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Inverse Domination Number of a Graph: A Survey

S. R. Jayaram, M. A. Sriraj and P. Siva Kota Reddy*

ABSTRACT: A subset S of the vertex set of a graph G is a dominating set if every vertex in the complement of S is adjacent to some vertex of S. The minimum cardinality among all minimal dominating sets is the domination number of the graph G. For a minimal dominating set D of G if the graph does not have any isolated vertices, then $V - D$ contains a dominating set. Such a dominating set in $V - D$ is called an inverse dominating set of G and the minimum cardinality among all inverse dominating sets is called the inverse domination number. This paper surveys existing results on inverse dominating sets, inverse domination number and lists some open problems on these concepts. The Kulli-Sigarkanti conjecture on inverse domination number is settled.

Key Words: Graph, dominating set, independent domination number, inverse domination number, Kulli-Sigarkanti conjecture.

Contents

1. Introduction

A graph $G = (V(G), E(G))$ consists of the vertex set $V(G)$ and the edge set $E(G)$, with respective cardinalities n and m. In this context G is also called an (n, m) graph. In this paper by a graph we mean a simple graph without multiple edges, self-loops, finite and undirected. The notations and definitions of concepts are as in West [19]. The degree of a vertex is the number of edges incident to it. If the degree of a vertex is zero, then the vertex is called an isolated vertex or briefly an isolate. In this paper graph does not have any isolated vertex. A subset S of the vertex set is called a dominating set if every vertex in $V(G) - S$ is adjacent to some vertex in G. The notion of a dominating set was introduced by Berge [1] and Ore [17] independently. The cardinality of a minimum dominating set of the graph is the domination number of the graph and is denoted by $\gamma(G)$. These concepts are well studied by many including Haynes and Hedetniemi [10,11], Cockayne and Hedetniemi [4], Walikar et al. [18].

The mathematical study of dominating sets in graphs started around 1960. Historically it began in 1862 by De Jaenisch [6] who found the number of queens to dominate an $n \times n$ chessboard. Some problems studied in the class of recreational mathematics are covering, independent covering and independence. Some interesting applications of dominating sets and numbers are sets of representatives, school bus routing, computer communication networks, (r, d) configurations and social network theory to mention a few.

In 1991, Kulli and Sigarkanti [15] introduced the notion of inverse dominating set and inverse domination number of a graph and obtained the same for some special classes of graphs including paths and cycles. The said paper also characterizes graphs for which the sum of the domination number and inverse domination number is equal to the order of the graph.

For a minimum dominating set D of G, if $V(G) - D$ contains a dominating set D' then this dominating set D' is called an inverse dominating set of G. The cardinality of a minimum inverse dominating set is called the inverse domination number of the graph G and is denoted by $\gamma^{-1}(G)$ or $\gamma'(G)$. For simplicity

* Corresponding author.
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we shall denote it by $\gamma'(G)$.

The notion of inverse set and the corresponding inverse parameter has been introduced for many graph invariants such as inverse vertex covering number [16], inverse vertex independence number [2] and the inverse neighbourhood number [3]. A subset S of vertices of the graph is called a vertex covering number if $\langle S \rangle$ consists of all the edges of G, where $\langle S \rangle$ denotes the induced subgraph of G. In this case it is interesting to note $\langle V - S \rangle$ is a totally disconnected graph. A subset S of vertices of a graph G is independent if no two vertices of S are adjacent in G. If S is also a maximal independent set then S is also a minimal dominating set and is usually called an independent dominating set. In [7] it is proved that the independence number of the graph is an upper bound for the inverse domination number of a graph without isolates. This proof contains an error. This error leads to the mention of this result as a conjecture of Kulli and Sigarkanti. In this paper, this conjecture is proved and some open problems concerning the bounds for inverse domination number are listed.

Theorem 1.1 (Kulli [15]) *For any graph G, $\gamma'(G) \leq \alpha(G)$, equality holds for the complete graph.*

Proof: (As in [15]) Let G be a graph with inverse domination number $\gamma'(G)$ and vertex independence number $\alpha(G)$. Let D be a dominating set of G, as G does not have an isolated vertex $V - D$ contains a dominating set say $S' \subset V - D$. Consider two cases namely

a. case 1: $V - D - S = \Phi$ then $V - D = S$ and is an independent inverse dominating set and hence, $\gamma'(G) \leq \alpha(G)$, else,

b. case 2: $V - D - S \neq \Phi$.

In this case every vertex in $V - D - S$ is adjacent to at least one vertex in S. If every vertex in D is adjacent to at least one vertex in S, then S is an inverse dominating set of G. Else let $D' \subseteq D$ be a set of vertices in D such that no vertex of D' is adjacent to any vertex of S. Since D is a minimal dominating set, every vertex in D' must be adjacent to at least one vertex $V - D - S$. Let $S' \subseteq V - D - S$ be such that every vertex of D' is adjacent to at least one vertex in S'. Clearly $|S'| \leqslant |D'|$ and $S \cup S'$ is an inverse dominating set, hence,

$$\gamma'(G) \leq |S \cup S'| \leq |S \cup D'| = \alpha(G).$$

This proof has an error as observed in [7,9]. It is easy to see that for the cycle on 4 vertices ({1, 2, 3, 4}, {12, 23, 34, 41}) any subset consisting of adjacent pair of vertices say {2, 3} is a minimal dominating set and the complement of this set namely is {1, 4} contains $S = \{1\}$ as a maximal independent set and fails to be an inverse dominating set. This is also true for the path on 6 vertices P_6 and the complete bipartite graph $K_{m,m}$. □

