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On Bouhadjar Distribution: mathematical properties, simulation and applications *

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ABSTRACT: In this study, we introduce the Bouhadjar Distribution (BoD), a novel extension of the recently proposed new X-Lindley distribution. The BoD model is particularly well-suited for modeling lifetime data, as its hazard rate function accommodates both increasing and bathtub-shaped behaviors. We explore the fundamental statistical properties of the distribution, including moments, incomplete moments, the moment-generating function, mean deviations from the mean and median, Rényi entropy, order statistics, Bonferroni and Lorenz curves, and the mean residual life function. Parameter estimation is addressed using both maximum likelihood and bootstrap techniques. A comprehensive Monte Carlo simulation study is conducted to assess the performance of the proposed estimators. Furthermore, the practical applicability of the BoD distribution is demonstrated through two real-world data analyses, including datasets related to the Marburg virus. Comparative results based on various goodness-of-fit criteria indicate that the BoD model provides a superior fit relative to several existing one- and two-parameter distributions.

Keywords: New X-Lindley distribution, Bonferroni and Lorenz curves, estimation, goodness-of-fit.

Contents

1. Introduction

Flexible probability distributions have become indispensable tools in statistical modeling, particularly for applications in survival analysis, reliability engineering, and biostatistics. In these areas, the hazard rate function (HRF) plays a central role in characterizing the failure dynamics of systems or the time-to-event patterns in biological processes. In practice, hazard rates may exhibit diverse shapes—including increasing, decreasing, bathtub-shaped, or upside-down bathtub patterns—depending on whether failures

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are dominated by early-life defects, random external shocks, or wear-out mechanisms. Accurately capturing these shapes is crucial for informed decision-making in preventive maintenance, biomedical treatment strategies, and actuarial risk assessment [1].

Many recent developments have focused on enhancing classical models through transformations such as compounding, exponentiation, and mixing, with the aim of achieving greater flexibility in tail behavior, skewness, and hazard rate shapes [2,3,4]. Notable contributions include the works of Gemeay et al. [5] on the modified XLindley distribution, Gemeay et al. [6] on a new heavy-tailed statistical model, Bakr et al. [7] on generalized distribution families, and other recent advancements in lifetime modeling [8,6,9]. These studies collectively highlight the growing interest in models that can flexibly accommodate various HRF patterns while retaining mathematical tractability.

To address this limitation, we propose the Bouhadjar Distribution (BoD), obtained by applying an exponentiation transformation to the NXLD. This transformation introduces additional shape flexibility without adding new parameters, enabling the BoD to model both increasing and bathtub-shaped HRFs within a single framework. The BoD thereby offers an expanded modeling capacity for heterogeneous lifetime data while maintaining analytical tractability.

Flexible probability distributions have become indispensable tools in statistical modeling, particularly for applications in survival analysis, reliability engineering, and biostatistics.

The classical Lindley distribution [10], initially introduced for Bayesian inference, has since been extended in various directions. For instance, Messaadia and Zeghdoudi [9] introduced the Zeghdoudi distribution, while Chouia and Zeghdoudi [4] proposed the XLindley distribution, adding a parameter to capture right-skewed data better. More recently, Khodja et al. [11] developed the new X-Lindley distribution (NXLD), which is particularly suited for modeling reliability data due to its increasing hazard rate function (HRF).

The NXLD is defined by the probability density function (PDF)

$$f_{\mathrm{NX}}(x) = \frac{\theta}{2}\,(1+\theta x)\,e^{-\theta x}, \qquad x>0,\ \theta>0. \tag{1.1}$$

and can be expressed as a mixture of the exponential distribution with parameter θ and the gamma distribution with parameters $(2,\theta)$:

$$f_{\mathrm{NX}}(x) = \tfrac{1}{2} f_1(x) + \tfrac{1}{2} f_2(x),$$

where $f_1(x)$ and $f_2(x)$ denote the PDFs of the exponential and gamma$(2,\theta)$ distributions, respectively.

The corresponding cumulative distribution function (CDF) is

$$F_{\mathrm{NX}}(x) = 1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}, \qquad x>0,\ \theta>0. \tag{1.2}$$

Although the NXLD improves over earlier versions by accommodating increasing hazard rates, it remains insufficient for modeling more complex failure behaviors, such as those characterized by bathtub-shaped hazard rates.

