1$ or $(x,y)\in\Omega_2$ 50 if $(x,y)\notin\Omega_1$ and $(x,y)\notin\Omega_2$	Initial susceptible population
$E_0(x,y)$	0 if $(x,y)\in\Omega$	Initial exposed population
$I_0(x,y)$	10 if $(x,y)\in\Omega_1$ or $(x,y)\in\Omega_2$ 0 if $(x,y)\notin\Omega_1,\Omega_2$	Initial infected population
$W_0(x,y)$	0 if $(x,y)\in\Omega$	Initial concentration of pathogens
$R_0(x,y)$	0 if $(x,y)\in\Omega$	Initial recovered population
d_i, $i=1,\cdots,5$	0.6	Diffusion coefficients for the susceptible, exposed, infected, recovered, and vaccinated populations
Λ	0.8	Recruitment rate of the population
β_1	0.1072	Infection rate for susceptible individuals
β_2	0.1072	Infection rate for vaccinated individuals
α	0.04	Rate at which vaccinated individuals develop immunity
σ	0.4	Transition rate from the latent phase E to the infected phase I
δ	0.25	Death rate induced by the disease
γ	0.25	Recovery rate after infection
ξ	0.333	the decay rate of the pathogen in the water
μ_i for $i=1,\cdots,5$	0.02	Natural death rates for susceptible, vaccinated, exposed, infected, and recovered individuals

Figures 3 to 10 show that the simultaneous application of the three control strategies u_1, u_2, and u_3 leads to a faster decrease in the number of susceptible, exposed, and infected individuals, as well as a more rapid increase in the number of recovered individuals, along with a significant reduction in the concentration of pathogens, compared to scenarios without control or with only one control law. The numerical results show that the system's behavior is generally similar whether the infection starts at the center or in one of the corners of the domain. However, slightly better outcomes are observed when the infection begins at the center. In this case, the number of infectious and infected individuals is higher than when the infection starts in a corner.

Overall, we conclude that our control strategies are effective regardless of the initial location of the infection, with a slight improvement observed when it begins at the center. The combined strategy, implementing all three controls, proves to be the most effective.

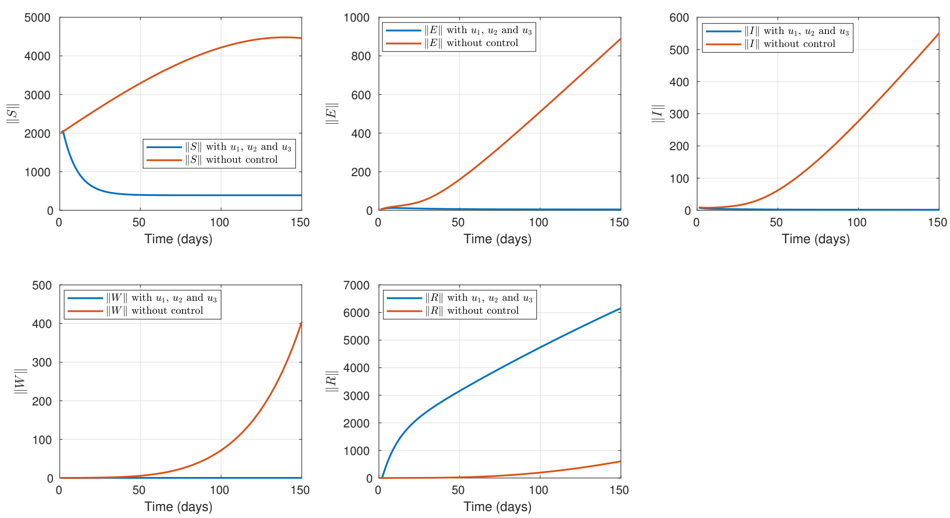

Figure 3: Dynamics of the different compartments under our three control laws u_1, u_2, and u_3; and without control, when the infection starts at the center.