Theorem 1.2 *The complete bipartite graph $K_{m,m}$, $m \geq 2$, is a counter example for the proof of Theorem 1.1.*

Proof: Let $K_{m,m}, m \geq 2$ be the bipartite graph with partite sets $V_1 = \{1, 2, 3, \ldots, m\}$ and $V_2 = \{m+1, m+2, m+3, \ldots, 2m\}$. The vertices of any edge, say $D = \{1, m+1\}$ form a minimal dominating set and $V - D = \{2, 3, 4, \ldots m, m+2, m+3, \ldots 2m\}$. $S = \{2, 3, 4, \ldots m\}$ or $S = \{m+2, \ldots, 2m\}$ are maximal independent sets of $V - D$ but S is not a dominating set. □

The error in the proof of the Theorem was noticed by many including Domke et al. [7] and Haynes et al. [10,11]. A second paper on the inverse domination number was published by Domke et al. [7]. In 2004, Domke et al. [7] formally stated this result of Kulli and Sigarkanti [15] as a conjecture:

Conjecture 1 (Domke et al. [7]) *If G is a graph with no isolated vertex, then $\gamma'(G) \leq \alpha(G)$.*

Now this Conjecture is restated as a Theorem and proved, thus settling the Kulli-Sigarkanti [15] and Domke et al. [7] conjecture.

Theorem 1.3 *For a graph G without isolated vertices $\gamma'(G) \leq \alpha(G)$.*

Proof: Let G be a graph without any isolated vertex. Let S be a maximal independent set of G, S is also a dominating set since every maximal independent set is a minimal dominating set. Clearly $V - S$ contains a dominating set of G, say D with $|D| = \gamma'(G)$ and $\gamma'(G) \leqslant |V - S|, D$ can be viewed as an inverse dominating set of G with respect to the dominating set S and the domination number of a graph is bounded above by the vertex independence number, hence $|D| \leq |S|$, and

$$\gamma'(G) \leq \alpha(G).$$

This Theorem yields the following results as Corollaries obtained in [12] for special families of graphs including claw-free graphs, bipartite graphs, split graphs, very well covered graphs, chordal graphs and cactus graphs. □

Corollary 1.1 (see 18) *If G is a graph with $\delta(G) \geq 1$ satisfying $\gamma'(G) = i(G)$, then $\gamma'(G) \leq \alpha(G)$.*

Corollary 1.2 *If G is a claw-free graph with $\delta(G) \geq 1$, then $\gamma'(G) \leq \alpha(G)$.*

Corollary 1.3 *If G is a graph with $\delta(G) \geq 1$ and $\alpha(G) = \Gamma(G)$, then $\gamma'(G) \leq \alpha(G)$.*

Corollary 1.4 *If G is a bipartite graph, a chordal graph, a circular arc graph, a permutation graph, or a comparability graph with $\delta(G) \geqslant 1$, then $\gamma'(G) \leq \alpha(G)$.*

Corollary 1.5 *If G is a split graph with no isolated vertex, then $\gamma'(G) \leq a(G)$.*

Corollary 1.6 (see [13]) *If $G \neq K_1$ is a generalized cactus graph, then $\gamma'(G) \leq \alpha(G)$.*

2. Bounds for the Inverse domination Number and Open Problems

In pursuit of settling the conjecture Krop et al. [14] obtained the following upper bound for the inverse domination number:

Theorem 2.1 *If G is a graph with no isolated vertices and G is not a clique, then $\gamma'(G) \leq 3/2a(G) - 1$.*

Kulli and Sigarkanti [15] characterized graphs in particular trees having equal domination number and inverse domination numbers. In [14], this work is extended by characterizing graphs with $\gamma(G)+\gamma'(G) = n$ and $\gamma(G) + \gamma'(G) = n - 1$. In [5], graphs having constant sum of domination and inverse domination numbers are studied and the following open problems are listed:

Problem 1 Characterize graphs with minimum degree1 and sum of the domination number and inverse domination numbers equal to $n - 1$.

Problem 2 Characterize graphs with minimum degree two or more and with sum of the domination number and inverse domination number equal to $n - 2$.

The inverse domination number is bonded below by the domination number, hence the following open problems:

Problem 3 Characterize graphs for which $\gamma'(G) = n - \Delta(G)$, where $\Delta(G)$ is the maximum degree of the graph.

Problem 4 Characterize graphs for which $n/(\Delta(G) + 1) + 1 = \gamma'(G)$. It is trivial to note this is true for a complete graph or a graph with a vertex of degree $n - 1$.

In [15], it is proved that the number of edges in the graph is at least $2n - 3\gamma(G)$, for G with equal domination and inverse domination number. Hence it is interesting to characterize graphs for which equality holds.

Problem 5 Characterize graphs with equal domination number and inverse domination number and $m = 2n - 3\gamma(G)$.

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S. R. Jayaram
Department of Mathematics
Mathematics Learning center
Shivamogga 577 205, India.
E-mail address: prof.srjayaram@gmail.com

and

M. A. Sriraj
Department of Mathematics
Vidyavardhaka College of Engineering
(Autonomous under Visvesvaraya Technological University, Belagavi-590 018, India.)
Mysuru-570 002, India.
E-mail address: masriraj@gmail.com; sriraj@vvce.ac.in

and

P. Siva Kota Reddy (Corresponding author)
Department of Mathematics
JSS Science and Technology University
Mysuru-570 006, India.

and

Universidad Bernardo O'Higgins
Facultad de Ingeniería, Ciencia y Tecnología
Departamento de Formación y Desarrollo Científico en Ingeniería
Av. Viel 1497, Santiago, Chile
E-mail address: pskreddy@jssstuniv.in