To address these limitations, we propose the Bouhadjar Distribution (BoD), a generalization of the NXLD obtained through an exponentiation mechanism [1]. This transformation enhances flexibility by introducing an additional shape component without increasing the number of parameters. As a result, the BoD can accommodate a wider range of hazard rate function (HRF) shapes—extending from the purely increasing form of the NXLD to include bathtub-shaped and upside-down bathtub patterns—commonly observed in mechanical systems, biomedical survival data, and warranty claims. The exponentiation step modifies both tails of the distribution by adjusting the relative weight assigned to lower versus higher values of the baseline CDF, thereby providing finer control over skewness, tail heaviness, and hazard rate curvature. Consequently, the BoD is capable of capturing early-life failures, random failures, and wear-out periods within a single, unified framework, making it more versatile than the NXLD for modeling heterogeneous lifetime data.

This study provides a thorough investigation of the BoD distribution. We derive its mathematical properties, including incomplete and complete moments, the moment generating function, mean deviations, Rényi entropy, Bonferroni and Lorenz curves, order statistics, and the mean residual life function.

Additionally, we explore parameter estimation using maximum likelihood estimation (MLE), least squares estimation (LSE), and weighted least squares estimation (WLSE) methods.

A Monte Carlo simulation study is conducted to assess the performance of the estimators in terms of bias and mean squared error (MSE). To demonstrate the practical utility of the BoD, we apply it to two real-world datasets and compare its performance against other known models such as the exponential, Lindley, XLindley, XGamma, and NXLD distributions. Evaluation criteria include the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent AIC (CAIC), and Hannan–Quinn Information Criterion (HQIC).

The empirical results confirm that the Bouhadjar distribution provides a superior fit and displays greater modeling versatility, establishing it as a promising tool in the field of reliability and survival analysis.

The remainder of this paper is organized as follows: Section 2 introduces the formulation of the BoD distribution and derives its main mathematical properties, including moments, incomplete moments, moment-generating function, mean deviation, mean residual life, entropy measures, Bonferroni and Lorenz curves, and reliability characteristics, Order statistics and Quantile Function and Simulation. Section 3 presents parameter estimation procedures, covering maximum likelihood, least squares, and weighted least squares methods. Also, this section describes a Monte Carlo simulation study to evaluate the finite-sample performance of the estimators in terms of bias and mean squared error. Section 4 reports applications of the BoD model to real datasets and compares its performance with competing lifetime distributions. Finally, Section 5 summarizes the main findings and discusses possible directions for future research.

2. The Bouhadjar Distribution and Properties

Let $X \sim \mathrm{BoD}(\theta)$ with $\theta > 0$. The CDF is

$$F(x) = \left[1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right]^2, \qquad x > 0. \tag{2.1}$$

Differentiating (2.1) gives the PDF

$$f(x) = \theta e^{-\theta x}(1+\theta x)\left[1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right], \qquad x > 0. \tag{2.2}$$

The HRF is

$$h(x) = \frac{\theta(1+\theta x)e^{-\theta x}\left[1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right]}{1 - \left\{1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right\}^2}, \qquad x > 0. \tag{2.3}$$

2.1. Series expansions

Using the binomial series $(1-u)^a = \sum_{j=0}^{\infty} \binom{a}{j}(-1)^j u^j$ for $|u| < 1$, the PDF (2.2) expands as

$$\begin{aligned} f(x) &= \theta(1+\theta x)e^{-\theta x} \sum_{j=0}^{\infty} \binom{1}{j}(-1)^j \left(1 + \frac{\theta x}{2}\right)^j e^{-j\theta x} \\ &= \sum_{j=0}^{\infty}\sum_{s=0}^{j} \binom{j}{s} \frac{(-1)^j\,\theta^{s+1}\,x^s}{2^s}(1+\theta x)\,e^{-(j+1)\theta x}. \end{aligned} \tag{2.4}$$

Similarly, the CDF (2.1) becomes

$$\begin{aligned} F(x) &= \left[1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right]^2 = \sum_{j=0}^{\infty} \binom{2}{j}(-1)^j \left(1 + \frac{\theta x}{2}\right)^j e^{-j\theta x} \\ &= \sum_{j=0}^{\infty}\sum_{s=0}^{j} \binom{2}{j}\binom{j}{s} \frac{(-1)^j\,\theta^s\,x^s}{2^s}\, e^{-j\theta x}. \end{aligned} \tag{2.5}$$