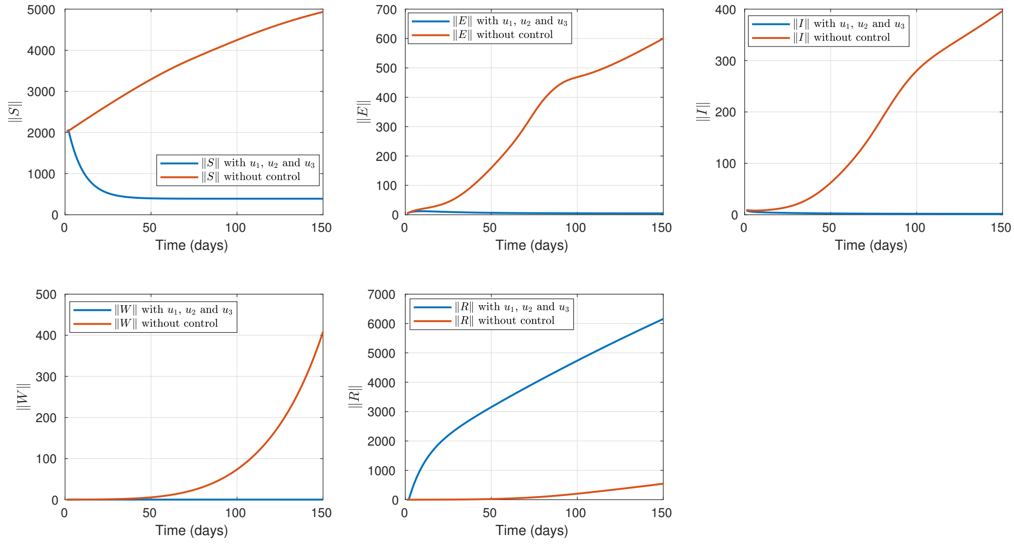

Figure 4: Dynamics of the different compartments under our three control laws u_1, u_2, and u_3; and without control, when the infection starts at the corner.

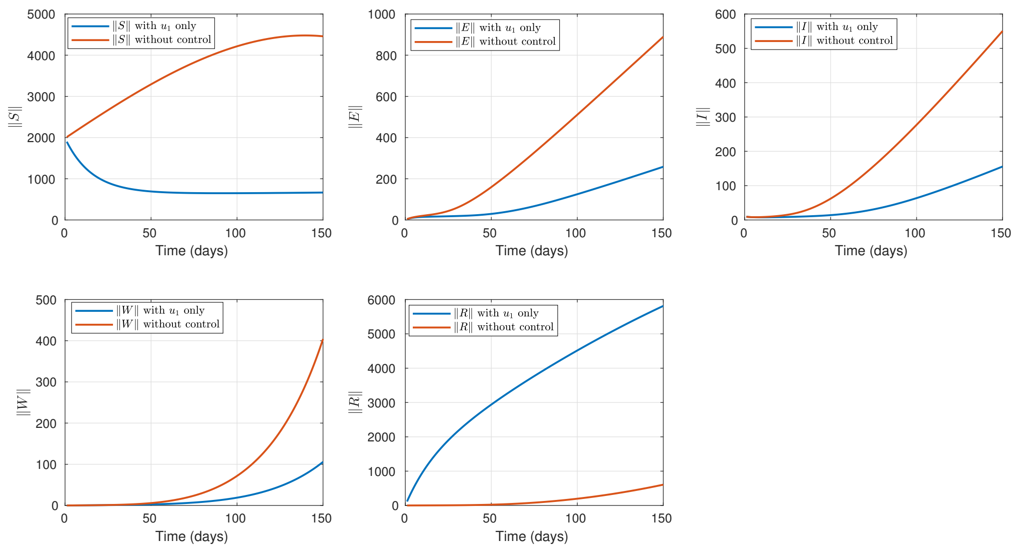

Figure 5: Dynamics of the different compartments under the control law u_1 only; and in the absence of control, when the infection starts at the center.