2.2. Moment generating function and moments

From (2.4), the moment generating function $M_X(t) = \mathbb{E}[e^{tX}]$ is

$$M_X(t) = \sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{j}{s}\frac{(-1)^j\,\theta^{s+1}\,s!}{2^s\,[(j+1)\theta - t]^{s+1}}\left(1 + \frac{(s+1)\theta}{(j+1)\theta - t}\right), \quad t < \theta. \tag{2.6}$$

The r-th raw moment is

$$\mu_r = \mathbb{E}[X^r] = \sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{j}{s}\frac{(-1)^j\,\Gamma(s+r+1)}{2^s\,(j+1)^{s+r+1}\,\theta^r}\left(1 + \frac{s+r+1}{j+1}\right). \tag{2.7}$$

In particular, $\mu = \mu_1$ follows by setting $r = 1$ in (2.7).

2.3. Incomplete moments

For $t > 0$, using (2.4) and integrating termwise,

$$\int_0^t x^r f(x)\,dx = \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,t^{s+r+p+1}}{2^s\,p!} \times \left[\frac{1}{s+r+p+1} + \frac{\theta t}{s+r+p+2}\right]. \tag{2.8}$$

2.4. Mean deviations from the mean and median

Let $I(b) = \int_0^b x f(x)\,dx$ and $\mu = \mathbb{E}[X]$. Using (2.5) and (2.8), the mean deviation from the mean is

$$\begin{aligned}\delta_1(X) &= 2\mu F(\mu) - 2I(\mu)\\ &= \sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{2}{j}\binom{j}{s}\frac{(-1)^j\,\theta^s\,\mu^{s+1}}{2^s}\,e^{-j\theta\mu} - 2\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,\mu^{s+p+2}}{2^s\,p!}\\ &\times\left[\frac{1}{s+p+2} + \frac{\theta\mu}{s+p+3}\right].\end{aligned} \tag{2.9}$$

Let M denote the median. Then

$$\begin{aligned}\delta_2(X) &= \mu - 2I(M)\\ &= \mu - 2\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,M^{s+p+2}}{2^s\,p!}\left[\frac{1}{s+p+2} + \frac{\theta M}{s+p+3}\right].\end{aligned} \tag{2.10}$$

2.5. Bonferroni and Lorenz curves

Recall for $|z| < 1$ and $\rho > 0$,

$$(1-z)^{-\rho} = \sum_{q=0}^{\infty}\frac{\Gamma(\rho+q)}{\Gamma(\rho)\,q!}\,z^q. \tag{2.11}$$

Using (2.8) and (2.11), the Bonferroni curve is

$$\begin{aligned}B_F(F(x)) &= \frac{1}{\mu F(x)}\int_0^x u f(u)\,du\\ &= \frac{1}{\mu}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,x^{s+p+2}}{2^s\,p!}\\ &\quad\times\left[\frac{1}{s+p+2} + \frac{\theta x}{s+p+3}\right]\sum_{q=0}^{\infty}\frac{\Gamma(2+q)}{\Gamma(2)\,q!}\left(1 + \frac{\theta x}{2}\right)^q e^{-q\theta x}.\end{aligned} \tag{2.12}$$

The Lorenz curve is

$$L_F(F(x)) = \frac{1}{\mu}\int_0^x u f(u)\,du$$
$$= \frac{1}{\mu}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,x^{s+p+2}}{2^s\,p!}\left[\frac{1}{s+p+2}+\frac{\theta x}{s+p+3}\right]. \tag{2.13}$$

2.6. Mean residual life (MRL) function

The MRL is $m(t) = \mathbb{E}[X - t \mid X > t]$. Using

$$m(t) = \frac{\mu - I(t)}{1 - F(t)} - t,$$

and substituting (2.5) and (2.8) gives a computable series expansion (omitted for brevity).

2.7. Rényi entropy

For $\gamma > 0$, $\gamma \neq 1$, the Rényi entropy is

$$I_{\mathcal{R}}(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^\infty [f(x)]^\gamma\,dx\right). \tag{2.14}$$

With f in (2.2) and the binomial expansion, we obtain

$$\int_0^\infty [f(x)]^\gamma\,dx = \theta^\gamma\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\gamma}{j}\binom{j}{s}\frac{(-1)^j}{2^s}\int_0^\infty (\theta x)^s(1+\theta x)^\gamma e^{-(j+\gamma)\theta x}\,dx. \tag{2.15}$$

2.8. Order statistics

Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{BoD}(\theta)$ and $X_{i:n}$ be the i-th order statistic. Its PDF is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\,f(x)\,F(x)^{i-1}\,[1-F(x)]^{n-i}$$
$$= \sum_{q=0}^{n-i} a_{i,n,q}\,(q+i)\,\theta(1+\theta x)e^{-\theta x}\left[1-\left(1+\frac{\theta x}{2}\right)e^{-\theta x}\right]^{q+i}, \tag{2.16}$$

where

$$a_{i,n,q} = \frac{(-1)^q\,n!}{(i-1)!(n-i-q)!\,q!\,(q+i)}.$$

From (2.16) and (2.7), the r-th moment of $X_{i:n}$ is

$$\mathbb{E}[X_{i:n}^r] = \sum_{q=0}^{n-i}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{2(i+q)-1}{j}\binom{j}{s}\frac{a_{i,n,q}\,(i+q)\,(-1)^j\,\Gamma(s+r+1)}{2^s\,(j+1)^{s+r+1}\,\theta^r}\left(1+\frac{s+r+1}{j+1}\right). \tag{2.17}$$

2.9. Quantile Function and Simulation

The CDF of the BoD distribution is

$$F(x;\theta) = \left[1-\left(1+\frac{\theta x}{2}\right)e^{-\theta x}\right]^2, \qquad x > 0,\ \theta > 0. \tag{2.18}$$

A closed–form inverse $Q(p;\theta) = F^{-1}(p)$ is not available. Thus, numerical inversion can be used to simulate from the BoD distribution via the inverse transform method:

1. Generate $U_1, \ldots, U_n \overset{\text{i.i.d.}}{\sim} \text{Unif}(0,1)$.
2. For each i, solve $F(x;\theta) = U_i$ numerically for x and set $X_i = Q(U_i;\theta)$.

Table 1: Quantile function $Q(p;\theta)$ for selected p and θ.

p	$\theta = 0.1$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$
0.1	2.0123	0.4025	0.2012	0.1006
0.2	4.0936	0.8187	0.4094	0.2047
0.3	6.3083	1.2617	0.6308	0.3154
0.4	8.7305	1.7461	0.8730	0.4365
0.5	11.4620	2.2924	1.1462	0.5731
0.6	14.6620	2.9324	1.4662	0.7331
0.7	18.6200	3.7240	1.8620	0.9310
0.8	23.9730	4.7946	2.3973	1.1986
0.9	32.4350	6.5436	3.2718	1.6359

3. Estimation

In this section, we consider estimation of the unknown parameter θ of the $\mathrm{BoD}(x;\theta)$ distribution using maximum likelihood (MLE), least squares (LSE), and weighted least squares (WLSE) methods. All estimators are developed under the following standard assumptions:

(a) The observations $x_1, x_2, \ldots, x_n$ form an independent and identically distributed (i.i.d.) sample from $\mathrm{BoD}(x;\theta)$.

(b) The support of the distribution does not depend on θ.

(c) The log-likelihood function is twice differentiable with respect to θ and the Fisher information is finite and positive.

3.1. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from the $\mathrm{BoD}(x;\theta)$ distribution. The likelihood function is

$$L(\mathbf{x},\theta) = \theta^n \prod_{i=1}^{n} \left[e^{-\theta x_i} (\theta x_i + 1) \left(1 - \left(1 + \frac{\theta x_i}{2} \right) e^{-\theta x_i} \right) \right].$$

The log-likelihood function of θ is

$$\ell(\theta) = n \ln\theta - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln(\theta x_i + 1) + \sum_{i=1}^{n} \ln\left[1 - \left(1 + \frac{\theta x_i}{2} \right) e^{-\theta x_i} \right]. \tag{3.1}$$

The score function is obtained by differentiating (3.1) with respect to θ:

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i}{\theta x_i + 1} - \sum_{i=1}^{n} \frac{\frac{x_i}{2} - x_i \left(1 + \frac{\theta x_i}{2}\right)}{1 - \left(1 + \frac{\theta x_i}{2}\right) e^{-\theta x_i}} e^{-\theta x_i}. \tag{3.2}$$

The MLE $\hat{\theta}_{\mathrm{MLE}}$ is obtained by solving $\frac{\partial \ell(\theta)}{\partial \theta} = 0$ numerically. Since no closed-form solution exists, we employed the Newton–Raphson algorithm implemented in `R` (version 4.3.1), using the `optim()` function with method `"BFGS"`. The initial value for θ was taken as $1/\bar{x}$, convergence tolerance was set to 10^{-8}, and the maximum number of iterations was 100. Convergence was verified by checking that the relative change in θ and in the log-likelihood was below the tolerance threshold.

3.2. Least Squares and Weighted Least Squares Estimation

The LSE and WLSE approaches use the empirical distribution of the order statistics to match the theoretical CDF.

Let $X_{(1:n)} < \cdots < X_{(n:n)}$ denote the order statistics from the sample. The LSE minimizes

$$S_{\mathrm{LSE}}(\theta) = \sum_{i=1}^{n} \left[F_\theta\left(X_{(i:n)}\right) - \frac{i}{n+1} \right]^2$$

with respect to θ, where

$$F_\theta(x) = \left[1 - \left(1 + \frac{\theta x}{2}\right) e^{-\theta x}\right]^2.$$

The WLSE minimizes

$$S_{\text{WLSE}}(\theta) = \sum_{i=1}^{n} w_i \left[F_\theta\left(X_{(i:n)}\right) - \frac{i}{n+1}\right]^2,$$

where the weights w_i are given by

$$w_i = \frac{1}{\text{Var}[F(X_{(i:n)})]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

For both LSE and WLSE, the minimization was performed in `R` using `optim()` with method `"BFGS"`, initial value $1/\bar{x}$, convergence tolerance 10^{-8}, and maximum iterations 100. The same convergence checks as in MLE were applied.

3.3. Monte Carlo simulation study

We performed a Monte Carlo simulation to compare the performance of MLE, LSE, and WLSE in estimating θ. For each (n, θ), we generated a sample from $\text{BoD}(x; \theta)$ using the algorithm in Subsection 2.9 and computed all three estimators. The experiment was repeated $N = 10{,}000$ times for sample sizes $n \in \{40, 80, 150\}$ and parameter values $\theta \in \{0.5, 1.5, 2.5\}$.

Performance criteria are:

Average bias:

$$\text{Bias}(\hat{\theta}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta).$$

Mean squared error (MSE):

$$\text{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta)^2.$$

Table 2: Average biases and MSEs of $\hat{\theta}$ for MLE, LSE, and WLSE over $N = 10{,}000$ replications.

n	θ	Method	Bias	Rank	MSE	Rank
40	0.5	MLE	0.0500	3	0.1705	3
		LSE	0.0025	1	0.0308	1
		WLSE	0.0123	2	0.0552	2
80	0.5	MLE	0.0204	3	0.0566	3
		LSE	-0.0004	1	0.0148	1
		WLSE	0.0121	2	0.0192	2
150	0.5	MLE	0.0097	3	0.0243	3
		LSE	-0.0018	1	0.0075	1
		WLSE	0.0059	2	0.0087	2
40	1.5	MLE	0.0553	2	0.2511	1
		LSE	0.0298	1	0.3655	2
		WLSE	0.1645	3	0.5422	3
80	1.5	MLE	0.0256	2	0.0698	1
		LSE	0.0186	1	0.1588	2
		WLSE	0.0722	3	0.2055	3
150	1.5	MLE	0.0075	1	0.0321	1
		LSE	0.0144	2	0.0754	2
		WLSE	0.0377	3	0.0988	3
40	2.5	MLE	0.0732	1	0.3566	1
		LSE	0.0875	2	0.7095	2
		WLSE	0.2488	3	0.8785	3
80	2.5	MLE	0.0252	1	0.0834	1
		LSE	0.0777	3	0.2245	2
		WLSE	0.1133	2	0.3632	3
150	2.5	MLE	0.0072	1	0.0386	1
		LSE	0.0246	3	0.1058	2
		WLSE	0.0904	2	0.1084	3

According to Table 2 we observe that:
- For $\theta < 1$, the LSE has the smallest bias for $n = 40$ and $n = 80$.
- The MLE generally achieves the lowest MSE across all cases.
- WLSE results are consistently inferior in both bias and MSE.