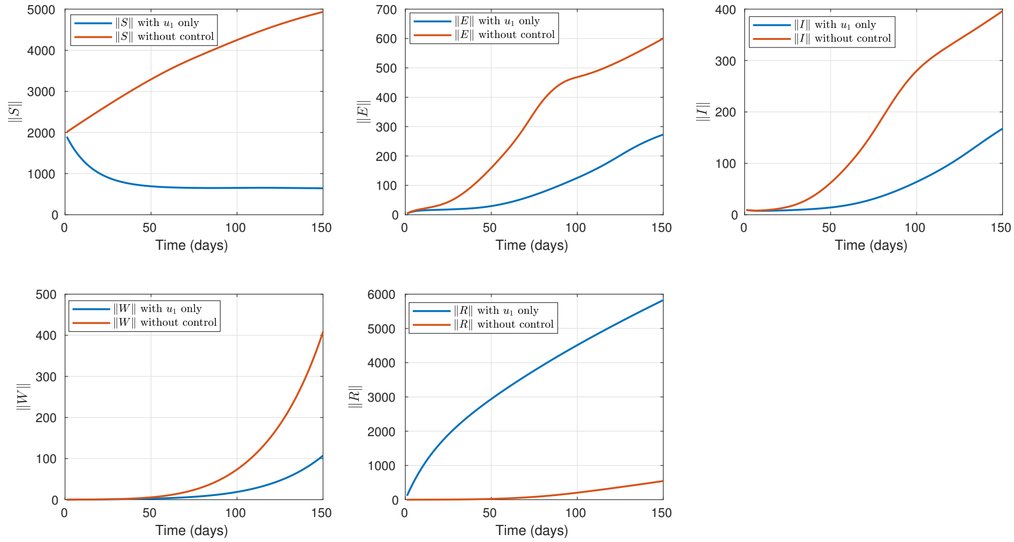

Figure 6: Dynamics of the different compartments under the control law u_1 only; and in the absence of control, when the infection starts at the corner.

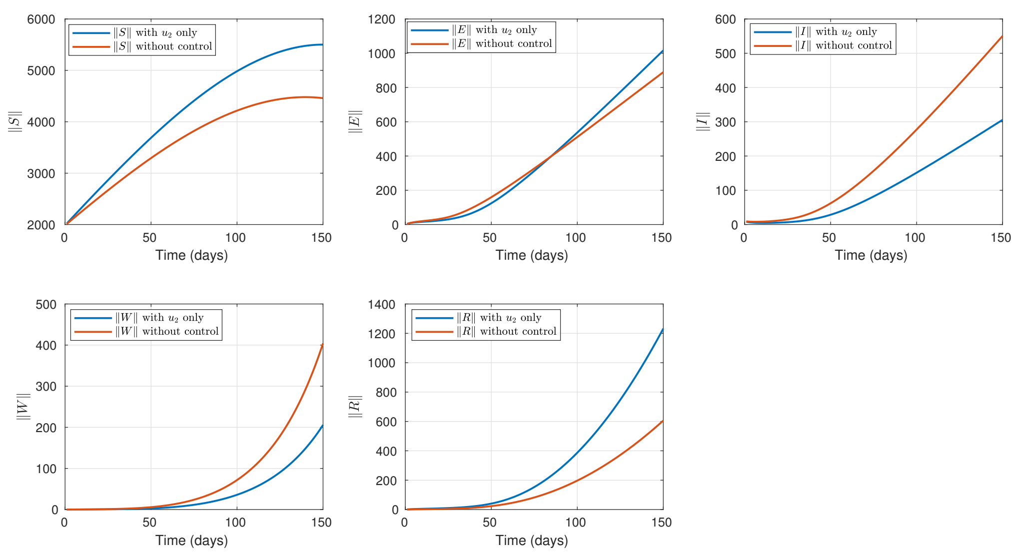

Figure 7: Dynamics of the different compartments under the control law u_2 only; and in the absence of control, when the infection starts at the center.

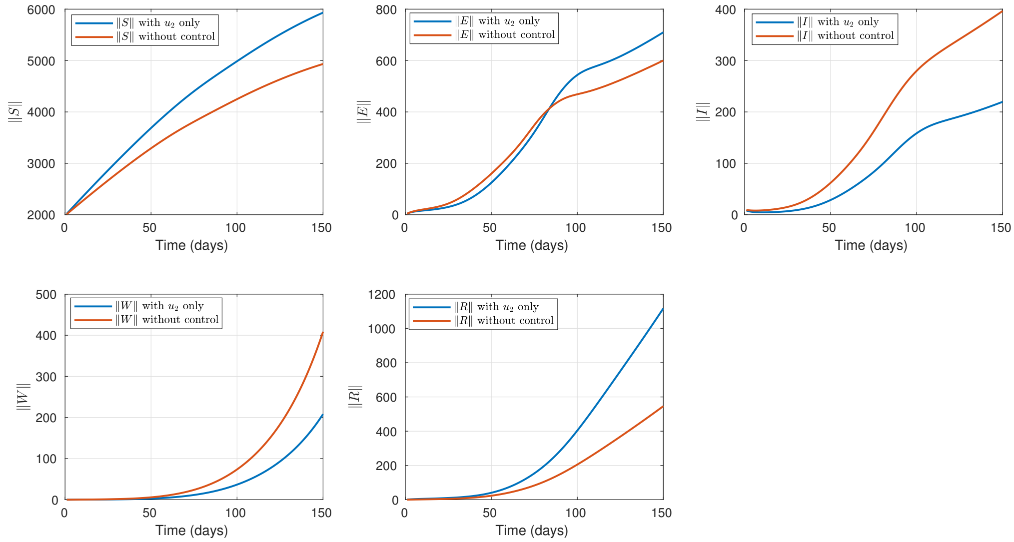

Figure 8: Dynamics of the different compartments under the control law u_2 only; and in the absence of control, when the infection starts at the corner.