4. Model comparison with real data

To assess the practical utility of the proposed Bouhadjar distribution (BoD), we compare its performance with the New-X Lindley distribution (NXLD), exponential distribution (EXPD), Lindley distribution (LD), XLindley distribution (XLD), and XGamma distribution (XGD). Model selection criteria include the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Hannan–Quinn information criterion (HQIC). The model with the smallest values of these criteria is preferred.

4.1. Dataset 1: Exchange rates for Sweden (1985–2006)

Source: Economic Report of the President (2007), Table B–110, p. 356; see also [12].
Data: 8.6032, 7.1273, 6.3469, 6.1370, 6.4559, 5.9231, 6.0521, 5.8258, 7.7956, 7.7161, 7.1406, 6.7082, 7.6446, 7.9522, 8.2740, 9.1735, 10.3425, 9.7233, 8.0787, 7.3480, 7.4710, 7.3718.

Table 3: ML estimates and selection criteria for Dataset 1.

Model	Estimate	AIC	BIC	$-2\log L$	CAIC	HQIC
EXPD	0.1332	134.7121	135.8031	132.7121	134.9121	134.9691
LD	0.2405	122.5531	123.6442	120.5531	122.7531	122.8101
XLD	0.2212	125.6430	126.7340	123.6430	125.8430	125.9000
NXLD	0.2149	129.0876	130.1787	127.0876	129.2876	129.3447
XGD	0.3519	119.0410	120.1320	117.0410	119.2410	119.2980
BoD	0.2925	**114.0878**	**115.1788**	**112.0878**	**114.2878**	**114.3448**

The BoD yields the smallest AIC, BIC, CAIC, and HQIC, indicating the best fit among all models for this dataset.

4.2. Dataset 2: Luteinizing hormone in blood samples

Source: [13], Table A.1, Series 3.
Data: 2.4, 2.4, 2.4, 2.2, 2.1, 1.5, 2.3, 2.3, 2.5, 2.0, 1.9, 1.7, 2.2, 1.8, 3.2, 3.2, 2.7, 2.2, 2.2, 1.9, 1.9, 1.8, 2.7, 3.0, 2.3, 2.0, 2.0, 2.9, 2.9, 2.7, 2.7, 2.3, 2.6, 2.4, 1.8, 1.7, 1.5, 1.4, 2.1, 3.3, 3.5, 3.5, 3.1, 2.6, 2.1, 3.4, 3.0, 2.9.

Table 4: ML estimates and selection criteria for Dataset 2.

Model	Estimate	AIC	BIC	$-2\log L$	CAIC	HQIC
EXPD	0.4167	182.0450	183.9162	180.0450	182.1320	182.7521
LD	0.6667	166.2406	168.1118	164.2406	166.3275	166.9477
XLD	0.5772	174.4943	176.3655	172.4943	174.5812	175.2014
NXLD	0.6705	170.2710	172.1422	168.2710	170.3579	170.9781
XGD	0.9280	169.8614	171.7326	167.8614	169.9484	170.5686
BoD	0.9151	**139.0729**	**140.9441**	**137.0729**	**139.1599**	**139.7801**

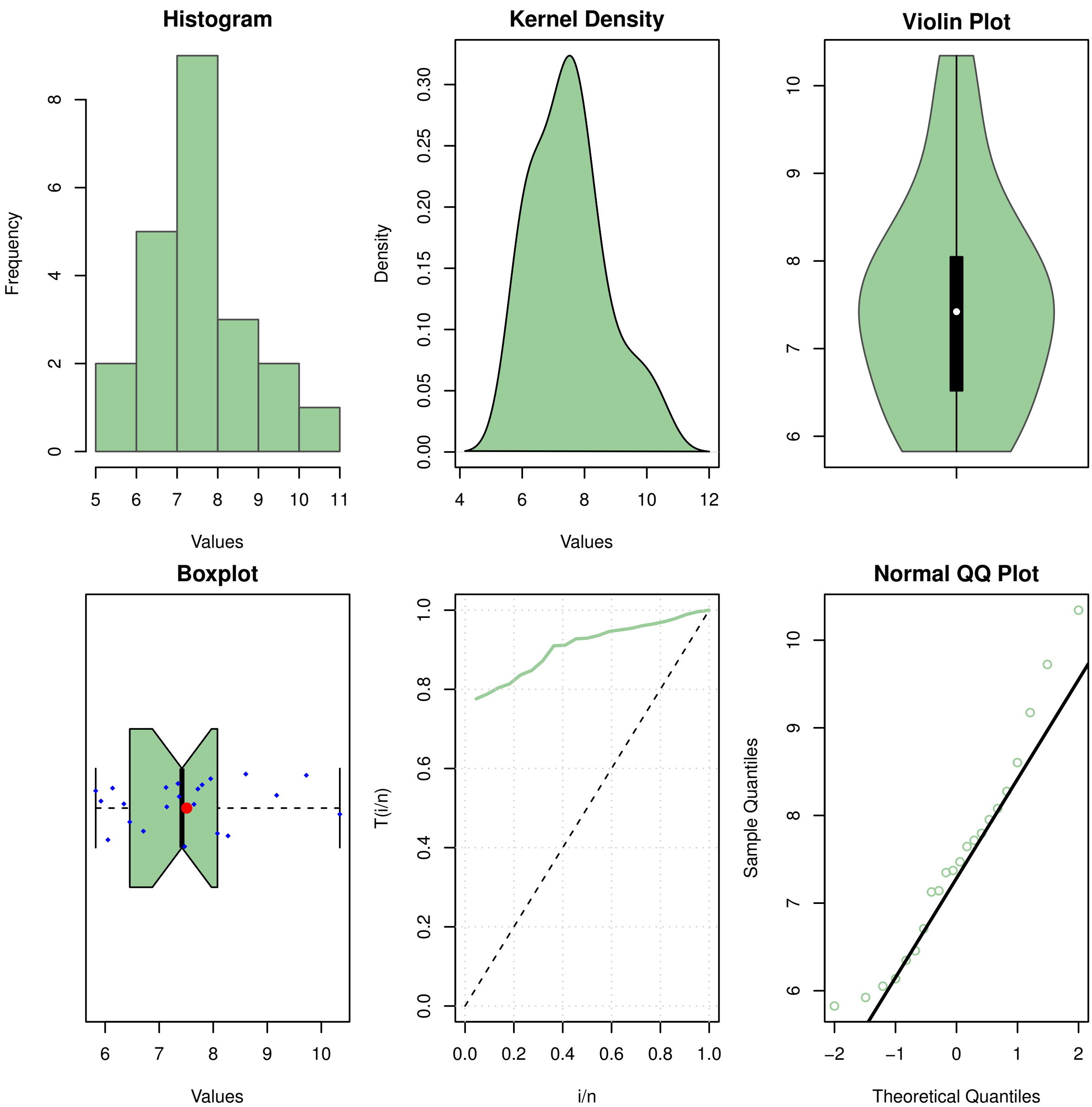

Figure 1: Exploratory Data Analysis of the Dataset 1

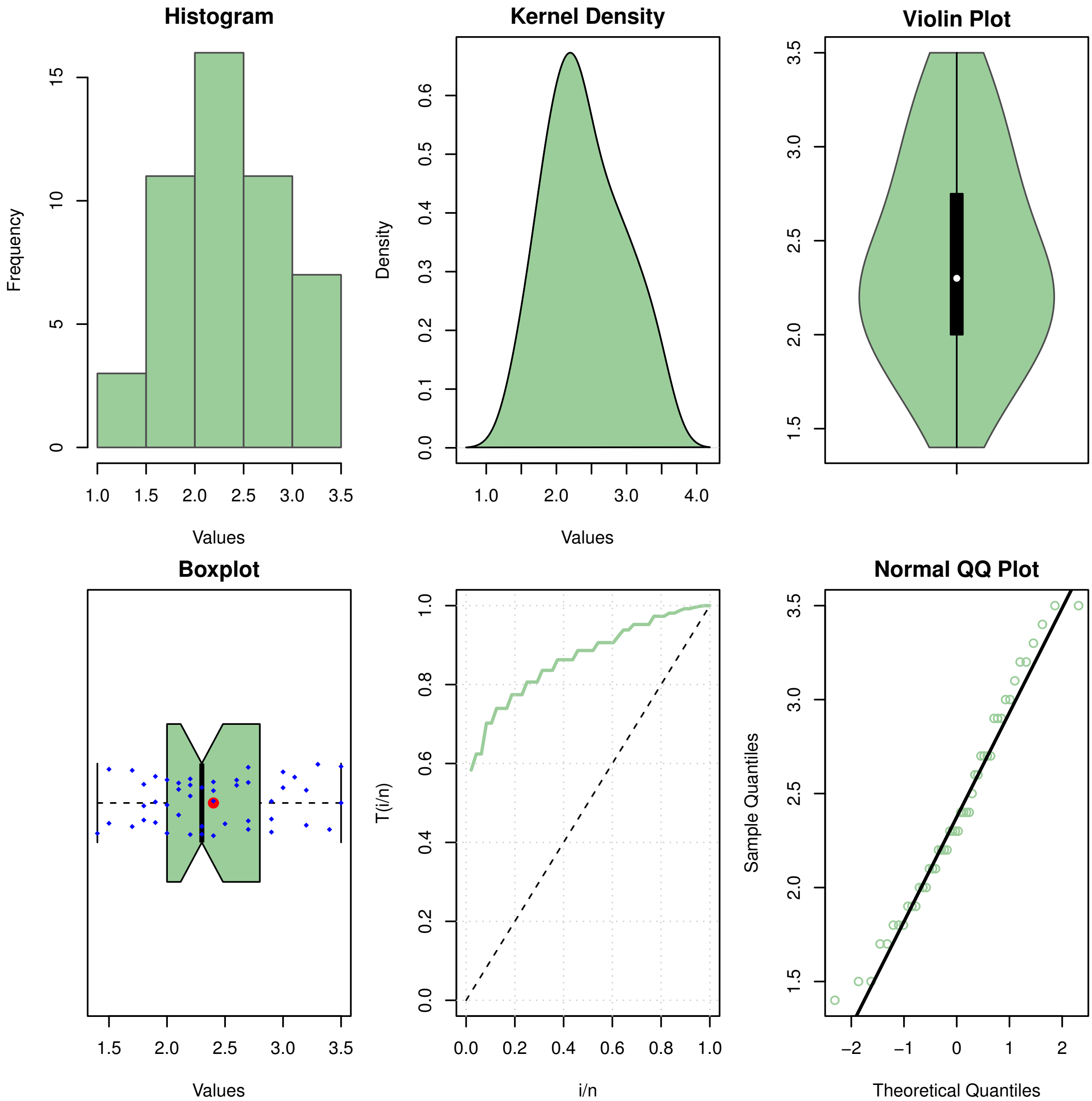

Figure 2: Exploratory Data Analysis of the Dataset 2

Figure 1 illustrates the distribution of exchange rates for Sweden (1985–2006) [12], whereas Figure 2 presents luteinizing hormone measurements [13]. In both cases, the BoD model achieves the smallest information criteria values, confirming its superior fit.

5. Conclusion

The Bouhadjar distribution, as an exponentiated extension of the NXLD [11], offers a flexible hazard rate function capable of modeling both increasing and bathtub-shaped patterns. Key findings include:

Theoretical results: Closed-form expressions were derived for the PDF, CDF, moments, Rényi entropy, Bonferroni and Lorenz curves, order statistics, and mean residual life.

Estimation: Parameters were estimated via MLE, LSE, and WLSE, with numerical methods applied to solve the score equations.

Simulation: MLE generally had the lowest MSE, while LSE performed best in bias for $\theta < 1$ and small n. WLSE performed worst overall.

Applications: In both datasets, the BoD outperformed competing models in all selection criteria.

The BoD thus represents a robust, flexible, and computationally tractable addition to the family of Lindley-type models, with potential applications in biomedical, engineering, and actuarial contexts.

Future work may explore:
- Two-parameter and compounded extensions.
- Bayesian inference and predictive modeling.
- Applications to censored or truncated survival data.

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