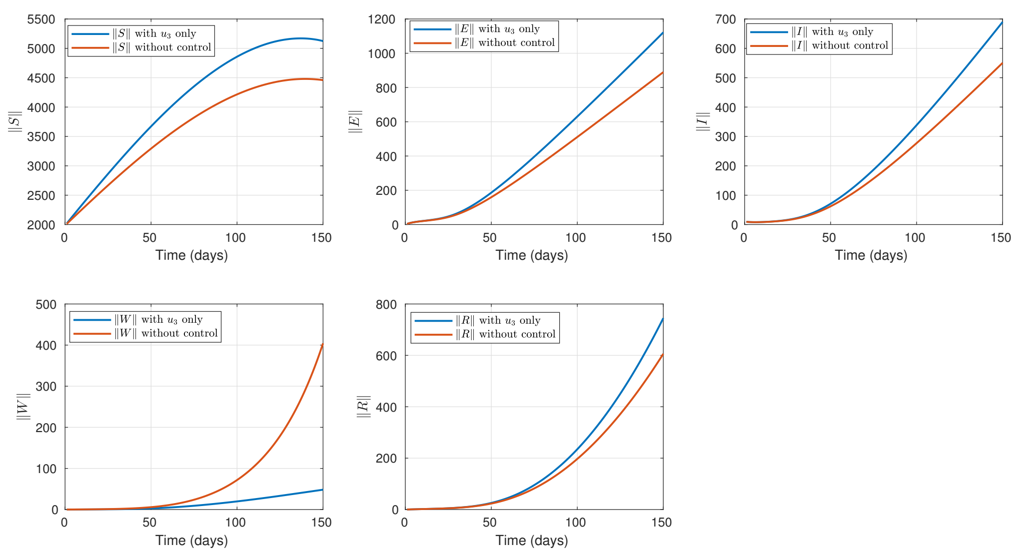

Figure 9: Dynamics of the different compartments under the control law u_3 only; and in the absence of control, when the infection starts at the center.

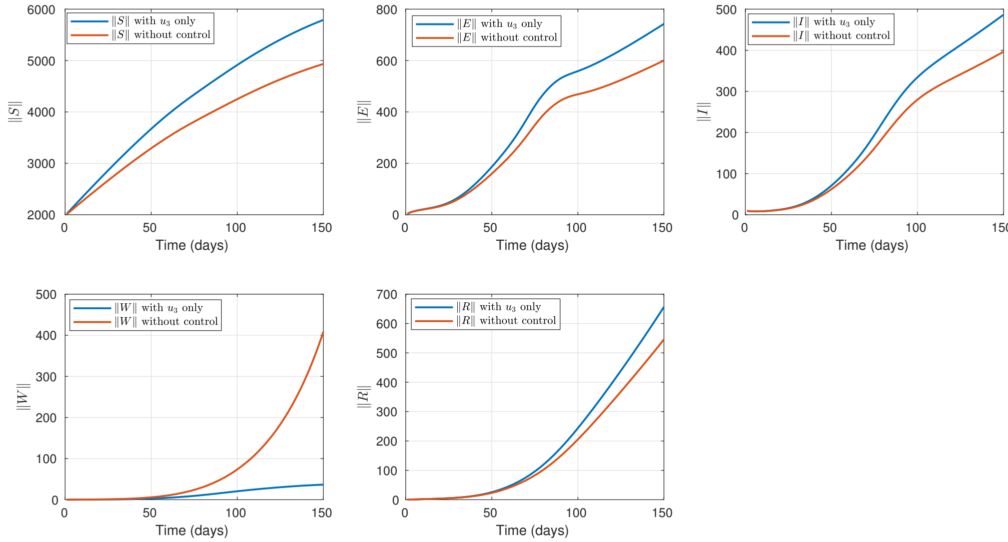

Figure 10: Dynamics of the different compartments under the control law u_3 only; and in the absence of control, when the infection starts at the corner.

Conflict of Interest

The authors declare no conflict of interest.

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